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ORIGINAL ARTICLE

# Inequalities for Humbert functions

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**Abstract** This paper is motivated by an open problem of Luke's theorem. We consider the problem of developing a unified point of view on the theory of inequalities of Humbert functions and of their general ratios are obtained. Some particular cases and refinements are given. Finally, we obtain some important results involving inequalities of Bessel and Whittaker's functions as applications.

**MATHEMATICS SUBJECT CLASSIFICATION:** 26D15; 30A10; 26D07; 26D20



## 1. Introduction

The two sided inequalities for generalized hypergeometric functions have been established by Luke [1–4] through Padé approximation. He later obtained the bounds of Appell's hypergeometric functions $F_1$, $F_2$ and $F_3$, through their Euler integral representations (see or instance Luke [4]). An extension to the framework of the classical families of inequalities for Gauss and Kummer's hypergeometric, Laguerre, Bessel and modified Bessel functions have been introduced in [5–19]. The reason of interest for this family of Humbert functions is due to their intrinsic mathematical importance and to the fact that these functions have applications in physics. Here we obtain some inequalities for Humbert, Bessel and Whittaker's functions. In what follows we observe that

$$1-z<e^{-z}<1-z+\frac{1}{2}z^2,\quad z>0,\tag{1.1}$$

and from Luke's results [4], can write that

$$1+z<e^{z}<1+2z,\quad 0<z<1,\tag{1.2}$$

then the function

$${}_0F_2(-,-;a,b;-z)=\sum_{k\geqslant 0}\frac{(-z)^k}{(a)_k(b)_k k!},\quad a,b>0,\quad z>0,\tag{1.3}$$

where

$$(a)_k=a(a+1)(a+2)\cdots(a+k-1)=\frac{\Gamma(a+k)}{\Gamma(a)};\quad k\geqslant 1;\quad (a)_0=1,$$

satisfies the inequalities

$$1-\frac{z}{ab}<{}_0F_2(-,-;a,b;-z)<1-\frac{z}{ab}+\frac{z^2}{2(a)_2(b)_2};\quad a,b>0,\ z>0,\tag{1.4}$$

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and

$$1+\frac{z}{ab}<{}_0F_2(-,-;a,b;z)<1+\frac{2z}{ab}\quad a,b>0,\ \ 0<z<1. \tag{1.5}$$

The proofs of (1.4) and (1.5) follow of help are the beta integral representation, confluence principle and the inequalities (1.1) and (1.2), respectively.

Thus, it follows from (1.4) that

$$\begin{aligned}1-\frac{z^3}{27(a+1)(b+1)}&<\ {}_0F_2\left(-,-;a+1,b+1;-\frac{z^3}{27}\right)\\ &<1-\frac{z^3}{27(a+1)(b+1)}\\ &\quad+\frac{z^6}{1458(a+1)(a+2)(b+1)(b+2)};\\ &\quad a,b>-1,\ z>0,\end{aligned} \tag{1.6}$$

and from (1.5), we get

$$\begin{aligned}1+\frac{z^3}{27(a+1)(b+1)}&<\ {}_0F_2\left(-,-;a+1,b+1;\frac{z^3}{27}\right)\\ &<1+\frac{2z^3}{27(a+1)(b+1)},\\ &\quad a,b>-1,\quad 0<z<1.\end{aligned} \tag{1.7}$$

In a similar way, we define the exponential functions as the following:

$$1-z-w<e^{-(z+w)}<1-z-w+\frac{1}{2}z^2 +zw+\frac{1}{2}w^2,\quad z+w>0, \tag{1.8}$$

$$1+z+w<e^{z+w}<1+2z+2w,\quad 0<z+w<1. \tag{1.9}$$

In general, a new generalized form of the hypergeometric functions ${}_0F_2$ is introduced by using the integral representation method

$${}_0F_4(-,-;a,b,c,d;-z,-w)=\sum_{h,k\geqslant 0}\frac{(-z)^h(-w)^k}{(a)_h(b)_h(c)_k(d)_k h!k!},\quad a,b,c,d>0,\ \ z>0,\ w>0, \tag{1.10}$$

$$\begin{aligned}&1-\frac{z}{(a+1)(b+1)}-\frac{w}{(c+1)(d+1)}\\ &<\ {}_0F_4(-,-;a+1,b+1,c+1,d+1;-z,-w)\\ &<1-\frac{z}{(a+1)(b+1)}-\frac{w}{(c+1)(d+1)}+\frac{z^2}{2(a)_2(b)_2}\\ &+\frac{zw}{(a+1)(b+1)(c+1)(d+1)}+\frac{w^2}{2(c)_2(d)_2};\\ &a,b,c,d>-1,\ z>0,\ w>0,\end{aligned} \tag{1.11}$$

and

$$\begin{aligned}&1+\frac{z}{(a+1)(b+1)}+\frac{w}{(c+1)(d+1)}\\ &\quad<\ {}_0F_4(-,-;a+1,b+1,c+1,d+1;z,w)\\ &\quad<1+\frac{2z}{(a+1)(b+1)}+\frac{2w}{(c+1)(d+1)}\\ &\quad a,b,c,d>-1,\quad 0<z<1,\ 0<w<1.\end{aligned} \tag{1.12}$$

In the next section, we will discuss further of the Humbert functions of one variable of the inequalities properties investigated.

## 2. Inequalities for Humbert functions of one variable

The Humbert function $J_{a,b}(z)$ of one variable $z$ for non-negative integers $a$, $b$ be defined in the form [20,21]:

$$\begin{aligned}J_{a,b}(z)&=\frac{1}{\Gamma(a+1)\Gamma(b+1)}\left(\frac{z}{3}\right)^{a+b}\ {}_0F_2\left(-,-;a+1,b+1;-\frac{z^3}{27}\right)\\ &=\sum_{k\geqslant 0}\frac{(-1)^k z^{a+b+3k}}{3^{a+b+3k}\Gamma(a+k+1)\Gamma(b+k+1)k!},\end{aligned} \tag{2.1}$$

where

$${}_0F_2\left(-,-;a+1,b+1;-\frac{-z^3}{27}\right)=\sum_{k\geqslant 0}\frac{(-1)^k z^{3k}}{3^{3k}(a+1)_k(b+1)_k k!}.$$

From (2.1) and (1.6), we find that

$$\begin{aligned}1-\frac{z^3}{27(a+1)(b+1)}&<\Gamma(a+1)\Gamma(b+1)\left(\frac{3}{z}\right)^{a+b}J_{a,b}(z)\\ &<1-\frac{z^3}{27(a+1)(b+1)}\\ &\quad+\frac{z^6}{1458(a+1)(a+2)(b+1)(b+2)};\\ &\quad a,b>-1,\ z>0.\end{aligned} \tag{2.2}$$

Eq. (2.2) can be expressed as

$$\begin{aligned}&\left(\frac{3}{x}\right)^{a+b}\left(\frac{\Gamma(a+1)\Gamma(b+1)}{1-\frac{x^3}{27(a+1)(b+1)}+\frac{x^6}{1458(a+1)(a+2)(b+1)(b+2)}}\right)<\frac{1}{J_{a,b}(x)}\\ &\quad<\left(\frac{3}{x}\right)^{a+b}\left(\frac{\Gamma(a+1)\Gamma(b+1)}{1-\frac{x^3}{27(a+1)(b+1)}}\right);\quad a,b>-1,\\ &\quad 0<x\leqslant\max\left(1,3\sqrt[3]{(a+1)(b+1)}\right).\end{aligned} \tag{2.3}$$

Thus, combining (2.2) and (2.3) we can write

$$\begin{aligned}&\left(\frac{z}{x}\right)^{a+b}\left(\frac{1-\frac{z^3}{27(a+1)(b+1)}}{1-\frac{x^3}{27(a+1)(b+1)}+\frac{x^6}{1458(a+1)(a+2)(b+1)(b+2)}}\right)<\frac{J_{a,b}(z)}{J_{a,b}(x)}\\ &\quad<\left(\frac{z}{x}\right)^{a+b}\left(\frac{1-\frac{z^3}{27(a+1)(b+1)}+\frac{z^6}{1458(a+1)(a+2)(b+1)(b+2)}}{1-\frac{x^3}{27(a+1)(b+1)}}\right)\quad a,b>-1,\ z>0,\\ &<x\leqslant\max\left(1,3\sqrt[3]{(a+1)(b+1)}\right),\end{aligned} \tag{2.4}$$

and also, for $a,b>-2, 0<z\leqslant\max\left(1,3\sqrt[3]{(a+2)(b+2)}\right)$, we get

$$\begin{aligned}&\left(\frac{9(a+1)(b+1)}{z^2}\right)\left(\frac{1-\frac{z^3}{27(a+1)(b+1)}}{1-\frac{z^3}{27(a+2)(b+2)}+\frac{z^6}{1458(a+2)(a+3)(b+2)(b+3)}}\right)\\ &\quad<\frac{J_{a,b}(z)}{J_{a+1,b+1}(z)}\\ &\quad<\left(\frac{9(a+1)(b+1)}{z^2}\right)\left(\frac{1-\frac{z^3}{27(a+1)(b+1)}+\frac{z^6}{1458(a+1)(a+2)(b+1)(b+2)}}{1-\frac{z^3}{27(a+2)(b+2)}}\right).\end{aligned} \tag{2.5}$$

Our interest is to show that our inequality (2.2) gives inequality at $z=1$

$$\begin{aligned}1-\frac{1}{27(a+1)(b+1)}&<\Gamma(a+1)\Gamma(b+1)3^{a+b}J_{a,b}(1)\\ &<1-\frac{1}{27(a+1)(b+1)}\\ &\quad+\frac{1}{1458(a+1)(a+2)(b+1)(b+2)};\quad a,b>-1.\end{aligned} \tag{2.6}$$

As an example for $a = b = 2$, $z = 1$, we have from (2.2)

$0.003073718 < J_{2,2}(1) < 0.003073733.$

For $a = b = 1$, $z = 0.9$, from (2.2) we have

$0.0893925 < J_{1,1}(0.9) < 0.089393411.$

## 3. Inequalities for Humbert functions of two variables

It is the purpose of this section to introduce a new inequalities for Humbert functions of two variables, which represents a generalization of the exponential functions and hypergeometric functions ${}_0F_2$ as given by the relations (1.8)–(1.11) and (1.12). Now we define the Humbert functions of two variables as the following

$$J_{a,b,c,d}(z,w) = \frac{z^{a+b}w^{c+d}}{3^{a+b+c+d}\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)\Gamma(d+1)} \times {}_0F_4\left(-,-;a+1,b+1,c+1,d+1;-\frac{z^3}{27},-\frac{w^3}{27}\right) = \sum_{h,k\geqslant 0}\frac{(-1)^{h+k}z^{a+b+3h}}{3^{a+b+c+d+3h+3k}\Gamma(a+h+1)\Gamma(b+h+1)\Gamma(c+k+1)\Gamma(d+k+1)h!k!}; \quad a,b,c,d > -1, \tag{3.1}$$

and with the same manner in the preceding, from (1.7) for the Humbert functions of two complex variables, we have the inequalities

$$1-\frac{z^3}{27(a+1)(b+1)}-\frac{w^3}{27(c+1)(d+1)} < \frac{3^{a+b+c+d}\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)\Gamma(d+1)}{z^{a+b}w^{c+d}}J_{a,b,c,d}(z,w) < 1-\frac{z^3}{27(a+1)(b+1)}-\frac{w^3}{27(c+1)(d+1)} + \frac{z^6}{1458(a+1)(a+2)(b+1)(b+2)} + \frac{z^3w^3}{729(a+1)(b+1)(c+1)(d+1)} + \frac{w^6}{1458(c+1)(c+2)(d+1)(d+2)}; \quad a,b,c,d > -1, \quad z,w > 0, \tag{3.2}$$

For the set of values $z = w = 0.4$, $x = y = 0.8$, $a = b = c = d = 1$, gives from

$$0.062431845 < \frac{J_{1,1,1,1}(0.4,0.4)}{J_{1,1,1,1}(0.8,0.8)} < 0.062431876.$$

In conclusion, we shall now show that there exist operational relations between Humbert functions and the various type of Whittaker's and Bessel functions, more results could be obtained, but the details are omitted for reasons of brevity.

$$\frac{1}{1-\frac{x^3}{27(a+1)(b+1)}-\frac{y^3}{27(c+1)(db+1)}+\frac{x^6}{1458(a+1)(a+2)(b+1)(b+2)}+\frac{x^3y^3}{729(a+1)(b+1)(c+1)(d+1)}+\frac{y^6}{1458(c+1)(c+2)(d+1)(d+2)}} < \frac{x^{a+b}y^{c+d}}{3^{a+b+c+d}\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)\Gamma(d+1)J_{a,b,c,d}(x,y)} < \frac{1}{1-\frac{x^3}{27(a+1)(b+1)}-\frac{y^3}{27(c+1)(d+1)}}, \quad a,b,c,d > -1, \quad 0 < x \leqslant \max\left(1, 3\sqrt[3]{(a+1)(b+1)}\right), \quad 0 < y \leqslant \max\left(1, 3\sqrt[3]{(c+1)(d+1)}\right), \tag{3.3}$$

$$\left(\frac{z}{x}\right)^{a+b}\left(\frac{w}{y}\right)^{c+d}\left(\frac{1-\frac{z^3}{27(a+1)(b+1)}-\frac{w^3}{27(c+1)(d+1)}}{1-\frac{x^3}{27(a+1)(b+1)}-\frac{y^3}{27(c+1)(d+1)}+\frac{x^6}{1458(a+1)(a+2)(b+1)(b+2)}+\frac{x^3y^3}{729(a+1)(b+1)(c+1)(d+1)}+\frac{y^6}{1458(c+1)(c+2)(d+1)(d+2)}}\right) < \frac{J_{a,b,c,d}(z,w)}{J_{a,b,c,d}(x,y)} < \left(\frac{z}{x}\right)^{a+b}\left(\frac{w}{y}\right)^{c+d}\left(\frac{1-\frac{z^3}{27(a+1)(b+1)}-\frac{w^3}{27(c+1)(d+1)}+\frac{z^6}{1458(a+1)(a+2)(b+1)(b+2)}+\frac{z^3w^3}{729(a+1)(b+1)(c+1)(d+1)}+\frac{w^6}{1458(c+1)(c+2)(d+1)(d+2)}}{1-\frac{x^3}{27(a+1)(b+1)}-\frac{y^3}{27(c+1)(d+1)}}\right), \quad a,b,c,d > -1, \quad z,w > 0, \quad 0 < x \leqslant \max\left(1, 3\sqrt[3]{(a+1)(b+1)}\right), \quad 0 < y \leqslant \max\left(1, 3\sqrt[3]{(c+1)(d+1)}\right), \tag{3.4}$$

and

$$\left(\frac{9(a+1)(b+1)}{z^2}\right)\left(\frac{9(c+1)(d+1)}{w^2}\right) \times \left(\frac{1-\frac{z^3}{27(a+1)(b+1)}-\frac{w^3}{27(c+1)(d+1)}}{1-\frac{z^3}{27(a+2)(b+2)}-\frac{w^3}{27(c+2)(d+2)}+\frac{z^6}{1458(a+2)(a+3)(b+2)(b+3)}+\frac{z^3w^3}{729(a+2)(b+2)(c+2)(d+2)}+\frac{w^6}{1458(c+2)(c+3)(d+2)(d+3)}}\right) < \frac{J_{a,b,c,d}(z,w)}{J_{a+1,b+1,c+1,d+1}(z,w)} < \left(\frac{9(a+1)(b+1)}{z^2}\right)\left(\frac{9(c+1)(d+1)}{w^2}\right) \times \left(\frac{1-\frac{z^3}{27(a+1)(b+1)}-\frac{w^3}{27(c+1)(d+1)}+\frac{z^6}{1458(a+1)(a+2)(b+1)(b+2)}+\frac{z^3w^3}{729(a+1)(b+1)(c+1)(d+1)}+\frac{w^6}{1458(c+1)(c+2)(d+1)(d+2)}}{1-\frac{z^3}{27(a+2)(b+2)}-\frac{w^3}{27(c+2)(d+2)}}\right), \quad a,b,c,d > -2, \quad 0 < z \leqslant \max\left(1, 3\sqrt[3]{(a+2)(b+2)}\right), 0 < w \leqslant \max\left(1, 3\sqrt[3]{(c+2)(d+2)}\right). \tag{3.5}$$

## 4. Inequalities for Whittaker's and Bessel functions

The Whittaker's functions $M_{k,m}(z)$ defined by the relation [22]

$$x^{\frac{4}{3}r}J_{r+s,2s}\left(3x^{\frac{2}{3}}\right)\doteqdot\frac{(-1)^s\left(\frac{4}{p^2}\right)^{r-\frac{1}{2}}\Gamma\left(r+s+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(2s+1)}M_{r,s}\left(\frac{4}{p^2}\right);$$
$$\frac{1}{p^\alpha}\doteqdot\frac{x^\alpha}{\Gamma(\alpha+1)},\quad R\left(r+s+\frac{1}{2}\right)>0. \tag{4.1}$$

From (4.1), we can write the relation

$$M_{r,s}(x)\doteqdot\frac{(-1)^s\sqrt{\pi}\Gamma(2s+1)2^{\frac{2}{3}r}x^{\frac{1}{2}-\frac{1}{3}r}}{\Gamma\left(r+s+\frac{1}{2}\right)}J_{r+s,2s}\left(3\sqrt[3]{\frac{x}{2}}\right). \tag{4.2}$$

We put that $a = r + s$, $b = 2s$ and $z = 3\sqrt[3]{\frac{x}{2}}$ into (2.2), it follows that

$$\left(\frac{\sqrt{\pi}x^{\frac{1}{2}+s}}{(-1)^s\Gamma(r+s+1)\Gamma\left(r+s+\frac{1}{2}\right)2^{s-\frac{1}{3}r}}\right)\left(1-\frac{x}{2(r+s+1)(2s+1)}\right)<M_{r,s}(x).$$
$$<\left(\frac{\sqrt{\pi}x^{\frac{1}{2}+s}}{(-1)^s\Gamma(r+s+1)\Gamma\left(r+s+\frac{1}{2}\right)2^{s-\frac{1}{3}r}}\right)$$
$$\left(1-\frac{x}{2(r+s+1)(2s+1)}+\frac{x^2}{8(r+s+1)(r+s+2)(2s+1)(2s+2)}\right), \tag{4.3}$$

thus, we can deduce that

$$\frac{1}{\sqrt{\pi}}\frac{(-1)^s2^{s-\frac{1}{3}r}x^{-s-\frac{1}{2}}\Gamma(r+s+1)\Gamma\left(r+s+\frac{1}{2}\right)}{1-\frac{x}{2(r+s+1)(2s+1)}+\frac{x^2}{8(r+s+1)(r+s+2)(2s+1)(2s+2)}}<\frac{1}{M_{r,s}(x)}$$
$$<\frac{1}{\sqrt{\pi}}\frac{(-1)^s2^{s-\frac{1}{3}r}x^{-s-\frac{1}{2}}\Gamma(r+s+1)\Gamma\left(r+s+\frac{1}{2}\right)}{1-\frac{x}{2(r+s+1)(2s+1)}}. \tag{4.4}$$

For example, if $s>\frac{-1}{2}, r>\frac{-1}{2}$, then (4.3) gives

$$0.5 < M_{0,0}(1) < 053125,$$
$$1.852662388 < M_{1,0}(1) < 1.8728325,$$
$$0.16111111 < M_{0,2}(1) < 0.161168981,$$
$$0.027298289 < M_{1,2}(1) < 0.027304122.$$

Her has given the following form for the operational relations of this Bessel function of the first kind [20,21,23]

$$x^{\frac{2m-n}{3}}J_{m,n}\left(3\sqrt[3]{x}\right)\doteqdot\frac{1}{p^{\frac{2m-n}{2}}}J_n\left(-2\sqrt{\frac{1}{p}}\right), \tag{4.5}$$

and

$$x^{\frac{2n-m}{3}}J_{m,n}\left(3\sqrt[3]{x}\right)\doteqdot\frac{1}{p^{\frac{2n-m}{2}}}J_m\left(-2\sqrt{\frac{1}{p}}\right). \tag{4.6}$$

where $\frac{1}{p^\alpha}\doteqdot\frac{x^\alpha}{\Gamma(\alpha+1)}$, $R(\alpha+1)>0$ and $J_n(x)$ is one of Bessel function of the first kind an inequalities for it:

$$\left(1-\frac{z^2}{4(a+1)}\right)\frac{\left(\frac{z}{2}\right)^a}{\Gamma(a+1)}<J_a(z)$$
$$<\left(1-\frac{z^2}{4(a+1)}+\frac{z^4}{32(a+1)(a+2)}\right)\frac{\left(\frac{z}{2}\right)^a}{\Gamma(a+1)};$$
$$a>-1,\quad z>0, \tag{4.7}$$

and

$$\left(\frac{1}{1-\frac{x^2}{4(a+1)}+\frac{x^4}{32(a+1)(a+2)}}\right)\Gamma(a+1)\left(\frac{x}{2}\right)^{-a}<\frac{1}{J_a(x)}$$
$$<\left(\frac{1}{1-\frac{x^2}{4(a+1)}}\right)\Gamma(a+1)\left(\frac{x}{2}\right)^{-a},\quad a>-1,\quad 0<x$$
$$\leqslant\max\left(1,2\sqrt{a+1}\right). \tag{4.8}$$

As an example for $a = 0$, $a = 2$, $z = 1$ and from (4.7), we get

$$0.75 < J_0(1) < 0.765625,$$
$$0.114583333 < J_2(1) < 0.114908854.$$

If $a = 1$, $z = 0.1$, then (4.7) gives

$$0.0498375 < J_1(0.1) < 0.049937526041.$$

In [24], the Bessel functions $J_{a,b}(z, w)$ of two index of two variables defined by

$$J_{a,b}(z,w)=\frac{z^aw^b}{\Gamma(a+1)\Gamma(b+1)2^{a+b}}{}_0F_2\left(-,-;a+1,b+1;\frac{-z^2}{4},\frac{-w^2}{4}\right)$$
$$=\sum_{h,k\geqslant0}\frac{(-1)^{h+k}z^{2h+a}w^{2k+b}}{2^{a+b+2(h+k)}\Gamma(a+h+1)\Gamma(b+k+1)h!k!}, \tag{4.9}$$

where

$${}_0F_2\left(-,-;a+1,b+1;\frac{-z^2}{4},\frac{-w^2}{4}\right)$$
$$=\sum_{h,k\geqslant0}\frac{(-1)^{h+k}z^{2h}w^{2k}}{2^{2(h+k)}(a+1)_h(b+1)_kh!k!},\quad a,b>-1,\quad z,w>0.$$

It is evident that the ordinary Humbert functions of two variables are linked to the bessel functions of two index of two variables by

$$x^{\frac{2a-b}{3}}y^{\frac{2c-d}{3}}J_{a,b,c,d}\left(3\sqrt[3]{x},3\sqrt[3]{y}\right)\doteqdot\frac{1}{p^{\frac{2a-b}{2}}q^{\frac{2c-d}{2}}}J_{b,d}\left(-2\sqrt{\frac{1}{p}},-2\sqrt{\frac{1}{q}}\right). \tag{4.10}$$

From (4.9) and (4.10), we find that

$$\left(1-\frac{z^2}{4(a+1)}-\frac{w^2}{4(b+1)}\right)\left(\frac{z^aw^b}{2^{a+b}\Gamma(a+1)\Gamma(b+1)}\right)$$
$$<J_{a,b}(z,w)<\left(1-\frac{z^2}{4(a+1)}-\frac{w^2}{4(b+1)}+\frac{z^4}{32(a+1)(a+2)}\right.$$
$$\left.+\frac{z^2w^2}{16(a+1)(b+1)}+\frac{w^4}{32(b+1)(b+2)}\right)$$
$$\times\left(\frac{z^aw^b}{2^{a+b}\Gamma(a+1)\Gamma(b+1)}\right);\quad a,b>-1,\quad z,w>0. \tag{4.11}$$

For example, when $a = b = 2$, $z = w = 1$, form (4.11) we get

$$0.013020833 < J_{2,2}(1,1) < 0.1321072049.$$

For $a = b = 1$, $z = w = 0.9$, from (4.11) we have

$$0.16149375 < J_{1,1}(0.9, 0.9) < 0.1649536523.$$

Here, we have obtained inequalities for some special functions from a known result for hypergeometric functions. This approach opens new possibilities to deal with other members of Humbert, Whittaker's and Bessel functions as well as hypergeometric functions. Hence, new results and further applications can be obtained and will be discussed in a forthcoming paper.

### *4.1. Open problem*

How can one derive the above identity directly from the properties of ${}_0F_2$? Such derivation would immediately give another proof inequalities for the fundamental relations of generalized hypergeometric function.

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