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ORIGINAL ARTICLE

Second Order (*F*, α, ρ, *d*, *E*)-convex function and the Duality Problem

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Abstract A class of second order (*F*, α, ρ, *d*, *E*)-convex functions and their generalization on functions involved, weak, strong, and converse duality theorems are established for a second order Mond-Weir type dual problem.

MATHEMATICS SUBJECT CLASSIFICATION: 90C25; 90C30; 90C46; 90C29

1. Introduction

The importance of convex functions is well known in optimization theory for many mathematical models used in decision science, economic, management science, applied mathematics and engineering. The notion of convexity does no longer suffice. So it is possible to generalize the notion of convexity and to extend the validity of result to larger classes of optimization problems. Consequently, various generalizations of convex functions have been introduced in the literature. More specifically, the concept of (*F*, α)-convexity was introduced by Preda [1] which is an extension of *F*-Convexity defined by Hanson and Mond [2] and ρ-convexity given by Vial [3], Gualti and Islam [4] and Ahmed [5] established optimality conditions and duality results for multiobjective programming problems involving *F*-Convexity and (*F*, α)-convexity assumptions. In [6,7] Weir and Mond discussed the generalized convexity,duality in multiobjective programming and the proper efficiency of the duality for vector valued Optimization Problem,and in [8] Mond and Weir discussed the generalized concavity and duality in Optimization and economics. Also, the concepts of E-convex sets and E-convex function have been introduced by Youness in [9–11], they have some important application in various branches of mathematical sciences.

Youness in [11] introduced a class of E-convex sets and E-convex functions by relaxing the definition of convex sets and convex functions. This kind of generalized convexity is based on the effect of an operator $E: R^n \to R^n$ on the sets and the domain of functions, and also in [10] Youness discussed the optimality criteria of E-convex programming. Xiusu Chen [12] introduced a new concept of semi E-convex functions and discusses its properties. Emam and Youness in [13,14] introduced a new class of E-convex sets and E-convex functions which is called semi strongly E-convex sets and strongly E-convex

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functions by taking the images of two points x and y under an operator E: $R^n \to R^n$ besides the two points themselves and strongly E-convex sets and strongly E-convex functions. In [15] Megahed and et al. introduced a combined interactive approach for solving E-convex multiobjective nonlinear programming and in [16] Youness discussed the characterization of efficient solutions of multiobjective E-convex programming problems.

In this paper we will discuss the connection between the concept of E-convex function [15] and the second order (F, α, ρ, d)-convex function [17] by introducing the concept second order (F, α, ρ, d, E)-convex functions and their generalization. These concepts are then used to develop weak, strong, and strict converse duality theorem for the second order Mond-Weir type dual problem. Where F: $M \times M \times R^n \to R$, α: $M \times M \to R^+ - \{0\}$, d: $M \times M \to R$, E: $R^n \to R^n$, and a real number ρ.

Definition 1.1. A functional F: $M \times M \times R^n \to R$ is said to be sub linear in its third component, if for all $x, \bar{x} \in M$

(i) $F(x,\bar{x},a+b) \leqslant F(x,\bar{x},a) + F(x,\bar{x},b), \forall a,b \in R^n$
(ii) $F(x,\bar{x},\beta a) = \beta F(x,\bar{x},a), \forall \beta \in R, \beta \geqslant 0, and, a \in R^n$

Definition 1.2 [11] E-Convex set. A set $M \subseteq R^n$ is said to be an E-convex set with respect to an operator E: $R^n \to R^n$ if $\lambda E(x) + (1-\lambda)E(y) \in M$, for each $x, y \in M$ and $0 \leqslant \lambda \leqslant 1$.

Definition 1.3 [11] E-convex Function. A real valued function f: $M \subseteq R^n \to R$ is said to be an E-convex function, with respect to an operator E: $R^n \to R^n$ on M. If M is an E-convex set and for each $x, y \in M$, $0 \leqslant \lambda \leqslant 1$,

$$f(\lambda E(x) + (1-\lambda)E(y)) \leqslant \lambda f(Ex) + (1-\lambda)f(Ey)$$

If $f(\lambda E(x) + (1-\lambda)E(y)) \geqslant \lambda f(E\,x) + (1-\lambda)f(E\,y)$, then f is called E-concave function on M.

The problem to be considered her is the following nonlinear programming problem:

$$(\mathrm{P}) \begin{cases} Minimize\ f(x) \\ \text{Subject to } M = \{x \in R^n : g_i(x) \leqslant 0,\ i = 1,2,\ldots,m\} \end{cases}$$

Where the function f and a set M are E-convex with respect to the map E: $R^n \to R^n$, and f and $g = (g_1, g_2, \ldots, g_m)$ are assumed to be twice differentiable function over M.

There is another problem (P_E problem) from the above problem is defined as

$$(P_E) \quad \begin{cases} Minimize (f \circ E)(x) \\ Subject\ to\ M' = \{x \in R^n : (g_i \circ E)(x) \leqslant 0, i = 1,2,\ldots,m\} \end{cases}$$

Where the function f and a set M are E-convex with respect to the map E: $R^n \to R^n$, and f and $g = (g_1, g_2, \ldots, g_m)$ are assumed to be twice differentiable function over M

Definition 1.4. A point $\bar{x}$ is an optimal solution of the problem (P) if and only if $f(E\bar{x}) \leqslant f(Ex) \forall x \in M$, M is an E-convex set.

Definition 1.5. Let E: $R^n \to R^n$ be an operator and f is E-convex function on an E-convex set M and $f \circ E$ is twice differentiable function on M, then f is said to be second order (F, α, ρ, d, E)-convex function at $\bar{x}$ on M if for all $x \in M$, then there exists a vector $P \in R^n$, a real valued function α: $M \times M \to R^+ - \{0\}$, a real valued function d: $M \times M \to R$ and a real number ρ such that $f(Ex) - f(E\bar{x}) + \frac{1}{2}P'\nabla^2 f(E\bar{x})P \geqslant F(x,\bar{x};\alpha(x,\bar{x})\{\nabla f(E\bar{x}) + \nabla^2 f(E\bar{x})P\}) + \rho d^2(x,\bar{x})$

Definition 1.6. Let E: $R^n \to R^n$ be an operator and f is E-convex function on an E-convex set M and $f \circ E$ is twice differentiable function on M, then f is said to be second order (F, α, ρ, d, E)-pseudoconvex function at $\bar{x}$ on M if for all $x \in M$, then there exists a vector $P \in R^n$, a real valued function α: $M \times M \to R^+ - \{0\}$, a real valued function d: $M \times M \to R$ and a real number ρ such that $f(Ex) < f(E\bar{x}) - \frac{1}{2}P'\nabla^2 f(E\bar{x})P \Rightarrow F(x,\bar{x};\alpha(x,\bar{x})\{\nabla f(E\bar{x}) + \nabla^2 f(E\bar{x})P\}) < -\rho d^2(x,\bar{x})$

Definition 1.7. Let E: $R^n \to R^n$ be an operator and f is E-convex function on an E-convex set M and $f \circ E$ is twice differentiable function on M, then f is said to be strictly second order (F, α, ρ, d, E)-pseudoconvex function at $\bar{x}$ on M if for all $x \in M$, then there exists a vector $P \in R^n$, a real valued function α: $M \times M \to R^+ - \{0\}$, a real valued function d: $M \times M \to R$ and a real number ρ such that

$$F(x,\bar{x};\alpha(x,\bar{x})\{\nabla f(E\bar{x}) + \nabla^2 f(E\bar{x})P\}) \geqslant -\rho d^2(x,\bar{x})$$
$$\Rightarrow f(Ex) > f(E\bar{x}) - \tfrac{1}{2}P'\nabla^2 f(E\bar{x})P$$

Or equivalently

$$f(Ex) \leqslant f(E\bar{x}) - \tfrac{1}{2}P'\nabla^2 f(E\bar{x})P$$
$$\Rightarrow F(x,\bar{x};\alpha(x,\bar{x})\{\nabla f(E\bar{x}) + \nabla^2 f(E\bar{x})P\}) < -\rho d^2(x,\bar{x})$$

Definition 1.8. Let E: $R^n \to R^n$ be an operator and f is E-convex function on an E-convex set M and $f \circ E$ is twice differentiable function on M, then f is said to be second order (F, α, ρ, d, E)-quasiconvex function at $\bar{x}$ on M if for all $x \in M$, then there exists a vector $P \in R^n$, a real valued function α: $M \times M \to R^+ - \{0\}$, a real valued function d: $M \times M \to R$ and a real number ρ such that

$$f(Ex) \leqslant f(E\bar{x}) - \tfrac{1}{2}P'\nabla^2 f(E\bar{x})P$$
$$\Rightarrow F(x,\bar{x};\alpha(x,\bar{x})\{\nabla f(E\bar{x}) + \nabla^2 f(E\bar{x})P\}) \leqslant -\rho d^2(x,\bar{x})$$

Definition 1.9. Let E: $R^n \to R^n$ be an operator and f is E-convex function on an E-convex set M and $f \circ E$ is twice differentiable function on M, then f is said to be strong second order (F, α, ρ, d, E)-pseudoconvex function at $\bar{x}$ on M if for all $x \in M$, then there exists a vector $P \in R^n$, a real valued function α: $M \times M \to R^+ - \{0\}$, a real valued function d: $M \times M \to R$ and a real number ρ such that

$$f(Ex) \leqslant f(E\bar{x}) - \tfrac{1}{2}P'\nabla^2 f(E\bar{x})P$$
$$\Rightarrow F(x,\bar{x};\alpha(x,\bar{x})\{\nabla f(\bar{x}) + \nabla^2 f(\bar{x})P\}) \leqslant -\rho d^2(x,\bar{x})$$

Example 1. Consider the function f: $M(=R_+) \to R$ such that $f(x) = x^2 - 2x$. If

$$F(x,\bar{x},\alpha) = \alpha(x-\bar{x}) - 4x$$

define the functions $d(x,\bar{x}) = x - \bar{x}$,

$$\alpha(x,\bar{x}) = \frac{x+\bar{x}+1}{2}$$

And the operator. $E(x) = x^2$

Then for $\rho = 0$, f is second order (F, α, ρ, d, E)-convex at $\bar{x} = 0$ with respect to p, $-\infty < p \leqslant 1$

Example 2. Consider the function f: $M \to R$, $M = \{x: 0 \leqslant x \leqslant 7\}$ such that $f(x) = x^3$.

$$F(x, \bar{x}, \alpha) = \alpha(x - \bar{x}) - 4x$$

If we define the functions $d(x, \bar{x}) = x - \bar{x}$,

$$\alpha(x, \bar{x}) = \frac{x + \bar{x} + 1}{2}$$

and the operator $E(x) = x^2$, then for $\rho = 0$, f is second order (F, α, ρ, d, E)-quasiconvex function at $\bar{x} = 0$ with respect to p, $0 < p \leqslant \infty$

Theorem 1.10 (*Fritz John Necessary Condition for Optimality*). *Let the objective and the constraint functions of problem P_E are continuously differentiable at the point $\bar{x} \in M$ and $I = \{i : (g_i \circ E)(\bar{x}) = 0\}$ is the index set.A necessary conditions for $\bar{x}$ to be a local solution of the problem P_E is that there exist vector $\mu_\circ \geqslant 0$ and $0 \leqslant \mu \in R^m$ for which $(\mu_\circ, \mu) \neq (0, 0)$ such that*

1- $\mu_0 \nabla(f \circ E)\bar{x} + \sum_{i=1}^{k} \mu_i \nabla(g_i \circ E)\bar{x} = 0.$
2- $\mu_i (g_i \circ E)\bar{x} = 0$, for all $i = 1, 2, \ldots, m$.

Proof. Let $\bar{x} \in M$ be a local solution for problem P_E. To prove this theorem we must prove that there does not exist any other vector $d \in R^n$ such that

$$\nabla(f \circ E)(x)^T d < 0 \tag{1.1}$$

$$\nabla(g_i \circ E)(\bar{x})^T d \leqslant 0 \qquad \textit{for all } i \in I(\bar{x}) \tag{1.2}$$

By contradiction, we assume that there exists some $d^* \in R$ such that $\nabla(f \circ E)(\bar{x})^T d^* < 0$. Since the functions $(f \circ E)$ are continuously differentiable, then

$$(f \circ E)(x) = (f \circ E)(\bar{x}) + \nabla(f \circ E)(\bar{x})^T (x - \bar{x}) + \|x - \bar{x}\| \alpha(\bar{x}, x - \bar{x}).$$

Let $d^* = x - \bar{x}$, then

$$(f \circ E)(x) = (f \circ E)(\bar{x}) + \nabla(f \circ E)(\bar{x})^T d^* + \|d^*\| \alpha(\bar{x}, d^*).$$

Since $\nabla(f \circ E)(\bar{x})^T d^* < 0$ and $\alpha(\bar{x}, d) \to 0$ at $\bar{x} \to 0$, then $(f \circ E)(x) < (f \circ E)(\bar{x})$.

This contradicts the optimality of $\bar{x}$, and then there does not exist any $d \in R^n$ such that

$$\nabla(f \circ E)(\bar{x})^T d < 0 \text{ and } \nabla(g_i \circ E)(\bar{x})^T d \leqslant 0 \text{ for all } i \in I(\bar{x}).$$

Now, from Motzkin's theorem [18,19] there exists multipliers $\mu_0 \geqslant 0$ for and $\mu_i \geqslant 0$ for $i \in I(\bar{x})$ such that

$$\mu_0 \nabla(f \circ E)\bar{x} + \sum_{i \notin I(\bar{x})}^{m} \mu_i \nabla(g_i \circ E)(\bar{x}) = 0$$

We obtain statement (1.1) of □

By setting $\mu_i = 0$ for all $i \in \{1, \ldots, m\} \setminus I(\bar{x})$. If $(g_i \circ E)(\bar{x}) < 0$ for some $i = 1, 2, \ldots, m$, then according to the above setting $\mu_i = 0$, statement (2) of is valid.

2. Second order Mond-Weir type duality

In this section, we consider the following Mond-Weir type second order dual associated with the problem (P_E) and establish weak, strong, and strict converse duality theorems under generalized second order (F, α, ρ, d, E)-convexity:

$$(\text{MDE}) \quad \begin{cases} \textit{Maximize } f(Eu) - \frac{1}{2}p^t \nabla^2 f(Eu)p \\ \textit{subjectb to } \nabla\lambda f(Eu) + \nabla^2 \lambda f(Eu)p + \nabla\gamma^t g(Eu) \\ \qquad + \nabla^2 \gamma^t g(Eu)p = 0 \\ \gamma^t g(Eu) - \frac{1}{2}p^t \nabla^2 \gamma^t g(Eu)p \geqslant 0 \\ \gamma \geqslant 0, \ \lambda \geqslant 0 \end{cases}$$

Theorem 2.1 (*weak duality*). *Suppose that for all feasible x in the problem P and all feasible (u, γ, λ, p) in MDE*

(1) $\gamma^t g(.)$ is second order (F, α, ρ, d, E)-quasiconvex at u, and assume that any one of the following consider holds
(2) $\lambda \geqslant 0$, and $f(.)$ is strong second order $(F, \alpha_1, \rho_1, d, E)$-pseudoconvex at u with $\alpha^{-1}\rho + \alpha_1^{-1}\rho_1\lambda \geqslant 0$.
(3) $\lambda^t f(.)$ is strictly second order $(F, \alpha_2, \rho_2, d, E)$-pseudoconvex at u with $\alpha^{-1}\rho + \alpha_2^{-1}\rho_2 \geqslant 0$

Then the following can't hold

$$f(Ex) \leqslant f(Eu) - \frac{1}{2}p^t \nabla^2 f(Eu)p \tag{2.1}$$

Proof. Let x be any feasible solution in P_E and (u, γ, λ, p) be any feasible solution in (MDE) problem. Then we have

$$\gamma g(Ex) \leqslant 0 \text{ and } \gamma g(Eu) - \frac{1}{2}p\nabla^2 \gamma^t g(Eu)p \geqslant 0$$

Using second order (F, α, p, d, E)-quasiconvexity of $\gamma^t g(.)$ at u, we get $F(x, u, \alpha(x, u)\{\nabla \gamma^t g(Eu) + \nabla^2 \gamma^t g(Eu)p\}) \leqslant -pd^2(x, u)$

Since $\alpha(x, u) > 0$, the above inequality with the sublinearity of F yields

$$F(x, u, \{\nabla\gamma^t g(Eu) + \nabla^2 \gamma^t g(Eu)p\}) \leqslant -\alpha^{-1}(x, u)pd^2(x, u) \tag{2.2}$$

The first dual constraint and the sublinearity of F give

$$F(x, u, \nabla\lambda f(Eu) + \nabla^2 \lambda f(Eu)p) \geqslant -F(x, u, \nabla\gamma^t g(Eu) + \nabla^2 \gamma^t g(Eu)p) \tag{2.3}$$

The inequalities (2.2) and (2.3) imply

$$F(x, u, \nabla\lambda f(Eu) + \nabla^2 \lambda f(Eu)p) \geqslant \alpha^{-1}(x, u)pd^2(x, u) \tag{2.4}$$

Now suppose contrary to the result that (2.1) holds, i.e.

$$f(Ex) \leqslant f(Eu) - \frac{1}{2}p^t \nabla^2 f(Eu)p \tag{2.5}$$

Which by virtue of (2.3), leads to

$$F(x, u, \nabla\lambda f(Eu) + \nabla^2 \lambda f(Eu)p) \leqslant -p_1 d^2(x, u) \tag{2.6}$$

On multiplying (2.6) by $\lambda \geqslant 0$ and using the sublinearity of F with $\alpha_1(x, u) > 0$,

We obtain

$$F(x,u,\nabla\lambda f(Eu)+\nabla^2\lambda f(Eu)p)\leqslant -\alpha_1^{-1}p_1\lambda d^2(x,u) \leqslant \alpha^{-1}(x,u)pd^2(x,u)$$

Which contradicts with (2.4). Hence (2.1) can't hold.

On the other hand, multiplying the inequality (2.5) by λ, we have

$$\lambda f(Ex)\leqslant \lambda f(Eu)-\frac{1}{2}p^t\nabla^2\lambda f(Eu)p \quad (2.7)$$

When hypothesis (3) holds, the inequality (2.7) implies

$$F(x,u,\alpha_2(x,u)\{\nabla\lambda f(Eu)+\nabla^2\lambda f(Eu)p\})<-p_2d^2(x,u) \quad (2.8)$$

Since F is sublinear and $\alpha_2(x,u)>0$,it follow from (2.8) that

$$F(x,u,\nabla\lambda f(Eu)+\nabla^2\lambda f(Eu)p)\leqslant -\alpha_1^{-1}p_1\lambda d^2(x,u) \leqslant \alpha_2^{-1}(x,u)p_2d^2(x,u) \leqslant \alpha^{-1}(x,u)pd^2(x,u)$$

Which contradicts with (2.4). Hence (2.1) can't hold. □

Theorem 2.2 (*Strong duality theorem*). *Let $\bar{x}$ be an optimal solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist $\bar{\lambda}\geqslant 0$ and $\bar{\gamma}\in R^m$ such that $(\bar{x},\bar{\gamma},\bar{\lambda},\bar{p}=0)$ is feasible for (MDE) and the corresponding values of (P) and (MDE) are equal.*

Proof. Since $\bar{x}$ is an optimal solution of (P) at which the Kuhn-Tucker constraint qualification is satisfied, then by Theorem 1.10, there exists $\bar{\lambda}\geqslant 0$ and $\bar{\gamma}\in R^m$, such that

$$\bar{\lambda}\nabla f(E\bar{x})+\bar{\gamma}\nabla g(E\bar{x})=0$$
$$\bar{\gamma}g(E\bar{x})=0$$
$$\bar{\gamma},\bar{\lambda}\geqslant 0$$

Therefore $(\bar{x},\bar{\gamma},\bar{\lambda},\bar{p}=0)$ is feasible for (MDE) and the corresponding values of (P) and (MDE) are equal from weak duality theorem $(\bar{x},\bar{\gamma},\bar{\lambda},\bar{p}=0)$ is optimal solution of (MDE) □

Theorem 2.3 (*Strict converse duality*). *Let $\bar{x}$ and $(\bar{u},\bar{\gamma},\bar{\lambda},\bar{p})$ is an optimal solution of (P_E) and (MDE) respectively, such that*

$$\lambda f(E\bar{x})=\lambda f(Eu)-\frac{1}{2}\bar{p}^t\nabla^2\bar{\lambda}f(Eu)\bar{p} \quad (2.9)$$

Suppose that any one of the following conditions is satisfied

(1) $\gamma^t g(.)$ is second order (F,α,ρ,d,E)-quasiconvex at $\bar{u}$ and $\lambda f(.)$ is strictly second order (F,α_1,ρ_1,d,E)-pseudoconvex at u with $\alpha^{-1}\rho+\alpha_1^{-1}\rho_1\geqslant 0$.
(2) $\gamma^t g(.)$ is strictly second order (F,α,ρ,d,E)-pseudoconvex at $\bar{u}$ and $\lambda^t f(.)$ is strictly second order (F,α_1,ρ_1,d,E)-quasiconvex at u with $\alpha^{-1}\rho+\alpha_1^{-1}\rho_1\geqslant 0$.

Then, $\bar{x}=\bar{u}$ that is, $\bar{u}$ is an optimal solution of (P_E)

Proof. We assume that $\bar{x}\neq\bar{u}$ and reach a contradiction. Since $\bar{x}$ and $(\bar{u},\bar{\gamma},\bar{\lambda},p)$ are respectively, the feasible solution of (P_E) and (MDE) we have

$$\gamma g(E\bar{x})\leqslant 0 \text{ and } \bar{\gamma}^t g(E\bar{u})-\frac{1}{2}\bar{p}^t\nabla^2\bar{\gamma}^t g(E\bar{u})\bar{p}\geqslant 0 \quad (2.10)$$

Using second order (F,α,ρ,d,E)-quasiconvexity of $\gamma^t g(.)$ at $\bar{u}$, we get

$$F(\bar{x},\bar{u},\alpha(\bar{x},\bar{u})\{\nabla\bar{\gamma}^t g(E\bar{u})+\nabla^2\bar{\gamma}^t g(E\bar{u})\bar{p}\})\leqslant -pd^2(\bar{x},\bar{u})$$

Since $\alpha(\bar{x},\bar{u})>0$, the inequality (2.10) along with the sublinearity of F yields

$$F(\bar{x},\bar{u},\nabla\gamma^t g(E\bar{u})+\nabla^2\gamma^t g(E\bar{u})\bar{p})\leqslant -\alpha^{-1}(\bar{x},\bar{u})pd^2(\bar{x},\bar{u}) \quad (2.11)$$

The first dual constraint and the sublinearity of F imply

$$F(\bar{x},\bar{u},\nabla\lambda f(E\bar{u})+\nabla^2\lambda f(E\bar{u})\bar{p})+F(\bar{x},\bar{u},\nabla\gamma^t g(E\bar{u})+\nabla^2\lambda^t g(E\bar{u})\bar{p}) \geqslant F(\bar{x},\bar{u},\nabla\lambda f(E\bar{u})+\nabla^2\lambda f(E\bar{u})\bar{p}+\nabla\gamma^t g(E\bar{u})+\nabla^2\lambda^t g(E\bar{u})\bar{p})=0 \quad (2.12)$$

The inequalities (2.11) and (2.12) and $\alpha^{-1}\rho+\alpha_1^{-1}\rho_1\geqslant 0$ imply

$$F(\bar{x},\bar{u},\nabla\lambda f(E\bar{u})+\nabla^2\lambda f(E\bar{u})\bar{p})\geqslant -\alpha_1^{-1}(\bar{x},\bar{u})p_1d^2(\bar{x},\bar{u}) \quad (2.13)$$

Use the strict second order (F,α_1,ρ_1,d,E)-pseudoconvexity of $\lambda f(.)$ with

$$\alpha(\bar{x},\bar{u})>0, \lambda f(E\bar{x})>\lambda f(E\bar{u})-\frac{1}{2}p^t\nabla^2\lambda^t f(E\bar{u})\bar{p}$$

Contradiction with (2.9)

When the hypothesis (2) holds, it follows from (2.10) that

$$F(\bar{x},\bar{u},\alpha(\bar{x},\bar{u})\{\nabla\bar{\gamma}^t g(E\bar{u})+\nabla^2\bar{\gamma}^t g(E\bar{u})\bar{p}\})\leqslant -pd^2(\bar{x},\bar{u})$$

Since $\alpha(\bar{x},\bar{u})>0$, the above inequality with the sublinearity of F gives

$$F(\bar{x},\bar{u},\nabla\gamma^t g(E\bar{u})+\nabla^2\gamma^t g(E\bar{u})\bar{p})\leqslant -\alpha^{-1}(\bar{x},\bar{u})pd^2(\bar{x},\bar{u})$$

Which on using first dual constraint with the sublinearity of F implies

$$F(\bar{x},\bar{u},\nabla\lambda^t f(E\bar{u})+\nabla^2\lambda^t f(E\bar{u})\bar{p})\geqslant -\alpha^{-1}(\bar{x},\bar{u})p_1d^2(\bar{x},\bar{u})$$

As $\alpha^{-1}\rho+\alpha_1^{-1}\rho_1\geqslant 0$, we obtain

$$F(\bar{x},\bar{u},\nabla\lambda^t f(E\bar{u})+\nabla^2\lambda^t f(E\bar{u})\bar{p})\geqslant -\alpha_1^{-1}(\bar{x},\bar{u})p_1d^2(\bar{x},\bar{u}) \quad (2.14)$$

The second order (F,α_1,ρ_1,d,E)-quasiconexity of $\lambda f(.)$ and (2.14) with $\alpha_1(\bar{x},\bar{u})>0$ yield $\lambda^t f(E\bar{x})>\lambda^t f(E\bar{u})-\frac{1}{2}p^t\nabla^2\lambda^t f(E\bar{u})\bar{p}$

Again contradiction (2.9) □

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