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ORIGINAL ARTICLE

On new inequalities of Hermite-Hadamard type for functions whose third derivative absolute values are quasi-convex with applications

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Abstract We establish some new inequalities of Hermite-Hadamard type for functions whose third derivatives absolute values are quasi-convex. Applications to special means have also been presented.

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1. Introduction

Let $f: \Phi \neq I \subseteq \mathbb{R} \to \mathbb{R}$ be a function defined on the interval I of real numbers. Then f is called convex if

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y),$$

for all $x, y \in I$ and $\lambda \in [0, 1]$. Geometrically, this means that if P, Q, and R are three distinct points on graph of f with Q between P and R, then Q is on or below chord PR. There are many results associated with convex functions in the area of inequalities, but one of those is the classical Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2},$$

for $a, b \in I$, with $a < b$.

Recently, many others [1–12] developed and discussed Hermite-Hadamarad's inequality in terms of refinements, counterparts, generalizations and new Hermite-Hadamarad's type inequalities.

The notion quasi-convex function which is the generalization of convex function is defined as:

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A function f: $[a, b] \to R$ is called a quasi-convex on $[a, b]$, if

$$f(\lambda x + (1-\lambda)y) \leqslant \max\{f(x), f(y)\}, \quad \forall x, y \in [a, b].$$

Any convex function is a quasi-convex function but converse is not true in general.(see for example [3]).

Recently, D.A. Ion [6] obtained established inequalities of the right-hand side of Hermite-Hadamard's type functions whose derivatives in absolute values are quasi-convex functions. These inequalities are defined as follows:

Theorem 1.1. *Let f: $I^0 \subseteq R \to R$ be a differentiable function on I^0 with $a, b \in I^0$ and $a < b$. If $|f'|$ is quasi-convex on $[a, b]$, then we have:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)}{4}\max\{|f'(a)|, |f'(b)|\}.$$

Theorem 1.2. *Let f: $I^0 \subseteq R \to R$ be a differentiable function on I^0 with $a, b \in I^0$ and $a < b$. If $|f'|^{p/(p-1)}$ is quasi-convex on $[a, b]$, then we have:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{b-a}{2(p+1)^{1/p}}\left(\max\{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}\}\right)^{(p-1)/p}.$$

Theorem 1.3. *Let f: $I^0 \subseteq R \to R$ be a differentiable function on I^0, $a, b \in I^0$ with $a < b$. If $|f''|$ is quasi-convex on $[a, b]$, then we have:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)^2}{12}\max\{|f''(a)|, |f''(b)|\}.$$

In this paper, we establish new refined inequalities of the right-hand side of Hermite-Hadamard result for the class of functions whose third derivatives at certain powers are quasi-convex functions. Applications to special means have also been presented.

2. Main results

Before proceeding toward our main theorem, we begin with the following lemma:

Lemma 2.1. [16]. *Let f: $I \subset R \to R$ be three times differentiable mapping on I^0 such that $a, b \in I^0$ with $a < b$ and $f''' \in L[a, b]$. Then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)] = \frac{(b-a)^3}{12}\int_0^1 \lambda(1-\lambda)(2\lambda-1)f'''(\lambda a + (1-\lambda)b)d\lambda$$

In the following, we present a new result of the upper Hermite-Hadamard inequality for quasi-convex functions.

Theorem 2.2. *Let f: $I \subseteq R \to R$ be three times differentiable mapping on I^0 such that $f''' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'''|$ is quasi-convex on $[a, b]$, then we have the following inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right| \leqslant \frac{(b-a)^3}{192}\max\{|f'''(a)|, |f'''(b)|\}. \quad (2.2.1)$$

Proof. Using Lemma 2.1 and quasi-convexity of $|f'''|$, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right| \\
&\leqslant \frac{(b-a)^3}{12}\int_0^1 \lambda(\lambda-1)|(2\lambda-1)||f'''(\lambda a + (1-\lambda)b)|d\lambda \\
&\leqslant \frac{(b-a)^3}{12}\max\{|f'''(a)|, |f'''(b)|\}.\int_0^1 \lambda(\lambda-1)|(2\lambda-1)|d\lambda \\
&= \frac{(b-a)^3}{192}\max\{|f'''(a)|, |f'''(b)|\}.
\end{aligned}$$

The proof is completed. □

In the following theorem, we establish the corresponding version for powers of the absolute value of the second derivative:

Theorem 2.3. *Let f: $I \subset R \to R$ be three times differentiable mapping on I^0, such that $f''' \in L[a, b]$ with $a,b \in I$ and $a < b$. If $|f'''|^{p/(p-1)}$ is quasi-convex on $[a, b]$, and $p > 1$, then we have the following inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right| \leqslant \frac{(b-a)^3}{96}\left(\frac{1}{p+1}\right)^{1/p}\left(\max\{|f'''(a)|^q, |f'''(b)|^q\}\right)^{1/q}, \quad (2.2.2)$$

where $q = p/(p-1)$.

Proof. By using Lemma 2.1 and well known Holder's integral inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right| \\
&\leqslant \frac{(b-a)^3}{12}\int_0^1 \lambda(\lambda-1)|(2\lambda-1)||f'''(\lambda a + (1-\lambda)b)|d\lambda \\
&\leqslant \frac{(b-a)^3}{12}\left(\int_0^1 \lambda^p(1-\lambda)^p|(2\lambda-1)|d\lambda\right)^{1/p}\left(\int_0^1 |f'''\lambda a + (1-\lambda)b|^q d\lambda\right)^{1/q}
\end{aligned}$$

By using the fact

$$\int_0^1 \lambda^p(1-\lambda)^p|2\lambda-1|d\lambda = \frac{1}{2^{2p+1}(p+1)}$$

and the quasi-convexity of $|f'''|^q$, we get the desired result. The proof is completed. □

Theorem 2.4. *Let f: $I \subset R \to R$ be three times differentiable mapping on I^0 such that $f''' \in L[a, b]$, with $a, b \in I$ and $a < b$. If $|f'''|^q$ is quasi-convex on $[a, b]$ and $q \geqslant 1$, then we have the following inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \frac{b-a}{12}[f'(b) - f'(a)]\right| \leqslant \frac{(b-a)^3}{192}\left(\max\{|f'''(a)|^q, |f'''(b)|^q\}\right)^{\frac{1}{q}}. \quad (2.2.3)$$

Proof. Using Lemma 1 and well known power-mean inequality, we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\frac{b-a}{12}[f'(b)-f'(a)]\right|\\&\leqslant\frac{(b-a)^3}{12}\int_0^1\lambda(\lambda-1)|(2\lambda-1)||f'''(\lambda a+(1-\lambda)b)|d\lambda\\&\leqslant\frac{(b-a)^3}{12}\left(\int_0^1\lambda(\lambda-1)|(2\lambda-1)|d\lambda\right)^{1-1/q}\left(\int_0^1\lambda(\lambda-1)|(2\lambda-1)||f'''(\lambda a+(1-\lambda)b)|^q d\lambda\right)^{1/q}\\&\leqslant\frac{(b-a)^3}{12}\left(\frac{1}{16}\right)^{1-1/q}\cdot\left(\frac{1}{16}\max\{|f'''(a)|^q,|f'''(b)|^q\}\right)^{1/q}\\&=\frac{(b-a)^3}{192}(\max\{|f'''(a)|^q,|f'''(b)|^q\})^{1/q}.\end{aligned}$$

where we use the fact

$$\int_0^1\lambda(1-\lambda)|(2\lambda-1)|d\lambda=\frac{1}{16}$$

The proof is completed. □

3. Application to some special means

We now consider the applications of our theorem to the special means.

For positive numbers $a>0$ and $b>0$, define

$A\ (a,b)=\frac{a+b}{2}$

and

$$L_p(a,b)=\begin{cases}\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p\neq -1,0;\\ \frac{b-a}{\ln b-\ln a}, & p=-1;\\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & p=0;\end{cases}$$

We known that A and L_P are respectively called the arithmetic and generalized logarithmic means of two positive numbers a and b

By applying Hermite-Hadamard type inequalities established in Section 2, we are in position to construct some inequalities for special means A and L_p.

Consider the following function:

$$f(x)=\frac{x^{\alpha+3}}{(\alpha+1)(\alpha+2)(\alpha+3)} \tag{3.1.1}$$

For $0<\alpha\leqslant 1$ and $x>0$. Since $f'''(x)=x^\alpha$ and $(\lambda x+(1-\lambda)y)^\alpha\leqslant\lambda^\alpha x^\alpha+(1-\lambda)^\alpha y^\alpha$ For all $x,\ y>0$ and $\lambda\in[0,1]$. then $f'''(x)=x^\alpha$ is α-convex function on R_0 and

$$\frac{f(a)+f(b)}{2}=\frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)}A(a^{\alpha+3},b^{\alpha+3}),$$

$$\frac{1}{b-a}\int_a^b f(x)dx=\frac{1}{(\alpha+1)(\alpha+2)(\alpha+3)}L_{\alpha+3}^{\alpha+3}(a^{\alpha+4},b^{\alpha+4}),$$

$$f'(b)-f'(a)=\frac{1}{12(\alpha+1)}L_{\alpha+1}^{\alpha+1}(a^{\alpha+2},b^{\alpha+2})$$

Theorem 3.1. *For positive number a and b such that $b>a$ and $0<\alpha\leqslant 1$, we have*

$$\left|\begin{matrix}12A(a^{\alpha+3},b^{\alpha+3})-12L_{\alpha+3}^{\alpha+3}(a^{\alpha+4},b^{\alpha+4})\\-(b-a)^2(\alpha+2)(\alpha+3)L_{\alpha+1}^{\alpha+1}(a^{\alpha+2},b^{\alpha+2})\end{matrix}\right|$$
$$=\frac{(b-a)^3}{16}(\alpha+1)(\alpha+2)(\alpha+3)\max\{|a^\alpha|,|b^\alpha|\}.$$

Proof. The assertion follows from inequality (2.2.1) applied to the mapping (3.1.1). □

Theorem 3.2. *For positive number a and b such that $b>a$ and $0<\alpha\leqslant 1$, we have*

$$\left|\begin{matrix}12A(a^{\alpha+3},b^{\alpha+3})-12L_{\alpha+3}^{\alpha+3}(a^{\alpha+4},b^{\alpha+4})\\-(b-a)^2(\alpha+2)(\alpha+3)L_{\alpha+1}^{\alpha+1}(a^{\alpha+2},b^{\alpha+2})\end{matrix}\right|$$
$$\leqslant\frac{(b-a)^3}{8}\left[\frac{1}{p+1}\right]^{1/p}(\alpha+1)(\alpha+2)(\alpha+3)\max\{|a^\alpha|^q,|b^\alpha|^q\}^{\frac{1}{q}}.$$

Proof. The assertion follows from inequality (2.2.2) applied to the mapping (3.1.1). □

Theorem 3.3. *For positive number a and b such that $b>a$ with $0<\alpha\leqslant 1$ and $q>1$ we have*

$$\left|\begin{matrix}12A(a^{\alpha+3},b^{\alpha+3})-12L_{\alpha+3}^{\alpha+3}(a^{\alpha+4},b^{\alpha+4})\\-(b-a)^2(\alpha+2)(\alpha+3)L_{\alpha+1}^{\alpha+1}(a^{\alpha+2},b^{\alpha+2})\end{matrix}\right|$$
$$\leqslant\frac{(b-a)^3}{16}(\alpha+1)(\alpha+2)(\alpha+3)\max\{|a^\alpha|^q,|b^\alpha|^q\}^{\frac{1}{q}}.$$

Proof. The assertion follows from inequality (2.2.3) applied to the mapping (3.1.1). □

References

[1] M. Alomari, M. Darus, U.S. Kirmaci, Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, Comp. Math. Appl. 59 (2010) 225–232.

[2] M. Alomari, M. Darus, Some Ostrowski type inequalities for quasi-convex functions with applications to special means, RGMIA 13 (2) (2010). Article No. 3. Preprint.

[3] M. Alomari, M. Darus, On the Hadamard's inequality for log-convex functions on the coordinates, J. Ineq. Appl. Volume 2009, Article ID 283147, 13 pp. doi:http://dx.doi.org/10.1155/2009/283147.

[4] S.S. Dragomir, Two mappings in connection to Hadamard's inequalities, J. Math. Anal. Appl. 167 (1992) 49–56.

[5] S.S. Dragomir, Y.J. Cho, S.S. Kim, Inequalities of Hadamard's type for Lipschitzian mappings and their applications, J. Math. Anal. Appl. 245 (2000) 489–501.

[6] D.A. Ion, Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, Ann. Univ. Craiova Math. Comp. Sci. Ser. 34 (2007) 82–87.

[7] U.S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula, Appl. Math. Comp. 147 (2004) 137–146.

[8] U.S. Kirmaci, M.E. Ozdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp. 153 (2004) 361–368.

[9] C.E.M. Pearce, J. Pecari'c, Inequalities for differentiable mappings with application to special means and quadrature formula, Appl. Math. Lett. 13 (2000) 51–55.

[10] G.S. Yang, D.Y. Hwang, K.L. Tseng, Some inequalities for differentiable convex and concave mappings, Appl. Math. Lett. 47 (2004) 207–216.

[11] J. Pecaric, F. Proschan, Y.L. Tong, Convex functions, partial ordering and statistical applications, Academic Press, New York, 1991.

[12] L. Chun, F. Qi, Integral inequalities for Hermite-Hadamard type for functions whose 3rd derivatives are s-convex, Appl. Math. 3 (2012) 1680–1885.