



ORIGINAL ARTICLE

# Numerical treatment for solving fractional Riccati differential equation

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## KEYWORDS

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**Abstract** This paper presents an accurate numerical method for solving fractional Riccati differential equation (FRDE). The proposed method so called fractional Chebyshev finite difference method (FCheb-FDM). In this technique, we approximate FRDE with a finite dimensional problem. The method is based on the combination of the useful properties of Chebyshev polynomials approximation and finite difference method. The Caputo fractional derivative is replaced by a difference quotient and the integral by a finite sum. By this method the given problem is reduced to a problem for solving a system of algebraic equations, and by solving this system, we obtain the solution of FRDE. Special attention is given to study the convergence analysis and estimate an error upper bound of the obtained approximate formula. Illustrative examples are included to demonstrate the validity and applicability of the proposed technique.

**MATHEMATICS SUBJECT CLASSIFICATION:** 65K10, 65G99, 35E99, 68U20

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## 1. Introduction

Ordinary and partial fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering [1]. Fractional calculus is a generalization of ordinary differentiation and integration to an arbitrary non-integer order. Many physical

processes appear to exhibit fractional order behavior that may vary with time or space. Most FDEs do not have exact solutions, so approximate and numerical techniques [2–8], must be used. Several numerical and approximate methods to solve FDEs have been given such as variational iteration method [6], homotopy perturbation method [9], Adomian decomposition method [10], homotopy analysis method [11] and collocation method [12–15].

The Riccati differential equation is named after the Italian Nobleman Count Jacopo Francesco Riccati (1676–1754). The book of Reid [16] contains the fundamental theories of Riccati equation, with applications to random processes, optimal control, and diffusion problems. Besides important engineering science applications that today are considered classical, such as stochastic realization theory, optimal control, robust

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stabilization, and network synthesis, the newer applications include such areas as financial mathematics [17]. The solution of this equation can be reached using classical numerical methods such as the forward Euler method and Runge–Kutta method. An unconditionally stable scheme was presented by Dubois and Saidi [18]. Bahnasawi et al. [19] presented the usage of Adomian decomposition method to solve the non-linear Riccati differential equation in an analytic form. Tan and Abbasbandy [11] employed the analytic technique called homotopy analysis method to solve the quadratic Riccati equation.

The fractional Riccati differential equation is studied by many authors and using different numerical methods. This problem is solved using by variational iteration method [20] and in [10] it solved using the Adomian decomposition method and others [21].

Clenshaw and Curtis [22] introduced a procedure for the numerical integration of a non-singular function  $y(x)$  by expanding the function in a series of Chebyshev polynomials and integration term by term. Elbarbary introduced Chebyshev finite difference approximation for the boundary value problems of integer derivatives [23,24]. The fractional derivatives of the function  $y(x)$  at the point  $x_k, 0 \leq k \leq N$  are expanded as a linear combination from the values of the function  $y(x)$  at the shifted Gauss–Lobatto points  $x_k = \frac{1}{2} - \frac{1}{2} \cos(\frac{\pi k}{N}), k = 0, 1, \dots, N$  associated with the interval  $[0, L]$ . In addition to the procedure for the numerical integration introduced by Clenshaw and Curtis to approximate the integral by a finite sum. Finally, we approximate the proposed problem and end up with a finite dimensional problem. The main characteristic of this new technique is that it gives a straight-forward algorithm in converting FRDE to a system of algebraic equations. The suggested method is more accurate in comparison to the finite difference and finite element methods as the approximation of the fractional derivatives is defined over the whole domain.

The main aim of the presented paper is concerned with an extension of the previous work on fractional differential equations and derive some general approximate formulae of Chebyshev-FDM and then we applied this approach to obtain the numerical solution of FRDE. Also, we present study of the convergence analysis of the proposed method.

In this article, we consider the fractional Riccati differential equation of the form

$$D^{(\alpha)}u(t) + u^2(t) - 1 = 0, \quad t > 0, \quad 0 < \alpha \leq 1, \quad (1)$$

the parameter  $\alpha$  refers to the fractional order of the time derivative. We also assume an initial condition

$$u(0) = u_0. \quad (2)$$

For  $\alpha = 1$ , Eq. (1) is the standard Riccati differential equation

$$\frac{du(t)}{dt} + u^2(t) - 1 = 0.$$

The exact solution to this equation is

$$u(t) = \frac{e^{2t} - 1}{e^{2t} + 1}.$$

The structure of this paper is arranged in the following way: In Section 2, we introduce some basic definitions about Caputo fractional derivatives and properties the shifted Chebyshev polynomials. Section 3, is assigned to study the existence and the uniqueness of the fractional Riccati differential equation. In

section 4, we give the basic formulation of the new operational matrix method using FCheb-FDM. In section 5, we introduce an error bound of the fractional derivatives. In section 6, we give the numerical implementation of the proposed method for solving FRDE to show the accuracy of the presented method. Finally, in section 7, the paper ends with a brief conclusion and some remarks.

## 2. Preliminaries and notations

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

### 2.1. The Caputo fractional derivatives

**Definition 1.** The Caputo fractional derivative operator  $D^{(\alpha)}$  of order  $\alpha$  is defined in the following form

$$D^{(\alpha)}f(x) = \frac{1}{\Gamma(m - \alpha)} \int_0^x \frac{f^{(m)}(\xi)}{(x - \xi)^{\alpha - m + 1}} d\xi, \quad \alpha > 0, \quad x > 0,$$

where  $m - 1 < \alpha \leq m, m \in \mathbb{N}$ .

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation

$$D^{(\alpha)}(\lambda p(x) + \mu q(x)) = \lambda D^{(\alpha)}p(x) + \mu D^{(\alpha)}q(x), \quad (3)$$

where  $\lambda$  and  $\mu$  are constants. For the Caputo's derivative we have

$$D^{(\alpha)} C = 0, \quad C \text{ is a constant}, \quad (4)$$

$$D^{(\alpha)} x^n = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < \lceil \alpha \rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & \text{for } n \in \mathbb{N}_0 \text{ and } n \geq \lceil \alpha \rceil. \end{cases} \quad (5)$$

We use the ceiling function  $\lceil \alpha \rceil$  to denote the smallest integer greater than or equal to  $\alpha$  and  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ . Recall that for  $\alpha \in \mathbb{N}$ , the Caputo differential operator coincides with the usual differential operator of integer order.

For more details on fractional derivatives definitions and its properties see [25].

### 2.2. The definition and properties of the shifted Chebyshev polynomials

The well known Chebyshev polynomials are defined on the interval  $[-1, 1]$  and can be determined with the aid of the following recurrence formula [5]

$$T_{n+1}(z) = 2z T_n(z) - T_{n-1}(z), \quad T_0(z) = 1, \quad T_1(z) = z, \\ n = 1, 2, \dots$$

It is well known that  $T_n(-1) = (-1)^n, T_n(1) = 1$ . The analytic form of the Chebyshev polynomials  $T_n(z)$  of degree  $n$  is given by

$$T_n(z) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i 2^{n-2i-1} \frac{n(n-i-1)!}{(i!(n-2i)!)} z^{n-2i}, \quad (6)$$

where  $\lfloor n/2 \rfloor$  denotes the integer part of  $n/2$ . The orthogonality condition is

$$\int_{-1}^1 \frac{T_i(z)T_j(z)}{\sqrt{1-z^2}} dz = \begin{cases} \pi, & \text{for } i = j = 0; \\ \frac{\pi}{2}, & \text{for } i = j \neq 0; \\ 0, & \text{for } i \neq j. \end{cases} \quad (7)$$

In order to use these polynomials on the interval  $[0, L]$  we define the so called shifted Chebyshev polynomials by introducing the change of variable  $z = \frac{2x}{L} - 1$ . So, the shifted Chebyshev polynomials are defined as

$$T_n^*(x) = T_n\left(\frac{2x}{L} - 1\right), \quad \text{where } T_0^*(x) = 1, \quad T_1^*(x) = \frac{2x}{L} - 1.$$

The analytic form of the shifted Chebyshev polynomial  $T_n^*(x)$  of degree  $n$  is given by

$$T_n^*(x) = n \sum_{k=0}^n (-1)^{n-k} \frac{(n+k-1)! 2^{2k}}{(n-k)! (2k)! L^k} x^k, \quad (8)$$

where  $T_n^*(0) = (-1)^n$ ,  $T_n^*(L) = 1$ . The orthogonality condition of these polynomials is

$$\int_0^L T_j^*(x) T_k^*(x) w(x) dx = \delta_{jk} h_k, \quad (9)$$

where the weight function  $w(x) = \frac{1}{\sqrt{Lx-x^2}}$ ,  $h_k = \frac{b_k}{2} \pi$ , with  $b_0 = 2$ ,  $b_k = 1$ ,  $k \geq 1$ .

The function  $y(x)$  which belongs to the space of square integrable in  $[0, L]$ , may be expressed in terms of shifted Chebyshev polynomials as

$$y(x) = \sum_{n=0}^{\infty} c_n T_n^*(x),$$

where the coefficients  $c_n$  are given by

$$c_n = \frac{1}{h_n} \int_0^L y(x) T_n^*(x) w(x) dx, \quad n = 0, 1, \dots \quad (10)$$

### 3. Existence and uniqueness

Let  $J = [0, T]$ ,  $T < \infty$  and  $C(J)$  be the class of all continuous functions defined on  $J$ , with the norm

$$\|u\| = \sup_{t \in J} |e^{-Nt} u(t)|, \quad N > 0,$$

which is equivalent to the sup-norm  $\|u\| = \sup_{t \in J} |u(t)|$ .

To study the existence and the uniqueness of the initial value problem of the fractional Riccati differential Eq. (1), we suppose that the solution  $u(t)$  belongs to the space  $B = \{u \in \mathcal{R} : |u| \leq b\}$ , for any constant  $b$ .

**Theorem 1.** *The initial value problem (1) has a unique solution  $u \in C(J)$ ,  $u' \in X = \{u \in L_1[0, T], \|u\| = \|e^{-Nt} u(t)\|_{L_1}\}$ .*

**Proof.** From the properties of fractional calculus, the fractional-order differential Eq. (1) can be written as [26]

$$I^{1-\alpha} \frac{du(t)}{dt} = 1 - u^2(t).$$

Operating with  $I^\alpha$  we obtain

$$u(t) = I^\alpha(1 - u^2(t)). \quad (11)$$

Now, let us define the operator  $F: C(J) \rightarrow C(J)$  by

$$Fu(t) = I^\alpha(1 - u^2(t)), \quad (12)$$

then

$$\begin{aligned} e^{-Nt}(Fu - Fv) &= e^{-Nt} I^\alpha[(1 - u^2(t)) - (1 - v^2(t))] \\ &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{-N(t-s)} (v(s) - u(s))(v(s) \\ &\quad + u(s)) e^{-Ns} ds \\ &\leq \|v - u\| \int_0^t \frac{s^{\alpha-1} e^{-Ns}}{\Gamma(\alpha)} ds, \end{aligned}$$

therefore, we obtain

$$\|Fu - Fv\| \leq \|u - v\|,$$

and the operator  $F$  given by Eq. (12) has a unique fixed point. Consequently the integral Eq. (11) has a unique solution  $u \in C(J)$ . Also we can deduce that [26]

$$I^\alpha(1 - u^2(t))|_{t=0} = 0.$$

Now from Eq. (11), we formally have

$$u(t) = \left[ \frac{t^\alpha}{\Gamma(\alpha+1)} (1 - u_0^2) + I^{\alpha+1}(0 - 2u(t)u'(t)) \right],$$

and

$$\frac{du}{dt} = \left[ \frac{t^{\alpha-1}}{\Gamma(\alpha)} (1 - u_0^2) + I^\alpha(-2u(t)u'(t)) \right],$$

$$e^{-Nt} u'(t) = e^{-Nt} \left[ \frac{t^{\alpha-1}}{\Gamma(\alpha)} (1 - u_0^2) + I^\alpha(-2u(t)u'(t)) \right],$$

from which we can deduce that  $u' \in C(J)$  and  $u' \in X$ . Now from Eq. (11), we get

$$\frac{du}{dt} = \frac{d}{dt} I^\alpha[1 - u^2(t)],$$

$$I^{1-\alpha} \frac{du}{dt} = I^{1-\alpha} \frac{d}{dt} I^\alpha[1 - u^2(t)] = \frac{d}{dt} I^{1-\alpha} I^\alpha[1 - u^2(t)],$$

$$D^\alpha u(t) = \frac{d}{dt} I[1 - u^2(t)] = 1 - u^2(t),$$

$$\text{and } u(0) = I^\alpha[1 - u^2(t)]|_{t=0} = 0.$$

Then the integral Eq. (11) is equivalent to the initial value problem (1) and the theorem is proved.  $\square$

### 4. Basic formulation of the operational matrix method using FCheb-FDM

The well known shifted Chebyshev polynomials of the first kind of degree  $n$  are defined on the interval  $[0, L]$  as in Eq. (8). We choose the grid (interpolation) points to be the Chebyshev-Gauss Lobatto points associated with the interval  $[0, L]$ ,  $x_r = \frac{L}{2} - \frac{L}{2} \cos\left(\frac{r\pi}{N}\right)$ ,  $r = 0, 1, \dots, N$ . These grids can be written as  $L = x_N < x_{N-1} < \dots < x_1 < x_0 = 0$ .

Now, we reformulate the introduced approximate formula of function  $u(x)$  [22] to use it on the shifted Chebyshev polynomials as follows,

$$u_N(x) \cong \sum_{n=0}^N {}''a_n T_n^*(x), \quad a_n = \frac{2}{N} \sum_{r=0}^N {}''u(x_r) T_n^*(x_r). \quad (13)$$

The summation symbol with double primes denotes a sum with both first and last term halved. The fractional derivative of the function  $u(x)$  at the point  $x_s$  is given in the following theorem.

**Theorem 2.** The Caputo fractional derivative of order  $\alpha$  for the function  $u(x)$  at the point  $x_s$  is approximated by the following formula

$$D^{(\alpha)} u_N(x_s) \cong \sum_{r=0}^N d_{s,r}^{(\alpha)} u(x_r), \quad \alpha > 0, \quad (14)$$

such that

$$d_{s,r}^{(\alpha)} = \frac{4\theta_r}{N} \sum_{n=\lceil \alpha \rceil}^N \sum_{j=0}^n \sum_{k=\lceil \alpha \rceil}^n \frac{n\theta_n}{b_j} \\ \times \frac{(-1)^{n-k} (n+k-1)! \Gamma(k-\alpha+\frac{1}{2})}{L^\alpha \Gamma(k+\frac{1}{2}) (n-k)! \Gamma(k-\alpha-j+1) \Gamma(k-\alpha+j+1)} \\ \times T_n^*(x_r) T_j^*(x_s),$$

where  $s, r = 0, 1, \dots, N$  with  $\theta_0 = \theta_N = \frac{1}{2}, \theta_i = 1 \forall i = 1, 2, \dots, N-1$ .

**Proof.** The fractional derivative of the approximate formula for the function  $u(x)$  in Eq. (13) is given by

$$D^{(\alpha)} u_N(x) \cong \sum_{n=0}^N n a_n D^{(\alpha)} T_n^*(x). \quad (15)$$

Employing Eqs. (4) and (5) in Eq. (8) we have

$$D^{(\alpha)} T_n^*(x) = 0, \quad n = 0, 1, \dots, \lceil \alpha \rceil - 1,$$

then,

$$D^{(\alpha)} u_N(x) \cong \sum_{n=\lceil \alpha \rceil}^N n a_n D^{(\alpha)} T_n^*(x). \quad (16)$$

Therefore, for  $n = \lceil \alpha \rceil, \dots, N$  and by using Eqs. (4), (5) and (8) we get

$$D^{(\alpha)} T_n^*(x) = n \sum_{k=0}^n (-1)^{n-k} \frac{(n+k-1)! 2^{2k}}{(n-k)! (2k)! L^k} D^{(\alpha)} x^k \\ = n \sum_{k=\lceil \alpha \rceil}^n (-1)^{n-k} \\ \times \frac{(n+k-1)! 2^{2k} k!}{(n-k)! (2k)! \Gamma(k-\alpha+1) L^k} x^{k-\alpha}. \quad (17)$$

Now,  $x^{k-\alpha}$  can be expressed approximately in terms of shifted Chebyshev series, so we have

$$x^{k-\alpha} \cong \sum_{j=0}^N c_{kj} T_j^*(x), \quad (18)$$

where  $c_{kj}$  is obtained from (10) with  $y(x) = x^{k-\alpha}$ . If only the first  $(N+1)$ -terms from shifted Chebyshev polynomials in Eq. (13) are considered, the approximate formula for the fractional derivative of the shifted Chebyshev polynomials introduced by Doha et al. [27] as follows

$$D^{(\alpha)} T_n^*(x) = \sum_{j=0}^N \sum_{k=\lceil \alpha \rceil}^n \\ \times \frac{(-1)^{n-k} 2n(n+k-1)! \Gamma(k-\alpha+\frac{1}{2})}{b_j L^\alpha \Gamma(k+\frac{1}{2}) (n-k)! \Gamma(k-\alpha-j+1) \Gamma(k-\alpha+j+1)} T_j^*(x). \quad (19)$$

From Eqs. (19) and (15), we have

$$D^{(\alpha)} u_N(x) = \frac{4}{N} \sum_{n=\lceil \alpha \rceil}^N \sum_{r=0}^N \sum_{j=0}^n \sum_{k=\lceil \alpha \rceil}^n \\ \times \frac{(-1)^{n-k} n(n+k-1)! \Gamma(k-\alpha+\frac{1}{2}) y(x_r) T_n^*(x_r) T_j^*(x)}{b_j L^\alpha \Gamma(k+\frac{1}{2}) (n-k)! \Gamma(k-\alpha-j+1) \Gamma(k-\alpha+j+1)}. \quad (20)$$

From Eq. (20), the fractional derivative of order  $\alpha$  for the function  $u(x)$  at the point  $x_s$  leads to the desired result.  $\square$

The coefficients  $d_{s,r}^{(\alpha)}$  which are defined in Theorem 2 are the elements of the  $s$ -th row of the matrix  $\mathbf{D}_\alpha$  which is defined in the following relation

$$[u^{(\alpha)}] = \mathbf{D}_\alpha [u],$$

where  $\mathbf{D}_\alpha$  is a square matrix of order  $(N+1)$  and the column matrices  $[u^{(\alpha)}]$  and  $[u]$  are given by  $u_r^{(\alpha)} = u^{(\alpha)}(x_r)$  and  $u_r = u(x_r)$ , respectively.

## 5. Error bound for the approximate fractional derivatives

In this section, we will find an error upper bound of the introduced approximate fractional derivative of the function  $u(x)$  which is defined in Eq. (14). To achieve this aim, we state and prove the following two theorems.

**Theorem 3.** [28] Suppose that  $H$  is a Hilbert space and  $U$  is a closed subspace of  $H$  such that  $\dim U < \infty$  and  $u_1, u_2, \dots, u_n$  is a basis for  $U$ . Let  $x$  be an arbitrary element in  $H$  and  $u_0$  be the unique best approximation to  $x$  out of  $U$ . Then

$$\|x - u_0\|_2^2 = \frac{G(x; u_1, u_2, \dots, u_n)}{G(u_1, u_2, \dots, u_n)},$$

where

$$G(x; u_1, u_2, \dots, u_n) = \begin{vmatrix} \langle x, x \rangle & \langle x, u_1 \rangle & \cdots & \langle x, u_n \rangle \\ \langle u_1, x \rangle & \langle u_1, u_1 \rangle & \cdots & \langle u_1, u_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle u_n, x \rangle & \langle u_n, u_1 \rangle & \cdots & \langle u_n, u_n \rangle \end{vmatrix}.$$

**Theorem 4.** The error upper bound for the approximated fractional derivative  $D^{(\alpha)}$  of the function  $u(x)$  is defined in the following manner

$$\|D^{(\alpha)} u(x) - D^{(\alpha)} u_N(x)\|_2 \leq \sum_{n=0}^N a_n \Omega_n \left( \frac{G(x^{k-\alpha}; T_0^*, \dots, T_N^*)}{G(T_0^*, \dots, T_N^*)} \right)^{\frac{1}{2}}, \quad (21)$$

where

$$\Omega_n = \sum_{k=\lceil \alpha \rceil}^n \frac{(-1)^{n-k} 2n(n+k-1)! \Gamma(k-\alpha+\frac{1}{2})}{b_j L^\alpha \Gamma(k+\frac{1}{2}) (n-k)! \Gamma(k-\alpha-j+1) \Gamma(k-\alpha+j+1)}. \quad (22)$$

**Proof.** Consider the following statement in which we call  $e_D^\alpha$  the error vector of the fractional derivative

$$e_D^\alpha := D^{(\alpha)} u(x) - D^{(\alpha)} u_N(x), \quad e_D^\alpha = [e_{D_0}^\alpha \quad e_{D_1}^\alpha \quad \cdots \quad e_{D_N}^\alpha]^T.$$

From the approximate formula (14) and according to Theorem 3, we have

$$\left\| x^{k-\alpha} - \sum_{j=0}^N c_{kj} T_j^*(x) \right\|_2 = \left( \frac{G(x^{k-\alpha}; T_0^*, \dots, T_N^*)}{G(T_0^*, \dots, T_N^*)} \right)^{\frac{1}{2}}, \quad (23)$$

also, according to Eqs. (20) and (23) we have

$$\begin{aligned} \|e_{D_j}^\alpha\|_2 &= \left\| D^\alpha T_n^*(x) - \sum_{j=0}^N \Omega_n T_j^*(x) \right\|_2 \\ &\leq \Omega_n \left( \frac{G(x^{k-\alpha}, T_0^*, \dots, T_N^*)}{G(T_0^*, \dots, T_N^*)} \right)^{\frac{1}{2}}. \end{aligned} \quad (24)$$

Since

$$D^{(\alpha)}u(x) - D^{(\alpha)}u_N(x) = \sum_{n=0}^N a_n \left[ D^{(\alpha)}T_n^*(x) - \sum_{j=0}^N \Omega_n T_j^*(x) \right]. \quad (25)$$

A combination from Eqs. (24) and (25) leads to the desired result.  $\square$

## 6. Implementation of FCheb-FDM for solving FRDE

In this section, we give a numerical algorithm using fractional Chebyshev finite difference formulation for solving the fractional Riccati differential equation of the form (1).

The procedure of the implementation is given by the following steps:

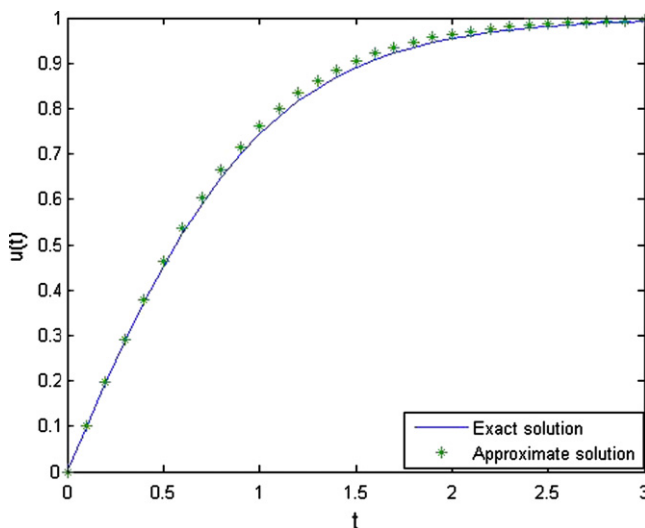
1. Approximate the function  $u(t)$  using the formula (13) and its Caputo fractional derivative  $D^{(\alpha)}u(t)$  using the presented formula (14) with  $N = 8$ , then the general form of FRDE (1) is transformed to the following approximated form

$$\sum_{r=0}^8 d_{r,s}^{(\alpha)} u(t_r) + \left( \sum_{n=0}^8 a_n T_n^*(t) \right)^2 - 1 = 0, \quad (26)$$

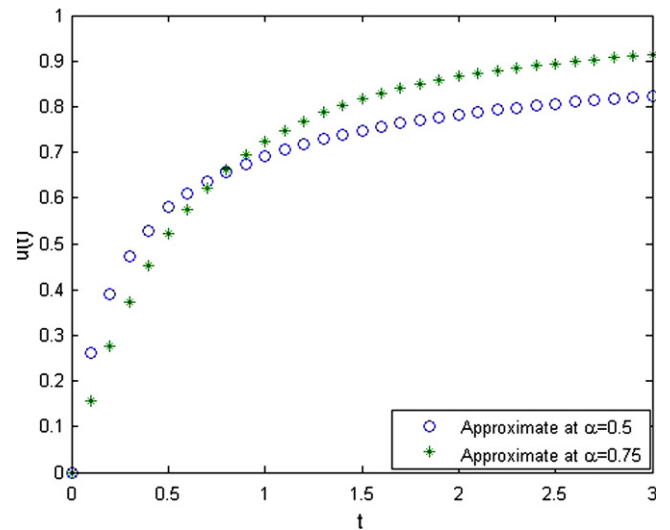
where  $a_n$  and  $d_{r,s}^{(\alpha)}$  are defined in (13) and (14), respectively.

2. The FCheb-FD approximation for the initial condition (2) is given by

$$\sum_{n=0}^8 a_n T_n^*(0) = u_0. \quad (27)$$



**Figure 1** The behavior of the approximate solution using FCheb-FDM and the exact solution at  $\alpha = 1$ ,  $u_0 = 0$ .



**Figure 2** The behavior of approximate solution with different values of  $\alpha$ .

The Eqs. (26) and (27) represent a system of non-linear algebraic equations which contains nine equations for the unknowns  $u(t_i)$ ,  $i = 0, 1, \dots, 8$ , where the grid points to be the Chebyshev–Gauss Lobatto points associated with the interval  $[0, 3]$ ,  $t_i = \frac{3}{2} - \frac{3}{2} \cos\left(\frac{\pi i}{8}\right)$ ,  $i = 0, 1, \dots, 8$ .

3. Solve the previous system using the Newton iteration method to obtain the unknowns  $u(t_i)$ ,  $i = 0, 1, \dots, 8$ .

Therefore, the approximate solution will take the same form (13).

The numerical results of the proposed problem (1) is given in Figs. 1 and 2 with different values of  $\alpha$  in the interval  $[0, 3]$ . Where in Fig. 1, we presented a comparison between the behavior of the exact solution and the approximate solution using the introduced technique at  $\alpha = 1$ ,  $u_0 = 0$ . But, in Fig. 2 we presented the behavior of the approximate solution at different values of  $\alpha$  ( $\alpha = 0.5, 0.75$ ).

From these figures we can conclude that the obtained numerical solutions are in excellent agreement with the exact solution.

## 7. Conclusion

In this article, we introduced a general fractional Chebyshev finite difference formulation and used it for solving FRDE. The proposed problem is transformed to a system of algebraic equations. The solution is expressed as a truncated Chebyshev series and so it can be easily evaluated for arbitrary values of  $t$  using any computer program without any computational effort. The error upper bound of the obtained solution is deduced. The numerical results show that the algorithm converges as the number of  $N$  terms is increased. From illustrative examples, it can be seen that this matrix approach can obtain very accurate and satisfactory results. In the end, from our numerical results using the proposed method, we can conclude that, the solutions are in excellent agreement with the exact solution. All computational results are made by Matlab program 8.

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