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ORIGINAL ARTICLE

On generalized Jordan *-derivation in rings

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Abstract Let $n \geqslant 1$ be a fixed integer and let R be an $(n+1)!$-torsion free *-ring with identity element e. If F, $d:R \to R$ are two additive mappings satisfying $F(x^{n+1}) = F(x)(x^*)^n + xd(x)(x^*)^{n-1} + x^2d(x)(x^*)^{n-2} + \cdots + x^nd(x)$ for all $x \in R$, then d is a Jordan *-derivation and F is a generalized Jordan *-derivation on R.

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1. Introduction

Throughout R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, where $n > 1$ is an integer, in case $nx = 0$, $x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. Recall that R is prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies $a = 0$. An additive mapping $d:R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs x, $y \in R$ and is called a Jordan derivation in case $d(x^2) = d(x)x + xd(x)$ is fulfilled for all $x \in R$. Every derivation is a Jordan derivation but the converse need not be true in general. A classical result of Herstein [8, Theorem 3.3] states that every Jordan derivation on a prime ring of characteristic different from two is a derivation. Bresar and Vukman proved this result briefly in [5]. Further, Cusack [4] has generalized this result for semiprime ring stating that every Jordan derivation on a 2-torsion free semiprime ring is a derivation (see also [4] for an alternate proof). In [3] Bresar has introduced the concept of a generalized derivation as follows: an additive mapping $F:R \to R$ is said to be generalized derivation if there exists an associated derivation $d:R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all pairs $x,y \in R$. An additive mapping $F:R \to R$ is said to be generalized Jordan derivation if there exists a Jordan derivation $d:R \to R$ such that $F(x^2) = F(x)x + xd(x)$ for all $x \in R$. In [1], Ashraf and the first author showed that in a 2-torsion free ring, which has a commutator nonzero divisor, every generalized Jordan derivation on R is generalized derivation. Recently, Vukman [9] has proved that every generalized Jordan derivation on a 2-torsion free semiprime ring is a generalized derivation.

Following [2], an additive mapping $d:R \to R$ is called Jordan triple derivation if $d(xyx) = d(x)yx + xd(y)x + xyd(x)$ holds for all x, $y \in R$. One can easily proved that any Jordan derivation on a 2-torsion free ring is a Jordan triple derivation (see example [2], where further reference can be found). Bresar

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[2] has proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Motivated by the definition of generalized Jordan derivation, Jing and Lu [10] introduced the concept of generalized Jordan triple derivation as follows: an additive mapping $F:R \to R$ is said to be a generalized Jordan triple derivation on R if there exists a Jordan triple derivation $d:R \to R$ such that $F(xyx) = F(x)yx + xd(y)x + xyd(x)$ holds for all $x,y \in R$. Inspired by the definition of generalized Jordan triple derivation, recently Dhara and Shrama [7] proved the following theorem:

Theorem 1.1. *Let $n \geqslant 1$ be a fixed integer and let R be an $(n+1)!$-torsion free any ring. If $F:R \to R$ and $d:R \to R$ are two additive mappings satisfying $F(x^{n+1}) = F(x)x^n + xd(x)(x)^{n-1} + x^2d(x)x^{n-2} + \cdots + x^nd(x)$ for all $x \in R$. Then d is a Jordan derivation and F is a generalized Jordan derivation.*

An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(x^*)^* = x$ and $(xy)^* = y^*x^*$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or *-ring. An additive mapping $d:R \to R$ is called a *-derivation (resp. Jordan *-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) for all $x, y \in R$. An additive mapping $F:R \to R$ is said to be a generalized *-derivation (resp. generalized Jordan *-derivation) if there exists a *-derivation (Jordan *-derivation) such that $F(xy) = F(x)y^* + xd(y)$ (resp. $F(x^2) = F(x)x^* + xd(x)$) for all $x, y \in R$.

If $F:R \to R$ and $d:R \to R$ are additive mappings satisfying

$$F(x^{n+1}) = F(x)(x^*)^n + xd(x)(x^*)^{n-1} + x^2d(x)(x^*)^{n-2} + \cdots + x^nd(x) \tag{1.1}$$

for all $x \in R$. In view of Theorem 1.1, it is natural to ask the additive mappings satisfying (1.1), implies that $F(x^2) = F(x)x^* + xd(x)$ and $d(x^2) = d(x)x^* + xd(x)$ for all $x \in R$. In a very recent paper Dhara et. al. [6] studied the identity $F(x^{m+n+1}) = F(x)x^{m+n} + x^mD(x)x^n$ for x in a non-central Lie ideal of a prime ring R, where both F and D are generalized derivations of R and then determined the relationship between the structure of F and D. In the present paper, we improve Theorem 1.1 in the setting of *-ring and also has the same flavour as [6]. In fact, it is shown that additive mappings F, $d:R \to R$ on an $(n+1)!$-torsion free *-ring R satisfying (1.1), implies $d(x^2) = d(x)x^* + xd(x)$ and $F(x^2) = F(x)x^* + xd(x)$ for all $x \in R$.

2. Main result

We begin our discussion with the following lemma which is essential for developing the proof of main theorem:

Lemma 2.1. *Let R be a ring with involution *, then $e^* = e$, where e is the identity element of R.*

Proof. Since $e = (e^*)^* = (ee^*)^* = ee^* = e^*$. This yields the required result. □

Theorem 2.1. *Let $n \geqslant 1$ be a fixed integer and let R be an $(n+1)!$-torsion free *-ring with identity element e. If F, $d:R \to R$ are two additive mappings satisfying $F(x^{n+1}) = F(x)(x^*)^n + xd(x)(x^*)^{n-1} + x^2d(x)(x^*)^{n-2} + \cdots + x^nd(x)$ for all $x \in R$, then d is a Jordan *-derivation and F is a generalized Jordan *-derivation on R.*

Proof. Given that

$$\begin{aligned} F(x^{n+1}) &= F(x)(x^*)^n + xd(x)(x^*)^{n-1} + x^2d(x)(x^*)^{n-2} + \cdots + x^nd(x) \\ &= F(x)(x^*)^n + \sum_{i=1}^{n} x^i d(x)(x^*)^{n-i} \text{ for all } x \in R. \quad \square \end{aligned} \tag{2.1}$$

Replacing x by e in (2.1), we get $F(e) = F(e)(e^*)^n + \sum_{i=1}^{n} e^i d(e)(e^*)^{n-i}$. But, by Lemma 2.1, we obtain $F(e) = F(e) + nd(e)$. This implies that $nd(e) = 0$ and since R is n-torsion free, we get $d(e) = 0$. Replacing x by $x + ke$ in (2.1), where k be any positive integer, we obtain

$$\begin{aligned} F((x+ke)^{n+1}) &= F(x+ke)((x+ke)^*)^n + \sum_{i=1}^{n}(x+ke)^i d(x+ke)((x+ke)^*)^{n-i} \\ &= (F(x)+F(ke))(x^*+ke)^n + \sum_{i=1}^{n}(x+ke)^i d(x+ke)(x^*+ke)^{n-i} \\ &= (F(x)+kF(e))(x^*+ke)^n + \sum_{i=1}^{n}(x+ke)^i d(x)(x^*+ke)^{n-i}. \end{aligned} \tag{2.2}$$

On expanding, we find that

$$\begin{aligned} &F\left(x^{n+1} + \binom{n+1}{1}x^nk + \binom{n+1}{2}x^{n-1}k^2 + \cdots + k^{n+1}e\right) \\ &= (F(x)+kF(e))\left\{(x^*)^n + \binom{n}{1}(x^*)^{n-1}k + \binom{n}{2}(x^*)^{n-2}k^2 + \cdots + k^ne + \sum_{i=1}^{n}\left\{x^i + \cdots + \binom{i}{i-2}x^2k^{i-2} \right.\right. \\ &\left. + \binom{i}{i-1}xk^{i-1} + k^ie\right\}d(x)\left\{(x^*)^{n-i} + \cdots + \binom{n-i}{n-i-2}(x^*)^2k^{n-i-2} \right. \\ &\left. + \binom{n-i}{n-i-1}x^*k^{n-i-1} + k^{n-i}e\right\} \end{aligned}$$

Now, using (2.1) we obtain

$$\begin{aligned} &F\left\{\binom{n+1}{1}x^nk + \binom{n+1}{2}x^{n-1}k^2 + \cdots + k^{n+1}e\right\} \\ &= kF(e)(x^*)^n + (F(x)+kF(e))\left\{\binom{n}{1}(x^*)^{n-1}k + \binom{n}{2}(x^*)^{n-2}k^2 + \cdots + \binom{n}{n-2}(x^*)^2k^{n-2} \right. \\ &\left. + \binom{n}{n-1}x^*k^{n-1} + k^ne\right\} + \sum_{i=1}^{n}x^id(x)\left\{\binom{n-i}{1}(x^*)^{n-i-1}k \right. \\ &\left. + \cdots + \binom{n-i}{n-i-2}(x^*)^2k^{n-i-2} + \binom{n-i}{n-i-1}x^*k^{n-i-1} + k^{n-i}e\right\} \end{aligned}$$

$$+\sum_{i=1}^{n}\left\{\binom{i}{1}x^{i-1}k+\binom{i}{2}x^{i-2}k^2+\cdots+\binom{i}{i-2}x^2k^{i-2}\right.$$
$$\left.+\binom{i}{i-1}xk^{i-1}+k^ie\right\}d(x)\left\{(x^*)^{n-i}+\binom{n-i}{1}(x^*)^{n-i-1}k+\cdots\right.$$
$$\left.+\binom{n-i}{n-i-2}(x^*)^2k^{n-i-2}+\binom{n-i}{n-i-1}x^*k^{n-i-1}+k^{n-i}e\right\}$$

This can be written as

$$kf_1(x^*,e)+k^2f_2(x^*,e)+\cdots+k^nf_n(x^*,e)=0 \quad \text{for all } x\in R,$$

where $f_i(x^*,e)$ are the coefficients of $k^i s$ for all $i = 1, 2, \ldots, n$. Now, replacing k by $1, 2, \ldots, n$ in turn and considering the resulting system of n homogeneous equations, we get that the resulting matrix of the system is a Van der Monde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^n \\ \vdots & \vdots & \cdots & \vdots \\ n & n^2 & \cdots & n^n \end{pmatrix}.$$

Since the determinant of the matrix is equal to the product of positive integers, each of which is less then n, and since R is $(n + 1)!$-torsion free, it follows immediately that $f_i(x^*, e) = 0$ for all $x\in R$ and $i = 1, 2, \ldots, n$. Now, $f_n(x^*, e) = 0$ implies that

$$(n+1)F(x)=F(x)+nF(e)x^*+nd(x) \quad \text{for all } x\in R.$$

This yields that $nF(x) = nF(e)x^* + nd(x)$. Since R is n-torsion free, we get

$F(x)=F(e)x^*+d(x)$ for all $x\in R$.

Again, $f_{n-1}(x^*,e) = 0$ gives that

$$n(n+1)F(x^2)=2nF(x)x^*+n(n-1)F(e)(x^*)^2+n(n+1)xd(x) + n(n-1)d(x)x^* \quad \text{for all } x\in R.$$

Since R is n-torsion free, then we obtain

$$(n+1)F(x^2)=2F(x)x^*+(n-1)F(e)(x^*)^2+(n+1)xd(x) + (n-1)d(x)x^* \quad \text{for all } x\in R.$$

Since we have that $F(x) = F(e)x^* + d(x)$. Using this in the above relation, we find that

$$\begin{aligned}(n+1)F(x^2)&=2\{F(e)x^*+d(x)\}x^*+(n-1)F(e)(x^*)^2\\&\quad+(n+1)xd(x)+(n-1)d(x)x^*\\&=2F(e)(x^*)^2+2d(x)x^*+nF(e)(x^*)^2-F(e)(x^*)^2\\&\quad+nxd(x)+xd(x)+nd(x)x^*-d(x)x^*\\&=(n+1)F(e)(x^*)^2+(n+1)d(x)x^*\\&\quad+(n+1)xd(x)\end{aligned}$$

Since R is $(n + 1)$-torsion free, then

$$F(x^2)=F(e)(x^*)^2+d(x)x^*+xd(x) \quad \text{for all } x\in R, \tag{2.3}$$

and also we have that $F(x) = F(e)x^* + d(x)$ for all $x\in R$. Replacing x by x^2 in the pervious relation, we obtain

$$F(x^2)=F(e)(x^*)^2+d(x^2) \tag{2.4}$$

Equating (2.4) and (2.3), we find that

$$d(x^2)=d(x)x^*+xd(x) \quad \text{for all } x\in R. \tag{2.5}$$

Now, by (2.3), we can write

$$\begin{aligned}F(x^2)&=F(e)(x^*)^2+d(x)x^*+xd(x)\\&=(F(e)x^*+d(x))x^*+xd(x) \quad \text{for all } x\in R.\end{aligned}$$

Using $F(x) = F(e)x^* + d(x)$ in the above, we get $F(x^2) = F(x)x^* + xd(x)$ for all $x\in R$. Hence, we get the required result.

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