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ORIGINAL ARTICLE

On integral operators for certain classes of *p*-valent functions associated with generalized multiplier transformations

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Abstract In this paper, we study new generalized integral operators for the classes of p-valent functions associated with generalized multiplier transformations.

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Introduction

Let $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \quad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. We write $\mathcal{A}(1) = \mathcal{A}$.

For two functions f and g, analytic in U, we say that the function f is subordinate to g, if there exists a Schwarz function w, i.e, $w \in \mathcal{A}(p)$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. This subordination is usually denoted by $f(z) \prec g(z)$. It is well-known that if the function g is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [1]). If f is subordinate to g, then g is superordinate to f.

Let $\mathcal{P}_k(p, \rho)$ be the class of functions $g(z)$ analytic in U satisfying the properties $g(0) = p$ and

$$\int_0^{2\pi} \left| \frac{\Re\{g(z)\} - \rho}{p - \rho} \right| d\theta \leqslant k\pi, \tag{1.2}$$

where $z = re^{i\theta}$, $k \geqslant 2$ and $0 \leqslant \rho < p$. This class was introduced by Aouf [2, with $\lambda = 0$].

We note that:

(i) $\mathcal{P}_k(1, \rho) = \mathcal{P}_k(\rho) (k \geqslant 2,\ 0 \leqslant \rho < 1)$(see [3]);
(ii) $\mathcal{P}_k(1, 0) = \mathcal{P}_k\ (k \geqslant 2)$ (see [4,5]);
(iii) $\mathcal{P}_2(p, \rho) = \mathcal{P}(p, \rho)\ (0 \leqslant \rho < p, p \in \mathbb{N})$, where $\mathcal{P}(p, \rho)$ is the class of functions with positive real part greater than α (see [6]);

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(iv) $\mathcal{P}_2(p,0)=\mathcal{P}(p)(p\in\mathbb{N})$, where $\mathcal{P}(p)$ is the class of functions with positive real part (see [6]).

The classes $\mathcal{R}_k(p,\rho)$ and $\mathcal{V}_k(p,\rho)$ are related to the class $\mathcal{P}_k(p,\rho)$ and can be defined as

$$f\in\mathcal{R}_k(p,\rho)\iff\frac{zf'(z)}{f(z)}\in\mathcal{P}_k(p,\rho)\quad(z\in U),\tag{1.3}$$

and

$$f\in\mathcal{V}_k(p,\rho)\iff\frac{(zf'(z))'}{f'(z)}\in\mathcal{P}_k(p,\rho)\quad(z\in U).\tag{1.4}$$

Using the concept of subordination, Aouf [2], with $\alpha=0$ introduced the class $\mathcal{P}[p,A,B]$ as follows:

For A and B, $-1\leqslant B<A\leqslant1$, a function h analytic in U with $h(0)=p$ belongs to the class $\mathcal{P}[p,A,B]$ if h is subordinate to $p\frac{1+Az}{1+Bz}$.

Let $\mathcal{P}_k[p,A,B](k\geqslant2,-1\leqslant B<A\leqslant1)$ denote the class of p-valent analytic functions h that are represented by

$$h(z)=\left(\frac{k}{4}+\frac{1}{2}\right)h_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)h_2(z)\ (z\in U;\ h_1,h_2\in\mathcal{P}[p,A,B]).\tag{1.5}$$

Now we define the following classes $\mathcal{R}_k[p,A,B]$ and $\mathcal{V}_k[p,A,B]$ of the class $\mathcal{A}(p)$ for $k\geqslant2$ and $-1\leqslant B<A\leqslant1$ as follows:

$$\mathcal{R}_k[p,A,B]=\left\{f:f\in\mathcal{A}(p)\text{ and }\frac{zf'(z)}{f(z)}\in\mathcal{P}_k[p,A,B],\ z\in U\right\},\tag{1.6}$$

and

$$\mathcal{V}_k[p,A,B]=\left\{f:f\in\mathcal{A}(p)\text{ and }\frac{(zf'(z))'}{f'(z)}\in\mathcal{P}_k[p,A,B],\ z\in U\right\}\tag{1.7}$$

Obviously, we know that

$$f(z)\in\mathcal{V}_k[p,A,B]\iff\frac{zf'(z)}{p}\in\mathcal{R}_k[p,A,B].\tag{1.8}$$

We note that $\mathcal{P}_k[1,A,B]=\mathcal{P}_k[A,B]$, $\mathcal{R}_k[1,A,B]=\mathcal{R}_k[A,B]$ and $\mathcal{V}_k[1,A,B]=\mathcal{V}_k[A,B]$ (see [7]).

Prajapat [8] defined a generalized multiplier transformation operator $\mathcal{J}_p^m(\lambda,\ell):\mathcal{A}(p)\to\mathcal{A}(p)$, as follows:

$$\mathcal{J}_p^m(\delta,\ell)f(z)=z^p+\sum_{j=p+1}^{\infty}\left(\frac{p+\ell+(j-p)\delta}{p+\ell}\right)^m a_kz^k$$
$$(\delta\geqslant0;\ \ell>-p;\ p\in\mathbb{N};\ m\in\mathbb{Z}=\{0,\pm1,\ldots\};\ z\in U).\tag{1.9}$$

It is readily verified from (1.3) that

$$\delta z(\mathcal{J}_p^m(\delta,\ell)f(z))'=(\ell+p)\mathcal{J}_p^{m+1}(\delta,\ell)f(z)-[\ell+p(1-\delta)]\mathcal{J}_p^m(\delta,\ell)f(z)\ (\delta>0).\tag{1.10}$$

By specializing the parameters m, δ, ℓ and p, we obtain the following operators studied by various authors:

(i) $\mathcal{J}_p^m(\delta,\ell)f(z)=I_p^m(\delta,\ell)f(z)$ ($\ell\geqslant0,p\in\mathbb{N},\delta\geqslant0$ and $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$) (see [9]);
(ii) $\mathcal{J}_p^m(1,\ell)f(z)=I_p(m,\ell)f(z)$ ($\ell\geqslant0,p\in\mathbb{N}$ and $m\in\mathbb{N}_0$) (see [10,11]);
(iii) $\mathcal{J}_p^m(\delta,0)f(z)=D_{\delta,p}^mf(z)$($\delta\geqslant0,p\in\mathbb{N}$ and $m\in\mathbb{N}_0$) (see [12]);
(iv) $\mathcal{J}_p^m(1,0)f(z)=D_p^mf(z)$ ($m\in\mathbb{N}_0$ and $p\in\mathbb{N}$) (see [13–15]);
(v) $\mathcal{J}_p^{-m}(\delta,\ell)f(z)=J_p^m(\delta,\ell)f(z)$ ($\ell\geqslant0,\delta\geqslant0$, $p\in\mathbb{N}$ and $m\in\mathbb{N}_0$) (see [16–18]);
(vi) $\mathcal{J}_p^{-m}(1,1)f(z)=D^mf(z)$ ($m\in\mathbb{Z}$) (see [19]);
(vii) $\mathcal{J}_1^m(1,\ell)f(z)=I_\ell^mf(z)$ ($\ell\geqslant0$ and $m\in\mathbb{N}_0$) (see [20,21]);
(viii) $\mathcal{J}_1^m(\delta,0)f(z)=D_\delta^mf(z)$ ($\delta\geqslant0$ and $m\in\mathbb{N}_0$) (see [22]);
(ix) $\mathcal{J}_1^m(1,0)f(z)=D^mf(z)$ ($m\in\mathbb{N}_0$) (see [23]);
(x) $\mathcal{J}_1^{-m}(\delta,0)f(z)=I_\delta^{-m}f(z)$ ($\delta\geqslant0$ and $m\in\mathbb{N}_0$) (see [24,25]);
(xi) $\mathcal{J}_1^{-m}(1,1)f(z)=I^mf(z)$ ($m\in\mathbb{N}_0$) (see [26]).

Let us consider the integral operators:

$$\mathcal{F}_{p,\delta,\ell}^{n,m}(z)=\int_0^z pt^{p-1}\left(\frac{\mathcal{J}_p^m(\delta,\ell)f_1(t)}{t^p}\right)^{\alpha_1}\ldots\left(\frac{\mathcal{J}_p^m(\delta,\ell)f_n(t)}{t^p}\right)^{\alpha_n}dt\tag{1.11}$$

and

$$\mathcal{G}_{p,\delta,\ell}^{n,m}(z)=\int_0^z pt^{p-1}\left(\frac{(\mathcal{J}_p^m(\delta,\ell)f_1(t))'}{pt^{p-1}}\right)^{\beta_1}\ldots\left(\frac{(\mathcal{J}_p^m(\delta,\ell)f_n(t))'}{pt^{p-1}}\right)^{\beta_n}dt,\tag{1.12}$$

where $f_i(z)\in\mathcal{A}(p)$ and α_i, $\beta_i>0$ for $i=\{1,2,\ldots,n\}$.

We note that:

(i) $\mathcal{F}_{p,\delta,\ell}^{n,0}(z)=F_p(z)$ and $\mathcal{G}_{p,\delta,\ell}^{n,0}(z)=G_p(z)$ (see [27,28]);
(ii) $\mathcal{F}_{p,\delta,0}^{n,m}(z)=F_n(z)$ (see [29,30]).

Also, we note that

(i) $$\mathcal{F}_{p,\delta,\ell}^{n,m}(z)=\mathcal{I}_{p,\delta,\ell}^{n,m}(z)=\int_0^z pt^{p-1}\left(\frac{I_p^m(\delta,\ell)f_1(t)}{t^p}\right)^{\alpha_1}\ldots\left(\frac{I_p^m(\delta,\ell)f_n(t)}{t^p}\right)^{\alpha_n}dt$$
$$(\ell\geqslant0;\ \delta\geqslant0;\ p\in\mathbb{N};\ m\in\mathbb{N}_0)\tag{1.13}$$

and

$$\mathcal{G}_{p,\delta,\ell}^{n,m}(z)=G_{p,\delta,\ell}^{n,m}(z)=\int_0^z pt^{p-1}\left(\frac{\left(I_p^m(\delta,\ell)f_1(t)\right)'}{pt^{p-1}}\right)^{\beta_1}\ldots\left(\frac{\left(I_p^m(\delta,\ell)f_n(t)\right)'}{pt^{p-1}}\right)^{\beta_n}dt$$
$$(\ell\geqslant0;\ \delta\geqslant0;\ p\in\mathbb{N};\ m\in\mathbb{N}_0);\tag{1.14}$$

(ii)
$$\mathcal{F}_{p,\delta,0}^{n,m}(z)=\mathcal{D}_{p,\delta}^{n,m}(z)=\int_0^z pt^{p-1}\left(\frac{D_{\delta,p}^mf_1(t)}{t^p}\right)^{\alpha_1}\ldots\left(\frac{D_{\delta,p}^mf_n(t)}{t^p}\right)^{\alpha_n}dt$$
$$(\delta\geqslant0;\ p\in\mathbb{N};\ m\in\mathbb{N}_0)\tag{1.15}$$

and

$$\mathcal{G}^{n,m}_{p,\delta,0}(z) = G^{n,m}_{p,\delta}(z) = \int_0^z pt^{p-1}\left(\frac{(D^m_{\delta,p}f_1(t))'}{pt^{p-1}}\right)^{\beta_1}\cdots\left(\frac{\left(D^m_{\delta,p}f_n(t)\right)'}{pt^{p-1}}\right)^{\beta_n} dt$$

$$(\delta \geqslant 0;\ p \in \mathbb{N};\ m \in \mathbb{N}_0); \tag{1.16}$$

(iii)

$$\mathcal{F}^{n,m}_{p,1,0}(z) = \mathcal{D}^{n,m}_p(z) = \int_0^z pt^{p-1}\left(\frac{D^m_p f_1(t)}{t^p}\right)^{\alpha_1}\cdots\left(\frac{D^m_p f_n(t)}{t^p}\right)^{\alpha_n} dt$$

$$(p \in \mathbb{N};\ m \in \mathbb{N}_0) \tag{1.17}$$

and

$$\mathcal{G}^{n,m}_{p,1,0}(z) = G^{n,m}_p(z) = \int_0^z pt^{p-1}\left(\frac{\left(D^m_p f_1(t)\right)'}{pt^{p-1}}\right)^{\beta_1}\cdots\left(\frac{\left(D^m_p f_n(t)\right)'}{pt^{p-1}}\right)^{\beta_n} dt$$

$$(p \in \mathbb{N};\ m \in \mathbb{N}_0). \tag{1.18}$$

1. Main results

Unless otherwise mentioned, we assume throughout this paper that:

$k \geqslant 2,\ \delta \geqslant 0,\ \ell > -p,\ p \in \mathbb{N},\ m \in \mathbb{Z},\ z \in U$ and $-1 \leqslant B < A \leqslant 1$.

Theorem 1. *Let $\mathcal{J}^m_p(\delta,\ell)f_i(z) \in \mathcal{R}_k[p,A,B]$, for $1 \leqslant i \leqslant n$ and $\sum_{i=1}^n \alpha_i = 1$. Then, the integral operator $\mathcal{F}^{n,m}_{p,\delta,\ell}(z)$ defined by (1.11), also belongs to the class $\mathcal{V}_k[p,A,B]$.*

Proof. From (1.11), we have

$$\left(\mathcal{F}^{n,m}_{p,\delta,\ell}(z)\right)' = pz^{p-1}\left[\left(\frac{\mathcal{J}^m_p(\delta,\ell)f_1(z)}{z^p}\right)^{\alpha_1}\cdots\left(\frac{\mathcal{J}^m_p(\delta,\ell)f_n(z)}{z^p}\right)^{\alpha_n}\right]. \tag{2.1}$$

Differentiating (2.1) logarithmically with respect to z and multiplying by z, we obtain

$$\frac{z(\mathcal{F}^{n,m}_{p,\delta,\ell}(z))''}{\left(\mathcal{F}^{n,m}_{p,\delta,\ell}(z)\right)'} = p - 1 + \sum_{i=1}^n \alpha_i\left(\frac{z(\mathcal{J}^m_p(\delta,\ell)f_i(z))'}{\mathcal{J}^m_p(\delta,\ell)f_i(z)} - p\right)$$

thus

$$1 + \frac{z(\mathcal{F}^{n,m}_{p,\delta,\ell}(z))''}{\left(\mathcal{F}^{n,m}_{p,\delta,\ell}(z)\right)'} = p + \sum_{i=1}^n \alpha_i\left(\frac{z\left(\mathcal{J}^m_p(\delta,\ell)f_i(z)\right)'}{\mathcal{J}^m_p(\delta,\ell)f_i(z)} - p\right)$$

or equivalently

$$\frac{\left(z\left(\mathcal{F}^{n,m}_{p,\delta,\ell}(z)\right)'\right)'}{\left(\mathcal{F}^{n,m}_{p,\delta,\ell}(z)\right)'} = \sum_{i=1}^n \alpha_i\left(\frac{z\left(\mathcal{J}^m_p(\delta,\ell)f_i(z)\right)'}{\mathcal{J}^m_p(\delta,\ell)f_i(z)}\right) = \left(\frac{k}{4}+\frac{1}{2}\right)\left(\sum_{i=1}^n \alpha_i p_i(z)\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(\sum_{i=1}^n \alpha_i q_i(z)\right), \tag{2.2}$$

where $p_i, q_i \in \mathcal{P}[p,A,B]$, for all $i = 1, 2, \ldots, n$. Since $\mathcal{P}[p,A,B]$ is a convex set (see [31]), it follows that

$$\frac{\left(z\left(\mathcal{F}^{n,m}_{p,\delta,\ell}(z)\right)'\right)'}{\left(\mathcal{F}^{n,m}_{p,\delta,\ell}(z)\right)'} = \left(\frac{k}{4}+\frac{1}{2}\right)H_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)H_2(z),$$

where $H_1, H_2 \in \mathcal{P}[p,A,B]$ and therefore,

$$\frac{\left(z\left(\mathcal{F}^{n,m}_{p,\delta,\ell}(z)\right)'\right)'}{\left(\mathcal{F}^{n,m}_{p,\delta,\ell}(z)\right)'} \in \mathcal{P}_k[p,A,B] \quad (z \in U).$$

This proves the result. □

Remark 1.

(i) Putting $p = 1$ and $m = 0$ in Theorem 1, we obtain the result obtained by Vijayvargy et al. [7], Theorem 2.3 (i);
(ii) Putting $p = 1, m = 0, n = 2, \alpha_1 = \alpha$ and $\alpha_2 = \beta$ in Theorem 1, we obtain the result obtained by Noor [32].

Putting $m = 0$ in Theorem 1, we obtain the following corollary:

Corollary 1. *Let $f_i(z) \in \mathcal{R}_k[p,A,B]$, for $1 \leqslant i \leqslant n$ and $\sum_{i=1}^n \alpha_i = 1$. Then, the integral operator $F_p(z)$ belongs to the class $\mathcal{V}_k[p,A,B]$.*

Putting $n = 1$, $\alpha_1 = 1$ and $f_1 = f$ in Theorem 1, we obtain the following corollary:

Corollary 2. *Let $\mathcal{J}^m_p(\delta,\ell)f(z) \in \mathcal{R}_k[p,A,B]$. Then, the integral operator $\int_0^z p\left(\frac{\mathcal{J}^m_p(\delta,\ell)f(t)}{t}\right)dt \in \mathcal{V}_k[p,A,B]$.*

Putting $m \in \mathbb{N}_0$ in Theorem 1, we obtain the following corollary:

Corollary 3. *Let $I^m_p(\delta,\ell)f_i(z) \in \mathcal{R}_k[p,A,B]$, for $1 \leqslant i \leqslant n$ and $\sum_{i=1}^n \alpha_i = 1$. Then, the integral operator $\mathcal{I}^{n,m}_{p,\delta,\ell}(z)$ defined by (1.13), also belongs to the class $\mathcal{V}_k[p,A,B]$.*

Putting $\ell = 0$ in Corollary 3, we obtain the following corollary:

Corollary 4. *Let $D^m_{\delta,p}f_i(z)f_i(z) \in \mathcal{R}_k[p,A,B]$, for $1 \leqslant i \leqslant n$ and $\sum_{i=1}^n \alpha_i = 1$. Then, the integral operator $\mathcal{D}^{n,m}_{p,\delta}(z)$ defined by (1.15), also belongs to the class $\mathcal{V}_k[p,A,B]$.*

Putting $\ell = 0$ and $\delta = 1$ in Corollary 3, we obtain the following corollary:

Corollary 5. *Let $D_p^m f_i(z) \in \mathcal{R}_k[p, A, B]$, for $1 \leqslant i \leqslant n$ and $\sum_{i=1}^n \alpha_i = 1$. Then, the integral operator $\mathcal{D}_p^{n,m}(z)$ defined by (1.17), also belongs to the class $\mathcal{V}_k[p, A, B]$.*

Theorem 2. *Let $\mathcal{J}_p^m(\delta, \ell) f_i(z) \in \mathcal{V}_k[p, A, B]$, for $1 \leqslant i \leqslant n$ and $\sum_{i=1}^n \beta_i = 1$. Then, the integral operator $\mathcal{G}_{p,\delta,\ell}^{n,m}(z)$ defined by (1.12), also belongs to the class $\mathcal{V}_k[p, A, B]$.*

Proof. From (1.12), we have

$$\left(\mathcal{G}_{p,\delta,\ell}^{n,m}(z)\right)' = pz^{p-1}\left[\left(\frac{(\mathcal{J}_p^m(\delta,\ell)f_1(z))'}{pz^{p-1}}\right)^{\beta_1} \cdots \left(\frac{\left(\mathcal{J}_p^m(\delta,\ell)f_n(z)\right)'}{pz^{p-1}}\right)^{\beta_n}\right]. \tag{2.3}$$

Differentiating (2.3) logarithmically and multiplying by z, we obtain

$$\frac{z\left(\mathcal{G}_{p,\delta,\ell}^{n,m}(z)\right)''}{\left(\mathcal{G}_{p,\delta,\ell}^{n,m}(z)\right)'} = p - 1 + \sum_{i=1}^n \beta_i \left(\frac{z\left(\mathcal{J}_p^m(\delta,\ell)f_i(z)\right)''}{\left(\mathcal{J}_p^m(\delta,\ell)f_i(z)\right)'} - (p-1)\right)$$

thus

$$1 + \frac{z\left(\mathcal{G}_{p,\delta,\ell}^{n,m}(z)\right)''}{\left(\mathcal{G}_{p,\delta,\ell}^{n,m}(z)\right)'} = \sum_{i=1}^n \beta_i \left(1 + \frac{z\left(\mathcal{J}_p^m(\delta,\ell)f_i(z)\right)''}{\left(\mathcal{J}_p^m(\delta,\ell)f_i(z)\right)'}\right),$$

or equivalently

$$\begin{aligned}\frac{\left[z\left(\mathcal{G}_{p,\delta,\ell}^{n,m}(z)\right)'\right]'}{\left[\mathcal{G}_{p,\delta,\ell}^{n,m}(z)\right]'} &= \sum_{i=1}^n \beta_i \left(\frac{\left[z\left(\mathcal{J}_p^m(\delta,\ell)f_i(z)\right)'\right]'}{\left[\mathcal{J}_p^m(\delta,\ell)f_i(z)\right]'}\right) \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)\left(\sum_{i=1}^n \beta_i p_i(z)\right) \\ &\quad - \left(\frac{k}{4} - \frac{1}{2}\right)\left(\sum_{i=1}^n \beta_i q_i(z)\right),\end{aligned} \tag{2.4}$$

where $p_i, q_i \in \mathcal{P}[p, A, B]$, for all $i = 1, 2, \ldots, n$. Since $\mathcal{P}[p, A, B]$ is a convex set (see [19]), it follows that

$$\frac{\left[z\left(\mathcal{G}_{p,\delta,\ell}^{n,m}(z)\right)'\right]'}{\left[\mathcal{G}_{p,\delta,\ell}^{n,m}(z)\right]'} = \left(\frac{k}{4} + \frac{1}{2}\right)H_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)H_2(z),$$

where $H_1,\ H_2 \in \mathcal{P}[p, A, B]$ and therefore,

$$\frac{\left[z\left(\mathcal{G}_{p,\delta,\ell}^{n,m}(z)\right)'\right]'}{\left[\mathcal{G}_{p,\delta,\ell}^{n,m}(z)\right]'} \in \mathcal{P}_k[p, A, B] \quad (z \in U).$$

This implies that $\mathcal{G}_{p,\delta,\ell}^{n,m}(z) \in \mathcal{V}_k[p, A, B]$. □

Remark 2. Putting $p = 1$ and $m = 0$ in Theorem 2, we obtain the result obtained by Vijayvargy et al. [7], Theorem 2.9 (i).

Putting $m = 0$ in Theorem 2, we obtain the following corollary:

Corollary 6. *Let $f_i(z) \in \mathcal{V}_k[p, A, B]$, for $1 \leqslant i \leqslant n$ and $\sum_{i=1}^n \beta_i = 1$. Then, the integral operator $G_p(z)$ also belongs to the class $\mathcal{V}_k[p, A, B]$.*

Putting $m \in \mathbb{N}_0$ in Theorem 2, we obtain the following corollary:

Corollary 7. *Let $I_p^m(\delta, \ell) f_i(z) \in \mathcal{V}_k[p, A, B]$, for $1 \leqslant i \leqslant n$ and $\sum_{i=1}^n \beta_i = 1$. Then, the integral operator $G_{p,\delta,\ell}^{n,m}(z)$ defined by (1.14), also belongs to the class $\mathcal{V}_k[p, A, B]$.*

Putting $\ell = 0$ in Corollary 7, we obtain the following corollary:

Corollary 8. *Let $D_{\delta,p}^m f_i(z) \in \mathcal{V}_k[p, A, B]$, for $1 \leqslant i \leqslant n$ and $\sum_{i=1}^n \beta_i = 1$. Then, the integral operator $G_{p,\delta}^{n,m}(z)$ defined by (1.16), also belongs to the class $\mathcal{V}_k[p, A, B]$.*

Putting $\ell = 0$ and $\delta = 1$ in Corollary 7, we obtain the following corollary:

Corollary 9. *Let $D_p^m f_i(z) \in \mathcal{V}_k[p, A, B]$, for $1 \leqslant i \leqslant n$ and $\sum_{i=1}^n \beta_i = 1$. Then, the integral operator $G_p^{n,m}(z)$ defined by (1.18), also belongs to the class $\mathcal{V}_k[p, A, B]$.*

Remark 3. By specializing the parameters k, A, B, ℓ, δ, p and m, we obtain various results for different operators defined in the introduction.

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