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Application of generalized Hadamard product on special classes of analytic univalent functions

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Abstract In this paper, the author established certain results concerning the quasi-Hadamard product for generalized subclasses of univalent functions with positive coefficients.

MATHEMATICS SUBJECT CLASSIFICATION: 30C45, 30C55, 30C80

1. Introduction

Let A denote the class of analytic univalent functions in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

A function $f(z) \in A$ is called starlike of order α if $f(z)$ satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \tag{1.2}$$

for $0 \leqslant \alpha < 1$ and $z \in U$. We denote by $S^*(\alpha)$ the class of all starlike functions of order α. Also, a function $f(z) \in A$ is called convex of order α if $f(z)$ satisfies

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \tag{1.3}$$

for $0 \leqslant \alpha < 1$ and $z \in U$. We denote by $C(\alpha)$ the class of convex functions of order α.

For $\beta > 1$ and $z \in U$, let $\mathfrak{M}(\beta)$ denotes the subclass of A consisting of functions $f(z)$ of the form (1.1) and satisfying the condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < \beta, \tag{1.4}$$

and let $\mathfrak{N}(\beta)$ denote the subclass of A consisting of functions $f(z)$ of the form (1.1) and satisfying the condition

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta, \tag{1.5}$$

it follows from (1.4) and (1.5) that

$$f(z) \in \mathfrak{N}(\beta) \Longleftrightarrow zf'(z) \in \mathfrak{M}(\beta). \tag{1.6}$$

The subclasses $\mathfrak{M}(\beta)$ and $\mathfrak{N}(\beta)$ and some related classes have been studied by several authors (e.g. [1–4]).

Furthermore, let V denote the subclass of analytic univalent functions of the form:

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$$f(z) = a_0 z + \sum_{n=2}^{\infty} |a_n| z^n \quad (a_0 > 0). \tag{1.7}$$

Also, let

$$f_i(z) = a_{0,i} z + \sum_{n=2}^{\infty} |a_{n,i}| z^n \quad (a_{0,i} > 0), \tag{1.8}$$

and

$$g_j(z) = b_{0,j} z + \sum_{n=2}^{\infty} |b_{n,j}| z^n \quad (b_{0,j} > 0), \tag{1.9}$$

the quasi-Hadamard product $(f_i * g_j)(z)$ of the functions $f_i(z)$ and $g_j(z)$ by

$$(f_i * g_j)(z) = a_{0,i} b_{0,j} z + \sum_{n=2}^{\infty} |a_{n,i}||b_{n,j}| z^n \quad (i,j \in \mathbb{N} = 1,2,3,\ldots).$$

Similarly, we can define the quasi-Hadamard product of more than two functions.

Also, let $V(\beta) = \mathfrak{M}(\beta) \cap V$ and $U(\beta) = \mathfrak{N}(\beta) \cap V$, following the results obtained by Uralegaddi et al. [5], we can obtain the following lemmas.

Lemma 1. *Let the function $f(z) \in V$, then $f(z) \in V(\beta)$ $\left(1 < \beta \leqslant \frac{4}{3}\right)$ if and only if*

$$\sum_{n=2}^{\infty} (n-\beta)|a_n| \leqslant (\beta - 1) a_0. \tag{1.10}$$

Lemma 2. *Let the function $f(z) \in V$, then $f(z) \in U(\beta)$ $(1 < \beta \leqslant \frac{4}{3})$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\beta)|a_n| \leqslant (\beta - 1) a_0. \tag{1.11}$$

Let $\varphi(z)$ be a fixed function of the form:

$$\varphi(z) = c_0 z + \sum_{n=2}^{\infty} c_n z^n \quad (c_0, c_n \geqslant 0). \tag{1.12}$$

Using the function defined by (1.12), we now define the following new classes.

Definition 1. A function $f(z) \in V_\varphi(c_n, \delta)$ $(c_n \geqslant c_2 > 0)$ if and only if

$$\sum_{n=2}^{\infty} c_n |a_n| \leqslant \delta a_0 \quad (\delta > 0). \tag{1.13}$$

Definition 2. A function $f(z) \in U_\varphi(c_n, \delta)$ $(c_n \geqslant c_2 > 0)$ if and only if

$$\sum_{n=2}^{\infty} n c_n |a_n| \leqslant \delta a_0 \quad (\delta > 0). \tag{1.14}$$

Also, we introduce the following class of analytic functions which plays an important role in the discussion that follows.

Definition 3. A function $f(z) \in V_\varphi^k(c_n, \delta)$ $(c_n \geqslant c_2 > 0)$ if and only if

$$\sum_{n=2}^{\infty} n^k c_n |a_n| \leqslant \delta a_0 \quad (\delta > 0), \tag{1.15}$$

where k is any fixed nonnegative real number.

For suitable choices of c_n, δ, k and $a_0 = 1$, we obtain:

(i) $V_\varphi^0((n-\beta),(\beta-1)) = V(\beta)$ $\left(1 < \beta \leqslant \frac{4}{3}\right)$ [5];
(ii) $V_\varphi^1(n(n-\beta),(\beta-1)) = U(\beta)$ $\left(1 < \beta \leqslant \frac{4}{3}\right)$ [5];
(iii) $V_\varphi^0((n-1) + |n - 2\beta + 1|, 2(\beta - 1)) = \mathfrak{M}(\beta)$ $(\beta > 1, a_0 = 1)$ ([6,7], with $k = 1$);
(iv) $V_\varphi^1(n\{(n-1) + |n - 2\beta + 1|\}, 2(\beta - 1)) = \mathfrak{N}(\beta)$ $(\beta > 1, a_0 = 1)$ ([6,7], with $k = 1$).

Evidently, $V_\varphi^0(c_n, \delta) = V_\varphi(c_n, \delta)$ and $V_\varphi^1(c_n, \delta) = U_\varphi(c_n, \delta)$. Further $V_\varphi^{\gamma_1}(c_n, \delta) \subset V_\varphi^{\gamma_2}(c_n, \delta)$ if $\gamma_1 > \gamma_2 \geqslant 0$, the containment being proper. Moreover, for any positive integer k, we have the following inclusion relation

$$V_\varphi^k(c_n,\delta) \subset V_\varphi^{k-1}(c_n,\delta) \subset \cdots \subset V_\varphi^2(c_n,\delta) \subset U(c_n,\delta) \subset V(c_n,\delta).$$

We also note that for nonnegative real number k, the class $V_\varphi^k(c_n, \delta)$ is nonempty as the function

$$f(z) = a_0 z + \sum_{n=2}^{\infty} n^{-k} \frac{\delta a_0}{c_n} \lambda_n z^n, \tag{1.16}$$

where $a_0 > 0$, $\lambda_n \geqslant 0$ and $\sum_{n=1}^{\infty} \lambda_n \leqslant 1$, satisfy the inequality (1.15).

The quasi-Hadamard product of two or more univalent functions has been recently defined and studied by Aouf [8], Darwish [9], Frasin [10], Frasin and Aouf [11] and Kumar [12].

The object of this paper is to establish a result concerning the quasi-Hadamard product of functions in the classes $V_\varphi^k(c_n, \delta)$, $U_\varphi(c_n, \delta)$, and $V_\varphi(c_n, \delta)$.

2. The main results

Theorem 1. *Let the functions $f_i(z)$ defined by (1.8) belong to the class $U_\varphi(c_n, \delta)$ for every $i = 1,2,\ldots,m$; and let the functions $g_j(z)$ defined by (1.9) belong to the class $V_\varphi(c_n, \delta)$ for every $j = 1,2,\ldots,q$. If $c_n \geqslant n\delta (n \in \mathbb{N})$, then the quasi-Hadamard product $f_1 * f_2 * \cdots * f_m * g_1 * g_2 * \cdots * g_q(z)$ belongs to the class $V_\varphi^{2m+q-1}(c_n, \delta)$.*

Proof. It is sufficient to show that

$$\sum_{n=2}^{\infty} \left[n^{2m+q-1} c_n \left(\prod_{i=1}^{m} |a_{n,i}| \cdot \prod_{j=1}^{q} |b_{n,j}| \right) \right] \leqslant \delta \left(\prod_{i=1}^{m} a_{0,i} \cdot \prod_{j=1}^{q} b_{0,j} \right).$$

Since $f_i(z) \in U_\varphi(c_n, \delta)$, we have

$$\sum_{n=2}^{\infty} n c_n |a_{n,i}| \leqslant \delta a_{0,i}, \tag{2.1}$$

for every $i = 1,2,\ldots,m$. Therefore,

$$|a_{n,i}| \leqslant n^{-1} \left(\frac{\delta}{c_n} \right) a_{0,i},$$

and hence

$$|a_{n,i}| \leqslant n^{-2} a_{0,i}, \tag{2.2}$$

the inequalities (2.1) and (2.2) hold for every $i = 1,2,\ldots,m$. Further, since $g_j(z) \in V_\varphi(c_n, \delta)$, we have

$$\sum_{n=2}^{\infty} c_n|b_{n,j}| \leqslant \delta b_{0,j}, \tag{2.3}$$

for every $j = 1,2,\ldots,q$. Hence, we obtain

$$|b_{n,j}| \leqslant n^{-1} b_{0,j}, \tag{2.4}$$

for every $j = 1,2,\ldots,q$.

Using (2.2) for $i = 1,2,\ldots,m$, (2.4) for $j = 1,2,\ldots,q-1$ and (2.3) for $j = q$, we have

$$\sum_{n=2}^{\infty}\left[n^{2m+q-1}c_n\left(\prod_{i=1}^{m}|a_{n,i}|\cdot\prod_{j=1}^{q}|b_{n,j}|\right)\right] \leqslant \sum_{n=2}^{\infty}\left[n^{2m+q-1}c_n\left(n^{-2m}n^{-(q-1)}\prod_{i=1}^{m}a_{0,i}\cdot\prod_{j=1}^{q-1}b_{0,j}\right)|b_{n,q}|\right] = \left(\prod_{i=1}^{m}a_{0,i}\cdot\prod_{j=1}^{q-1}b_{0,j}\right)\sum_{n=2}^{\infty}c_n|b_{n,q}| \leqslant \delta\left(\prod_{i=1}^{m}a_{0p,i}\cdot\prod_{j=1}^{q}b_{0,j}\right).$$

Hence, $f_1 * f_2 * \cdots * f_m * g_1 * g_2 * \cdots * g_q \in V_{\varphi}^{2m+q-1}(c_n,\delta)$.

We note that the required estimate can also be obtained by using (2.2) for $i = 1,2,\ldots,m-1$, (2.4) for $j = 1,2,\ldots,q$, and (2.1) for $i = m$. □

Taking into account the quasi-Hadamard product functions $f_1(z), f_2(z),\ldots,f_m(z)$ only, in the proof of Theorem 1 and using (2.2) for $i = 1,2,\ldots,m-1$, and (2.1) for $i = m$, we obtain

Corollary 1. *Let the functions $f_i(z)$ defined by (1.8) belong to the class $U_{\varphi}(c_n, \delta)$ for every $i = 1,2,\ldots,m$. If $c_n \geqslant n\delta$, $(n \in \mathbb{N})$, then the quasi-Hadamard product $f_1 * f_2 * \cdots * f_m(z)$ belongs to the class $V_{\varphi}^{2m-1}(c_n,\delta)$.*

Also taking into account the quasi-Hadamard product functions $g_1(z), g_2(z),\ldots,g_q(z)$ only, in the proof of Theorem 1 and using (2.4) for $j = 1,2,\ldots,q-1$, and (2.3) for $j = q$, we obtain

Corollary 2. *Let the functions $g_i(z)$ defined by (1.9) belong to the class $V_{\varphi}(c_n, \delta)$ for every $i = 1,2,\ldots,q$. If $c_n \geqslant n\delta$, $(n \in \mathbb{N})$. Then the quasi-Hadamard product $g_1 * g_2 * \cdots * g_q$ belongs to the class $V_{\varphi}^{q-1}(c_n,\delta)$.*

Remarks

(i) Putting $c_n = (n-\beta)\,(n \geqslant 2)$ and $\delta = (\beta-1)\left(1 < \beta \leqslant \frac{4}{3}\right)$ in the above results, we obtain results corresponding to the class $V(\beta)$.

(ii) Putting $c_n = n(n-\beta)$ $(n \geqslant 2)$ and $\delta = (\beta-1)\left(1 < \beta \leqslant \frac{4}{3}\right)$ in the above results, we obtain results corresponding to the class $U(\beta)$.

(iii) Putting $c_n = (n-1) + |n - 2\beta + 1|$ $(n \geqslant 2)$ and $\delta = 2(\beta-1)$ $(\beta > 1)$ in the above results, we obtain results corresponding to the class $\mathfrak{M}(\beta)$.

(iv) Putting $c_n = n\{(n-1) + |n - 2\beta + 1|\}$ $(n \geqslant 2)$ and $\delta = 2(\beta-1)$ $(\beta > 1)$ in the above results, we obtain results corresponding to the class $\mathfrak{N}(\beta)$.

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