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ORIGINAL ARTICLE

A critical remark on "Fixed point theorems for occasionally weakly compatible mappings"

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Abstract In the present paper, we show that under contractive conditions, the existence of a common fixed point and occasional weak compatibility are equivalent conditions. We also show that contractive conditions employed by Jungck and Rhoades [Fixed point theorems for occasionally weakly compatible mappings, Fixed Point Theory 7(2) (2006) 287–296; Fixed Point Theory 9 (2008) 383–384 (erratum)] do not provide a nontrivial setting for the application of occasional weak compatible mappings. Finally, we improve the results of Jungck and Rhoades by employing a proper setting.

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1. Introduction and preliminaries

In a recent work, Jungck and Rhoades [1] employed the notion of occasionally weakly compatible mappings introduced by Al-Thagafi and Shahzad [2] to prove fixed point theorems under contractive conditions for pair of mappings. A pair (f,S) of self-mappings defined on a nonempty set X is said to be occasionally weakly compatible [2] (in short owc) if there exists a point x in X which is a coincidence point of f and S at which f and S commute. A point x satisfying $fx = Sx$ is called a coincidence point of f and S. Thus, if f and S are owc mappings such that $fx = Sx$, $fSx = Sfx$ for some x then both x and $fx(=Sx)$ are coincidence points of f and S.

Now suppose that f and S satisfy some contractive condition. If f and S have a common fixed point, say z, then $z = fz = Sz$, $fSz = Sfz = z$ and f and S are, therefore, owc mappings. On the other hand, if f and S are owc mappings such that $fx = Sx$ and $fSx = Sfx$ for some x then, since contractive conditions exclude the existence of two coincidence points x,y for f and S such that $fx \neq fy$, we get $fx = ffx(=Sfx)$. This means that $fx = Sx$ is a common fixed point of f and S.

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Therefore, under contractive conditions the existence of a common fixed point and occasional weak compatibility are equivalent conditions, and consequently, proving existence of fixed points by assuming owc is equivalent to proving the existence of fixed points by assuming the existence of fixed points.

In view of this, proving fixed point theorems for owc mappings under contractive conditions reduces to a redundant exercise. We thus see that contractive conditions do not provide a proper setting for the application of the concept of owc and for proper applications of the notion of owc one should look to mappings satisfying nonexpansive condition, Lipschitz type condition or some other general condition. Moreover, owc mappings can be divided in two categories:

(1) Mappings commuting at all the coincidence points, and
(2) mappings commuting on a proper subset of the set of coincidences.

In the first case, the mappings are obviously pointwise R-weakly commuting [3] or equivalently weakly compatible [4]. In the second case, the mappings are noncompatible. Therefore, a proper setting for the application of owc should allow the existence of multiple fixed points or multiple coincidence points with distinct functional values and the classes of mappings that allow such possibility include:

(i) Noncompatible mappings satisfying nonexpansive or Lipschitz type conditions.
(ii) Weakly compatible mappings satisfying nonexpansive or Lipschitz type conditions and (E.A.) property [5].

Before proceeding further, we recall some relevant concepts and results.

Definition 1.1 [1]. Let X be a nonempty set. A symmetric on X is a mapping $r{:}X\times X\times[0,\infty)$ such that

$$r(x,y)=0 \text{ if } x=y,\quad \text{and}\quad r(x,y)=r(y,x)\ \forall x,y\in X. \tag{1}$$

Definition 1.2 [3]. Two self-mappings f and S of a metric space (X,d) are called pointwise R – weakly commuting on X if given x in X there exists $R>0$ such that $d(fSx,Sfx)\leqslant Rd(fx,Sx)$.

Definition 1.3 [4]. Two self-mappings f and S of a metric space (X,d) are said to be weakly compatible if they commute at their coincidence points, that is, if $fx=Sx$ for x in X, then $fSx=Sfx$.

Definition 1.4 [5]. Let f and S be two self-mappings of a metric space (X,d). The maps f and S satisfy the (E. A.) property if there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n=\lim_n Sx_n=t$ for some t in X.

In a recent work, Jungck and Rhoades [1] proved the following theorem in symmetric metric space.

Theorem 1.1 [1]. *Let (X,d) be a symmetric space with symmetric r and f, S self-maps of X such that $f(X)\subset S(X)$, f and S are owc, and*

$$\begin{aligned} r(fx,fy)\leqslant{}& ar(Sx,Sy)+b\max\{r(fx,Sx),r(fy,Sy)\}\\ &+c\max\{r(Sx,Sy),r(Sx,fx),r(Sy,fy)\} \end{aligned} \tag{2}$$

for all x, $y\in X$, where a, b, $c>0$, $a+b+c=1$ and $a+c<\sqrt{a}$. Then f and Shave a unique common fixed point.

It may be observed that under the contractive condition (2) of Theorem 1.1, assumption of owc and the existence of a unique common fixed point are equivalent conditions. To see this, first suppose that f and$S$ have a unique common fixed point z then, as already discussed above, f and S are owc.

On the other hand, if f and S are owc mappings, then there exists a u in X such that $fu=Su$ and $fSu=Sfu(=ffu=SSu)$. Condition (2) now straight away implies that

$$\begin{aligned} r(fu,ffu)\leqslant{}& ar(Su,Sfu)+b\max\{r(fu,Su),r(ffu,Sfu)\}\\ &+c\max\{r(Su,Sfu),r(Su,fu),r(Sfu,ffu)\}, \end{aligned}$$

that is, $fu=ffu=Sfu$ and fu is a common unique fixed point of f and S. We thus see that under the contractive condition (2) of Theorem 1.1, assumption of owc and the existence of a unique common fixed point are equivalent conditions.

This shows that contractive conditions do not provide a nontrivial setting for the application of owc. There can be a possible approach to remedy the situation and improve the results of Jungck and Rhoades [1].

(a). To replace the contractive condition by more general conditions that may hold for mappings satisfying contractive as well as nonexpansive and Lipschitz type conditions. We adopt this approach in the next theorem (Theorem 1.2) for improving results of Jungck and Rhoades [1].

Theorem 1.2. *Let (X,d) be a symmetric space with symmetric r and f and S are occasionally weakly compatible self-mappings of X satisfying*

$$r(fx,f^2x)\neq\max\{r(fx,Sfx),r(f^2x,Sfx),r(f^2x,S^2x)\}, \tag{iii}$$

whenever the right hand side is nonzero. Then, f and S have a common fixed point.

Proof. Since f and S are owc, there exists a point u in X such that $fu=Su$ and $fSu=Sfu$. This in turn yields $ffu=fSu=Sfu=SSu$. If $fu\neq f^2u$ then using (iii) we get $r(fu,f^2u)\neq$ $(fu,f^2u)\neq\max\{r(fu,Sfu),r(f^2u,Sfu),r(f^2u,S^2u)\}=r(fu,f^2u)$, a contradiction. Hence, $fu=fSu=SSu$ and fu is a common fixed point of f and S. □

Remark 1.1. Contractive conditions employed in Theorems 1, 3 and 5 respectively of Jungck and Rhoades [1] also imply the equivalence of assumption of owc and the existence of a unique common fixed point.

Remark 1.2. Theorem 1.2 also remains true if we replace condition (iii) by any one of the following:

$$r(fx,f^2x)\neq r(Sx,S^2x), \tag{3}$$

$$\begin{aligned} r(fx,f^2x)\neq\max\{&r(Sx,Sfx),r(fx,Sx),r(f^2x,Sfx),\\ &r(fx,Sfx),r(Sx,f^2x)\}, \end{aligned} \tag{4}$$

$$\begin{aligned} r(Sx,S^2x)\neq\max\{&r(fx,fSx),r(Sx,fx),r(S^2x,\\ &fSx),r(Sx,fSx),r(fx,S^2x)\}, \end{aligned} \tag{5}$$

$$r(x, fx) \neq \max\{r(x, Sx), r(fx, Sx)\}, \tag{6}$$

$$r(x, Sx) \neq \max\{r(x, fx), r(Sx, fx)\}, \tag{7}$$

$$r(fx, f^2x) \neq r(fx, S^2x) + r(S^2x, fSx) + r(fSx, f^2x), \tag{8}$$

whenever the right hand side is nonzero.

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