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ORIGINAL ARTICLE

On extended generalized ϕ-recurrent Sasakian manifolds

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Abstract The object of this paper is to introduce the notion of extended generalized ϕ-recurrency to Sasakian manifolds and study its various geometric properties with the existence by an interesting example. Among the results established here it is shown that an extended generalized ϕ-recurrent Sasakian manifold is an Einstein manifold. Further, we study extended generalized T–ϕ-recurrent Sasakian manifold and obtain the results which reveal the nature of its associated 1-forms. Finally, an example of a 3-dimensional extended generalized ϕ-recurrent Sasakian manifold which is neither ϕ-recurrent nor generalized ϕ-recurrent is constructed for illustration.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 53C25, 53D15

1. Introduction

The notion of local symmetry of a Riemannian manifolds began with the work of Cartan [1]. The notion of locally symmetry of a Riemannian manifold has been weakened by many authors in several directions such as recurrent manifolds by Walker [2], semi-symmetric manifold by Szabo [3], pseudo-symmetric manifold by Chaki [4], pseudo-symmetric manifold by Deszcz [5], weakly symmetric manifold by Tamassy and Binh [6], weakly symmetric manifold by Selberg [7]. However, the notion of pseudo-symmetry by Chaki and Deszcz are different and that of weak symmetry by Selberg and Tamassy and Binh are also different. As a weaker version of locally symmetry, in 1977 Takahashi [8] introduced the notion of local ϕ-symmetry on a Sasakian manifold. By extending this notion, De et al. [9] introduced and studied the notion of ϕ-recurrent Sasakian manifolds.

The notion of generalized recurrent manifolds was introduced by Dubey [10] and then studied by De and Guha [11]. A Riemannian manifold (M^n, g), $n > 2$, is called generalized recurrent if its curvature tensor R satisfies the condition

$$\nabla R = A \otimes R + B \otimes G, \tag{1.1}$$

where A and B are two non-vanishing 1-forms defined by $A(\circ) = g(\circ, \rho_1)$, $B(\circ) = g(\circ, \rho_2)$ and the tensor G is defined by

$$G(X, Y)Z = g(Y, Z)X - g(X, Z)Y \tag{1.2}$$

for all $X, Y, Z \in \chi(M)$; $\chi(M)$ being the Lie algebra of smooth vector fields and $ denotes the covariant differentiation with respect to the metric g. Here ρ_1 and ρ_2 are vector fields associated with 1-forms A and B respectively. Especially, if the 1-form B vanishes, then (1.1) turns into the notion of recurrent manifold introduced by Walker [2].

A Riemannian manifold (M^n, g) is called a generalized Ricci-recurrent [12] if its Ricci tensor S of type (0, 2) satisfies the condition

$$\nabla S = A \otimes S + B \otimes g, \tag{1.3}$$

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where A and B are defined in (1.1). In particular, if $B = 0$, then (1.3) reduces to the notion of Ricci-recurrent manifolds introduced by Patterson [13].

In 2007, Ozgur [14] studied generalized recurrent Kenmotsu manifold. Generalizing this notion recently, Basari and Murathan [15] introduced the notion of generalized ϕ-recurrency to Kenmotsu manifolds. Also, the notion of generalized ϕ-recurrency to Sasakian manifolds and Lorentzian α-*Sasakian* manifolds are respectively studied in [16,17]. By extending the notion of generalized ϕ-recurrency, Shaikh and Hui [18] introduced the notion of extended generalized ϕ-recurrency to β-Kenmotsu manifolds. Also, this notion has been further studied by Shaikh, Prakasha and Ahmad [19] for LP-Sasakian manifolds. As a continuation of this, here we plan to study extended generalized ϕ-recurrency to Sasakian manifolds.

The paper is organized as follows: Section 2 is concerned with some preliminaries about Sasakian manifolds. Section 3 deals with an extended generalized ϕ-recurrent Sasakian manifolds and we obtain a necessary and sufficient condition for such a manifold to be a generalized Ricci-recurrent. Further, it is shown that an extended generalized ϕ-recurrent Sasakian manifold is an Einstein manifold and in such a manifold the 1-forms A and B are related by $A + B = 0$. In Section 4, we give definition of extended generalized T–ϕ-recurrent Sasakian manifolds analogous to those of concircular and projective curvature tensors defined in [18] for β -Kenmotsu manifolds. Here, it is shown that an extended generalized T–ϕ-recurrent Sasakian manifold is an Einstein manifold. We also tabulated the nature of associated 1-forms A and B. In last section, the existence of an extended generalized ϕ-recurrent Sasakian manifold is ensured by an interesting example.

2. Preliminaries

A $(2n + 1)$-dimensional smooth manifold M is said to be an almost contact metric manifold [20] if it admits an $(1, 1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, which satisfy

$$\text{(a) } \phi\xi = 0, \quad \text{(b) } \eta(\phi X) = 0, \quad \text{(c) } \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\text{(a) } g(\phi X, Y) = -g(X, \phi Y), \quad \text{(b) } \eta(X) = -g(X, \xi)c)\eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for all $X, Y \in \chi(M)$. An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be Sasakian manifold if the following conditions hold [20,21]:

$$(\nabla_X\phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.4}$$

$$\nabla_X\xi = -\phi X. \tag{2.5}$$

In a Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$, the following relations hold [20–22]:

$$(\nabla_X\eta)Y = g(X, \phi Y), \tag{2.6}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.7}$$

$$R(\xi, X)Y = (\nabla_X\phi)Y, \tag{2.8}$$

$$S(X, \xi) = 2n\eta(X), \tag{2.9}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2n\eta(X)\eta(Y), \tag{2.10}$$

$$(\nabla_W R)(X, Y)\xi = g(\phi X, W)Y - g(\phi X, W)X + R(X, Y)\phi W, \tag{2.11}$$

for any vector fields $X, Y, Z \in \chi(M)$.

3. Extended generalized ϕ-recurrent Sasakian manifolds

Definition 3.1. A Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$, $n \geqslant 1$, is said to be an extended generalized ϕ-recurrent Sasakian manifold if its curvature tenor R satisfies the relation

$$\phi^2((\nabla_W R)X, Y)Z) = A(W)\phi^2(R(X, Y)Z) + B(W)\phi^2(G(X, Y)Z) \tag{3.1}$$

for all $X, Y, Z, W \in \chi(M)$, where A and B are two non-vanishing 1-forms such tha $A(X) = g(X, \rho_1)$, $B(X) = g(X, \rho_2)$. Here ρ_1 and ρ_2 are vector fields associated with 1-forms A and B respectively.

Now we begin with the following:

Theorem 3.1. *An extended generalized ϕ-recurrent Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$, $n \geqslant 1$, is generalized Ricci recurrent if and only if the sum of associated 1-forms A and B is zero.*

Proof. Let us consider an extended generalized ϕ-recurrent Sasakian manifold. Then by virtue of (2.1), we have from (3.1) that

$$\begin{aligned} -(\nabla_W R)(X, Y)Z + \eta(\nabla_W R)(X, Y)Z)\xi &= A(W)[-R(X, Y)Z + \eta(R(X, Y)Z)\xi] + B(W) \\ &\quad \times [-G(X, Y)Z + \eta(G(X, Y)Z)\xi], \end{aligned} \tag{3.2}$$

from which it follows that

$$\begin{aligned} -g((\nabla_W R)(X, Y)Z, U) + \eta((\nabla_W R)(X, Y)Z\eta(U) &= A(W)[-g(X, Y)Z, U) + \eta((R(X, Y)Z\eta(U)] + B(W) \\ &\quad \times [-g(G(X, Y)Z, U) + \eta(G(X, Y)Z)\eta(U)]. \end{aligned} \tag{3.3}$$

Let $\{e^i: i = 1, 2, \ldots, 2n + 1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Setting $X = U = e_i$ in (3.3) and taking summation over i, $1 \leqslant i \leqslant 2n + 1$, and then using (1.2), we get

$$\begin{aligned} -(\nabla_W S)(Y, Z) + g((\nabla_W R)(\xi, Y)Z, \xi) &= A(W)[-S(Y, Z) + \eta(R(\xi, Y)Z)] + B(W)[-(2n \\ &\quad - 1)g(Y, Z) - \eta(Y)\eta(Z)]. \end{aligned} \tag{3.4}$$

Using (2.7) and (2.11) and the relation $g((\$_W R)(X, Y)Z, U) = -g((\$_W R)(X, Y)U, Z)$, we have

$$g((\nabla_W R)(\xi, Y)Z, \xi) = 0. \tag{3.5}$$

By virtue of (2.8) and (3.5), it follows from (3.4) that

$$\begin{aligned} (\nabla_W S)(Y, Z) = A(W)S(Y, Z) + [(2n - 1)b(W) \\ - A(W)]g(Y, Z) + [A(W) \\ + B(W)\eta(Y)\eta(Z)]. \end{aligned} \tag{3.6}$$

If $A(W) + B(W) = (A + B)(W) = 0$, that is, the sum of associated 1-forms A and B is zero, then (3.6) reduces to

$$\nabla S = A \otimes S + \psi \otimes g, \tag{3.7}$$

where $\psi(W) = 2nB(W)$ for all $W \in \chi(M)$. This completes the proof. □

Theorem 3.2. *An extended generalized ϕ-recurrent Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$, $n \geqslant 1$, is an Einstein manifold and moreover the associated 1-forms A and B are related by $A + B = 0$.*

Proof. Setting $Z = \xi$ in (3.6) and using (2.2(b)) and (2.9), we obtain

$$(\nabla_W S)(Y, \xi) = 2n\{A(W) + B(W)\}\eta(Y). \quad (3.8)$$

Also we have

$$(\nabla_W S)(Y, \xi) = (\nabla_W S)(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi). \quad (3.9)$$

Using (2.6) and (2.9) in (3.9), it follows that

$$(\nabla_W S)(Y, \xi) = 2ng(Y, \phi W) - S(Y, \phi W). \quad (3.10)$$

By (3.8) and (3.10) we have

$$2ng(\phi W, Y) - S(\phi W, Y) = 2n\{A(W) + B(W)\}\eta(Y). \quad (3.11)$$

Agian setting Y by ξ in (3.11) we get

$$A(W) + B(W) = 0 \quad \text{for all } W. \quad (3.12)$$

By taking account of (3.12) in (3.11), we have

$$S(\phi W, Y) = 2ng(\phi W, Y). \quad (3.13)$$

Substituting Y by ϕY in (3.13) and using (2.3) and (2.10), we have

$$S(W, Y) = 2ng(W, Y). \quad (3.14)$$

From (3.12) and (3.14), the theorem follows. □

It is known that a Sasakian manifold is Ricci-semisymmetric if and only if it is an Einstein manifold. In fact, by Theorem 3.2, we have the following:

Corollary 3.1. *An extended generalized ϕ-recurrent Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$, $n \geqslant 1$, is Ricci-semisymmetric.*

Theorem 3.3. *In an extended generalized ϕ-recurrent Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$, $\frac{r-2n(2n-1)}{2}$ is an eigen value of the Ricci tensor S corresponding to the eigen vector ρ_1.*

Proof. Changing W, X, Y cyclically in (3.3) and adding them, we get by virtue of Bianchi identity and (3.12) that

$$\begin{aligned}
&A(W)[\{g(R(X, Y)Z, U) - g(G(X, Y)Z, U)\} \\
&\quad + \{\eta(R(X, Y)Z) - \eta(G(X, Y)Z)\}\eta(U)] \\
&A(X)[\{g(R(Y, W)Z, U) - g(G(Y, W)Z, U)\} \\
&\quad + \{\eta(R(Y, W)Z) - \eta(G(Y, W)Z)\}\eta(U)] \\
&A(Y)[\{g(R(W, X)Z, U) - g(G(W, X)Z, U)\} \\
&\quad + \{\eta(R(W, X)Z) - \eta(G(W, X)Z)\}\eta(U)] = 0.
\end{aligned} \quad (3.15)$$

Setting $Y = Z = e_i$ in (3.15) and taking summation over i, $1 \leqslant i \leqslant 2n + 1$, we get

$$\begin{aligned}
&A(W)[S(X, U) - 2ng(X, U)] - A(X)[S(U, W) - 2ng(U, W)] \\
&\quad - A(R(W, X)U) - A(R(W, X)\xi)\eta(U) - A(X)g(W, U) \\
&\quad + A(W)g(X, U) - \{A(X)\eta(W) - A(W)\eta(X)\} \\
&\quad = 0.
\end{aligned}$$

Again setting $X = U = e_i$ in above relation and taking summation over i, $1 \leqslant i \leqslant 2n + 1$, we have

$$S(W, \rho_1) = \frac{r - 2n(2n - 1)}{2} g(W, \rho_1).$$

This proves the theorem. □

Theorem 3.4. *A Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$, $n \geqslant 1$, is an extended generalized ϕ-recurrent if and only if the following relation holds:*

$$\begin{aligned}
(\nabla_W R)(X, Y)Z = {} & [\{g(Y, \phi W)g(X, Z) \\
& - g(X, \phi W)g(Y, Z)\} \\
& + g(R(X, Y)\phi W, Z)]\xi + A(W) \\
& \times [R(X, Y)Z - \eta(R(X, Y)Z)\xi] \\
& + B(W)[G(X, Y)Z - \eta(G(X,))Z)\xi].
\end{aligned} \quad (3.16)$$

Proof. Using (2.11) and the relation $g((\$_w R)(X, Y)Z, U) = -g((\$_w R)(X, Y)U, Z)$ in (3.2), we have (3.16). Conversely, applying ϕ^2 on both sides of (3.16), we get the relation (3.1). □

4. Extended generalized T–ϕ-recurrent Sasakian manifolds

In a $(2n + 1)$-dimensional Riemannian manifold M^{2n+1}, the T-curvature tensor [23,24] is given by

$$\begin{aligned}
T(X, Y)Z = {} & a_0 R(X, Y)Z + a_1 S(Y, Z)X + a_2 S(X, Z)Y \\
& + a_3 S(X, Y)Z + a_4 g(Y, Z)QX + a_5 g(X, Z)QY \\
& + a_6 g(X, Y)QZ + a_7 r(g(Y, Z)X - g(X, Z)Y),
\end{aligned} \quad (4.1)$$

where R, S, Q and r are the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature, respectively. In particular, T-curvature tensor is reduced to be quasi-conformal curvature tensor C_*, conformal curvature tensor C, conharmonic curvature tensor L, concircular curvature tensor V, pseudo-projective curvature tensor P_*, projective curvature tensor P, M-projective curvature tensor, W_i-curvature tensors ($i = 0, \ldots, 9$) and W_j^*-curvature tensors ($j = 0, 1$).

Analogous to the definitions of an extended generalized concircular ϕ-recurrency for β-Kenmotsu manifolds [18] and an extended generalized projective ϕ-recurrency for LP-Sasakian manifolds[19], here we define the following:

Definition 4.2. A Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$,$n \geqslant 1$, is said to be an extended generalized T–ϕ-recurrent if its T-curvature tensor satisfies the relation

$$\begin{aligned}
\phi^2((\nabla_W T)(X, Y)Z) = {} & A(W)\phi^2(T(X, Y)Z)) \\
& + B(W)\phi^2(G(X, Y)Z),
\end{aligned} \quad (4.2)$$

where A and B are defined as in (1.1).

In particular, an extended generalized T–ϕ-recurrent Sasakian $M^{2n+1}(\phi, \xi, \eta, g)$, $n \geqslant 1$, manifold is reduced to be

(1) an extended generalized C^*–ϕ-recurrent if

$$a_1 = -a_2 = a_4 = -a_5, \quad a_3 = a_6 = 0,$$
$$a_7 = -\frac{1}{2n+1}\left(\frac{a_0}{2n} + 2a_1\right),$$

(2) an extended generalized C–ϕ-recurrent if

$$a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2n-1},$$
$$a_3 = a_6 = 0, \quad a_7 = -\frac{1}{2n(2n-1)},$$

(3) an extended generalized L–ϕ-recurrent if

$$a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2n-1},$$
$$a_3 = a_6 = 0, \quad a_7 = 0,$$

(4) an extended generalized V–ϕ-recurrent if

$$a_0 = 1, \quad a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0,$$
$$a_7 = -\frac{1}{2n(2n+1)},$$

(5) an extended generalized P_*–ϕ-recurrent if

$$a_0 = 1, \quad a_1 = -a_2, \quad a_3 = a_4 = a_5 = a_6 = 0,$$
$$a_7 = -\frac{1}{2n(2n+1)}\left(\frac{a_0}{2n} + a_1\right),$$

(6) an extended generalized P–ϕ-recurrent if

$$a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{2n}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(7) an extended generalized M–ϕ-recurrent if

$$a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{4n}, \quad a_3 = a_6 = a_7 = 0,$$

(8) an extended generalized W_0–ϕ-recurrent if

$$a_0 = 1, \quad a_1 = -a_5 = -\frac{1}{2n}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

(9) an extended generalized W_0^*–ϕ- recurrent if

$$a_0 = 1, \quad a_1 = -a_5 = \frac{1}{2n}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

(10) an extended generalized W_1–ϕ-recurrent if

$$a_0 = 1, \quad a_1 = -a_2 = \frac{1}{2n}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(11) an extended generalized W_1^*–ϕ- recurrent if

$$a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{2n}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(12) an extended generalized W_2–ϕ-recurrent if

$$a_0 = 1, \quad a_4 = -a_5 = -\frac{1}{2n}, \quad a_1 = a_2 = a_3 = a_6 = a_7 = 0,$$

(13) an extended generalized W_3–ϕ-recurrent if

$$a_0 = 1, \quad a_2 = -a_4 = -\frac{1}{2n}, \quad a_1 = a_3 = a_5 = a_6 = a_7 = 0,$$

(14) an extended generalized W_4–ϕ-recurrent if

$$a_0 = 1, \quad a_5 = -a_6 = -\frac{1}{2n}, \quad a_1 = a_2 = a_3 = a_4 = a_7 = 0,$$

(15) an extended generalized W_5–ϕ-recurrent if

$$a_0 = 1, \quad a_2 = -a_5 = -\frac{1}{2n}, \quad a_1 = a_3 = a_4 = a_6 = a_7 = 0,$$

(16) an extended generalized W_6–ϕ-recurrent if

$$a_0 = 1, \quad a_1 = -a_6 = -\frac{1}{2n}, \quad a_2 = a_3 = a_4 = a_5 = a_7 = 0,$$

(17) an extended generalized W_7–ϕ-recurrent if

$$a_0 = 1, \quad a_1 = -a_4 = -\frac{1}{2n}, \quad a_2 = a_3 = a_5 = a_6 = a_7 = 0,$$

(18) an extended generalized W_8–ϕ-recurrent if

$$a_0 = 1, \quad a_1 = -a_3 = \frac{1}{2n}, \quad a_2 = a_4 = a_5 = a_6 = a_7 = 0,$$

(19) an extended generalized W_9–ϕ-recurrent if

$$a_0 = 1, \quad a_3 = -a_4 = \frac{1}{2n}, \quad a_1 = a_2 = a_5 = a_6 = a_7 = 0.$$

Theorem 4.5. *If a (2n + 1)-dimensional Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$, $n \geqslant 1$, is an extended generalized T–ϕ-recurrent such that $a_0 + 2na_1 + a_2 + a_3 \neq 0$, then M^{2n+1} is generalized Ricci-recurrent if and only if the following relation holds:*

$$\begin{aligned}
&\frac{[B(W) - A(W)\{2n(a_2 + a_3 + a_5 + a_6) - a_0 - ra_7\} - a_7 dr(W)]}{a_0 + 2na_1 + a_2 + a_3}\eta(Y)\eta(Z) \\
&- \frac{(a_5 + a_6)}{2(a_0 + 2na_1 + a_2 + a_3)}[g((\nabla_W Q)Y, Z) - \eta((\nabla_W Q)Y)\eta(Z) \\
&+ g((\nabla_W Q)Z, Y) - \eta((\nabla_W Q)Z)\eta(Y)] \\
&+ \frac{(a_2 + a_3)}{2(a_0 + 2na_1 + a_2 + a_3)}[\{S(\phi W, Z) - 2ng(\phi W, Z)\}\eta(Y) \\
&+ \{S(\phi W, Y) - 2ng(\phi W, Y)\}\eta(Z)] \\
&= 0.
\end{aligned} \tag{4.3}$$

Proof. Let us consider an extended generalized T–ϕ-recurrent Sasakian manifold. Then by virtue of (2.1), it follows from (4.2) that

$$\begin{aligned}
&-(\nabla_W T)(X, Y)Z + \eta((\nabla_W T)(X, Y)Z)\xi \\
&= A(W)[-T(X, Y)Z + \eta(T(X, Y)Z)\xi] + B(W)[-G(X, Y)Z \\
&\quad + \eta(G(X, Y)Z)\xi],
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&-g((\nabla_W T)(X, Y)Z, U) + \eta((\nabla_W T)(X, Y)Z)\eta(U) \\
&= A(W)[-g(T(X, Y)Z, U) + \eta(T(X, Y)Z)\eta(U)] \\
&\quad + B(W)[-g(G(X, Y)Z, U) + \eta(G(X, Y)Z)\eta(U)].
\end{aligned} \tag{4.4}$$

Let $\{e_i: i = 1, 2, \ldots, 2n + 1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Setting $X = U = e_i$ in (4.4) and taking summation over i, $1 \leqslant i \leqslant 2n + 1$, then using (1.2) and (4.1), we get

$$
\begin{aligned}
&-\{a_0+(2n+1)a_1+a_2+a_3\}(\nabla_W S)(Y,Z)-\{a_4\\
&\quad+2na_7\}dr(W)g(Y,Z)-a_5 g((\nabla_W Q)Y,Z)\\
&\quad-a_6 g((\nabla_W Q)Z,Y)+a_0 g((\nabla_W R)(\xi,Y)Z,\xi)\\
&\quad+a_1(\nabla_W S)(Y,Z)+a_2(\nabla_W S)(\xi,Z)\eta(Y)+a_3(\nabla_W S)\\
&\quad\times(Y,\xi)\eta(Z)+a_4 g(Y,Z)\eta((\nabla_W Q)\xi)\\
&\quad+a_5\eta((\nabla_W Q)Y)\eta(Z)+a_6\eta((\nabla_W Q)Z)\eta(Y)\\
&\quad+a_7 dr(W)\{g(Y,Z)-\eta(Y)\eta(Z)\}\\
&\quad=A(W)[-\{a_0+(2n+1)a_1+a_2+a_3+a_5\\
&\qquad+a_6\}S(Y,Z)-\{a_4+2na_7\}rg(Y,Z)\\
&\qquad+a_0\eta(R(\xi,Y)Z)+a_1 S(Y,Z)+\{a_2\\
&\qquad+a_6\}S(\xi,Z)\eta(Y)+\{a_3+a_5\}S(Y,\xi)\eta(Z)\\
&\qquad+a_4 S(\xi,\xi)g(Y,Z)+a_7 r\{g(Y,Z)-\eta(Y)\eta(Z)\}]\\
&\qquad+B(W)[-(2n-1)g(Y,Z)-\eta(Y)\eta(Z)]. \qquad (4.5)
\end{aligned}
$$

Using (2.8), (2.9) and (2.11) and the relation $g((\nabla_W R)(X,Y)Z,U)=-g((\nabla_W R)(X,Y)U,Z)$, we have

$$
\begin{aligned}
&\{a_0+2na_1+a_2+a_3\}(\nabla_W S)(Y,Z)\\
&\quad=A(W)\{a_0+2na_1+a_2+a_3+a_5+a_6\}S(Y,Z)\\
&\qquad+[(2n-1)B(W)+\{a_0+2na_7\}\{A(W)r-dr(W)\}\\
&\qquad-A(W)\{a_0+2na_4+ra_7\}+a_7 dr(W)]g(Y,Z)\\
&\qquad+[B(W)-A(W)\{2n(a_2+a_3+a_5+a_6)-a_0\\
&\qquad-ra_7\}-a_7 dr(W)]\eta(Y)\eta(Z)-a_5[g((\nabla_W Q)Y,Z)\\
&\qquad-\eta((\nabla_W Q)Y)\eta(Z)]-a_6[g((\nabla_W Q)Z,Y)\\
&\qquad-\eta((\nabla_W Q)Z)\eta(Y)]+a_2[S(\phi W,Z)\\
&\qquad-2ng(\phi W,Z)]\eta(Y)+a_3[S(\phi W,Y)\\
&\qquad-2ng(\phi W,Y)]\eta(Z). \qquad (4.6)
\end{aligned}
$$

Interchanging Y and Z in (4.6), and then subtracting the resultant from (4.6), we obtain by symmetric property of S that

$$
\begin{aligned}
(\nabla_W S)(Y,Z)&=A(W)\left[1+\frac{a_5+a_6}{a_0+2na_1+a_2+a_3}\right]S(Y,Z)\\
&+\frac{[(2n-1)B(W)+\{a_0+2na_7\}\{A(W)r-dr(W)\}-A(W)\{a_0+2na_4+ra_7\}+a_7 dr(W)]}{a_0+2na_1+a_2+a_3}g(Y,Z)\\
&+\frac{[B(W)-A(W)\{2n(a_2+a_3+a_5+a_6)-a_0-ra_7\}-a_7 dr(W)]}{a_0+2na_1+a_2+a_3}\eta(Y)\eta(Z)-\frac{(a_5+a_6)}{2(a_0+2na_1+a_2+a_3)}\\
&\times[g((\nabla_W Q)Y,Z)-\eta((\nabla_W Q)Y)\eta(Z)+g((\nabla_W Q)Z,Y)-\eta((\nabla_W Q)Z)\eta(Y)]+\frac{(a_2+a_3)}{2(a_0+2na_1+a_2+a_3)}\\
&\times[\{S(\phi W,Z)-2ng(\phi W,Z)\}\eta(Y)+\{S(\phi W,Y)-2ng(\phi W,Y)\}\eta(Z)]. \qquad (4.7)
\end{aligned}
$$

If the relation (4.3) holds, then the above relation can be reduced to

$$\nabla S=A_1\otimes S+B_1\otimes g,$$

where

$$A_1(W)=A(W)\left[1+\frac{a_5+a_6}{a_0+2na_1+a_2+a_3}\right]$$

and

$$B_1(W)=\frac{[2nB(W)+\{a_0+2na_7\}\{A(W)r-dr(W)\}-A(W)\{a_0+2na_4+ra_7\}+a_7 dr(W)]}{a_0+2na_1+a_2+a_3}.$$

This implies M^{2n+1} is generalized Ricci-recurrent. □

Theorem 4.6. *An extended generalized τ–ϕ-recurrent Sasakian manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n\geqslant 1$, such that*

$$\frac{2(a_0+2na_1)+a_2+a_3}{2(a_0+2na_1+a_2+a_3)}\neq 0$$

is an Einstein manifold.

Proof. Substituting $Z=\xi$ in (4.7) then using (2.2(b)) and (2.9) we get

$$
\begin{aligned}
&(\nabla_W S)(Y,\xi)\\
&=\left\{\frac{A(W)[2n\{a_0+2na_1-a_4\}+r\{a_0+2na_7\}]+2nB(W)-\{a_0+2na_7\}dr(W)}{a_0+2na_1+a_2+a_3}\right\}\eta(Y)\\
&+\frac{(a_2+a_3)}{2(a_0+2na_1+a_2+a_3)}\{S(\phi W,Y)-2ng(\phi W,Y)\}. \qquad (4.8)
\end{aligned}
$$

Replacing Y by ϕY in (4.8) and then using (2.1(b)) we have

$$(\nabla_W S)(\phi Y,\xi)=\frac{(a_2+a_3)}{2(a_0+2na_1+a_2+a_3)}\{S(\phi W,\phi Y)-2ng(\phi W,\phi Y)\}.$$

Using (3.10) we obtain from above relation that

$$\frac{2(a_0+2na_1)+a_2+a_3}{2(a_0+2na_1+a_2+a_3)}\{S(\phi W,\phi Y)-2ng(\phi W,\phi Y)\}=0. \qquad (4.9)$$

If $\frac{2(a_0+2na_1)+a_2+a_3}{2(a_0+2na_1+a_2+a_3)}\neq 0$, then by virtue of (2.3) and (2.10), relation (4.9) yields

$$S(Y,W)=2ng(Y,W). \quad □ \qquad (4.10)$$

Corollary 4.2. *Let M^{2n+1} be a $2n+1$-dimensional, $n\geqslant 1$, extended generalized T–ϕ-recurrent Sasakian manifold such that $a_0+2na_1+a_2+a_3\neq 0$. Then the associated 1-forms A and B are related by*

$$
\begin{aligned}
B(W)&=\left[-a_0-2na_1+a_4\left(1-\frac{r}{2n}\right)-ra_7\right]A(W)+\frac{1}{2n}\\
&\quad\times[a_4+2na_7]dr(W) \qquad (4.11)
\end{aligned}
$$

for any vector field $W\in\chi(M)$.

Consequently, we have the following:

Sasakian manifold	$B(W)=$
Extended generalized C_*–ϕ-recurrent	$\{a+(2n-1)b\}\left[\left(\frac{r}{2n(2n+1)}-1\right)A(W)-\frac{1}{2n(2n+1)dr(W)}\right]$
Extended generalized C–ϕ-recurrent	0
Extended generalized L–ϕ-recurrent	$\frac{1}{(2n)(2n-1)}\{A(W)r-dr(W)\}$
Extended generalized V–ϕ-recurrent	$\left[-1+\frac{r}{2n(2n+1)}\right]A(W)-\frac{1}{2n(2n+1)}dr(W)$
Extended generalized P_*–ϕ-recurrent	$\{a+(2n-1)b\}\left[\left(\frac{r}{2n(2n+1)}-1\right)A(W)-\frac{1}{2n(2n+1)dr(W)}\right]$
Extended generalized P–ϕ-recurrent	0
Extended generalized M–ϕ-recurrent	0
Extended generalized W_0–ϕ-recurrent	0
Extended generalized $W_0^*-\phi$-recurrent	$-2A(W)$
Extended generalized W_1–ϕ-recurrent	$-2A(W)$
Extended generalized $W_1^*-\phi$-recurrent	0
Extended generalized W_2–ϕ-recurrent	$\frac{1}{(2n)^2}[\{r-2n(2n+1)\}A(W)-dr(W)]$
Extended generalized W_3–ϕ-recurrent	$\frac{1}{(2n)^2}[dr(W)-\{r+2n(2n-1)\}A(W)]$
Extended generalized W_4–ϕ-recurrent	$-A(W)$
Extended generalized W_5–ϕ-recurrent	$-A(W)$
Extended generalized W_6–ϕ-recurrent	0
Extended generalized W_7–ϕ-recurrent	$\frac{1}{(2n)^2}[\{2n-r\}A(W)+dr(W)]$
Extended generalized W_8–ϕ-recurrent	0
Extended generalized W_9–ϕ-recurrent	$\frac{1}{(2n)^2}[\{r-2n(2n+1)\}A(W)+dr(W)]$

Proof. By plugging Y by ξ in (4.8), we have (4.11). □

It is also observed from the above corollary that, in an extended generalized T–ϕ-recurrent Sasakian manifold if T is equal to C, P, M, W_0, W_1^*, W_6, W_8, then the 1-form B vanishes (that is, $B=0$). Which is not possible. Hence we can state the following:

Theorem 4.7. *There exists no extended generalized* $\{C, P, M, W_0, W_1^*, W_6, W_8\}$*–$\phi$-recurrent Sasakian manifold.*

5. Example of extended generalized ϕ-recurrent Sasakian manifolds

Theorem 5.8. *There exists a 3-dimensional extended generalizedϕ-recurrent Sasakian manifold, which is neither ϕ-recurrent nor generalized ϕ-recurrent.*

Proof. We consider a 3-dimensional manifold $M=\{(x,y,z)\in\Re^3,(x,y,z)\neq 0\}$, where (x,y,z) are standard coordinates of $\Re^3$. The vector fields

$$E_1=\frac{\partial}{\partial x}-y\frac{\partial}{\partial z},\quad E_2=\frac{\partial}{\partial y},\quad E_3=\frac{1}{2}\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(E_1,E_3)=g(E_1,E_2)=g(E_2,E_3)=0$$
$$g(E_1,E_1)=g(E_2,E_2)=g(E_3,E_3)=1.$$

Let η be the 1-form defined by $\eta(U)=g(U,E_3)$ for any $U\in\chi(M)$. Let ϕ be the (1, 1) tensor field defined by

$$\phi(E_1)=E_2,\quad \phi(E_2)=-E_1,\quad \phi(E_3)=0.$$

So, using the linearity of ϕ and g, we have

$$\eta(E_3)=1,$$
$$\phi^2Z=-Z+\eta(Z)E_3,$$
$$g(\phi Z,\phi W)=g(Z,W)-\eta(Z)\eta(W),$$

for any $Z, W\in\chi(M)$. Then for $E_3=\xi$, the structure (ϕ,ξ,η,g) defines an almost contact metric structure on M. Let \$ be the Levi-Civita connection with respect to metric g. Then we have

$$[E_1,E_2]=2E_3,\quad [E_1,E_3]=0,\quad [E_2,E_3]=0.$$

Using the Koszula formula for the Riemannian metric g, we can easily calculate

$$\nabla_{E_1}E_3=-E_2,\quad \nabla_{E_3}E_3=0,\quad \nabla_{E_2}E_3=E_1.$$
$$\nabla_{E_2}E_2=0,\quad \nabla_{E1}E_2=E_3,\quad \nabla_{E_2}E_1=-E_3.$$
$$\nabla_{E_1}E_1=0,\quad \nabla_{E_3}E_2=E_1,\quad \nabla_{E_3}E_1=-E_2.$$

From the above it can be easily seen that (ϕ,ξ,η,g) is a Sasakian structure on M. Consequently $M^3(\phi,\xi,\eta,g)$ is a Sasakian manifold. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor R as follows:

$$R(E_1,E_2)E_1=3E_2,\quad R(E_1,E_2)E_2=-3E_1,$$
$$R(E_1,E_3)E_1=E_2,\quad R(E,E)E=E_1,$$
$$R(E,E)E=E_3,\quad R(E,E)E=E_2,$$

and the components which can be obtained from these by the symmetry properties.

Since $\{E_1,E_2,E_3\}$ forms a basis of the 3-dimensional Sasakian manifold, any vector field $X, Y, Z\in\chi(M)$ can be written as

$$X=a_1E_1+b_1E_2+c_1E_3,$$
$$Y=a_2E_1+b_2E_2+c_2E_3,$$
$$Z=a_3E_1+b_3E_2+c_3E_3,$$

where $a_i, b_i, c_i\in\Re^+$ (the set of all positive real numbers), $i=1,2,3$. Then

$$R(X,Y)Z = [(a_1c_2 - c_1a_2)c_3 - 3(a_1b_2 - b_1a_2)b_3]E_1 + [(b_1c_2 - c_1b_2)c_3 + 3(a_1b_2 - b_1a_2)a_3]E_2 - [(b_1c_2 - c_1b_2)b_3 + (a_1c_2 - c_1a_2)a_3]E_3, \quad (5.1)$$

$$G(X,Y)Z = (a_2a_3 + b_2b_3 + c_2c_3)(a_1E_1 + b_1E_2 + c_1E_3) - (a_1a_3 + b_1b_3 + c_1c_3)(a_2E_1 + b_2E_2 + c_2E_3). \quad (5.2)$$

By virtue of (5.1) we have the following:

$$(\nabla_{E_1}R)(X,Y)Z = 4[(a_1b_2 - b_1a_2)(a_3E_3 - c_3E_1) + (a_1c_2 - c_1a_2)(a_3E_2 - b_3E_1)], \quad (5.3)$$

$$(\nabla_{E_2}R)(X,Y)Z = 4[(a_1b_2 - b_1a_2)(b_3E_3 - c_3E_2) - (b_1c_2 - c_1b_2)(a_3E_2 + b_3E_1)], \quad (5.4)$$

$$(\nabla_{E_3}R)(X,Y)Z = 0. \quad (5.5)$$

From (5.1) and (5.2), we get

$$\phi^2(R(X,Y)Z) = u_1E_1 + u_2E_2 \quad \text{and} \quad \phi^2(G(X,Y)Z) = v_1E_1 + v_2E_2,$$

where

$$u_1 = -[(a_1c_2 - c_1a_2)c_3 - 3(a_1b_2 - b_1a_2)b_3],$$
$$u_2 = -[(b_1c_2 - c_1b_2)c_3 + 3(a_1b_2 - b_1a_2)a_3],$$
$$v_1 = a_2(b_1b_3 + c_1c_3) - a_1(b_2b_3 + c_2c_3),$$
$$v_2 = b_2(a_1a_3 + c_1c_3) - b_1(a_2a_3 + c_2c_3).$$

Also from (5.3)–(5.5), we obtain

$$\phi^2((\nabla_{E_i}R)(X,Y)Z) = p_iE_1 + q_iE_2 \quad \text{for } i = 1,2,3, \quad (5.6)$$

where

$$p_1 = 4[c_3(a_1b_2 - b_1a_2) + b_3(a_1c_2 - c_1a_2)], \quad q_1 = -4a_3(a_1c_2 - c_1a_2),$$
$$p_2 = 4b_3(b_1c_2 - c_1b_2), \quad q_2 = 4[c_3(a_1b_2 - b_1a_2)c_3 + a_3(b_1c_2 - c_1b_2)],$$
$$p_3 = 0, \quad q_3 = 0.$$

Let us now consider the 1-forms as

$$A(E_1) = \frac{v_2p_1 - v_1q_1}{u_1v_2 - u_2v_1}, \quad B(E_1) = \frac{u_1q_1 - u_2p_1}{u_1v_2 - u_2v},$$
$$A(E_2) = \frac{u_1q_1 - u_2p_1}{u_1v_2 - u_2v}, \quad B(E_2) = \frac{u_1q_2 - u_2p_2}{u_1v_2 - u_2v}, \quad (5.7)$$
$$A(E_3) = 0, \quad B(E_3) = 0,$$

where $v_2p_1 - v_1q_1 \neq 0$, $u_1q_1 - u_2p_1 \neq 0$, $u_1q_1 - u_2p_1 \neq 0$, $u_1q_2 - u_2p_2 \neq 0$, $u_1v_2 - u_2v_1 \neq 0$. From (3.1) we have

$$\phi^2((\nabla_{E_i}R)(X,Y)Z) = A(E_i)\phi^2(R(X,Y)Z + B(E_i)\phi^2(G(X,Y)Z, i = 1,2,3. \quad (5.8)$$

By virtue of (5.6)–(5.8), it can be easily shown that the manifold satisfies the relation (5.8). Hence the manifold under consideration is a 3-dimensional extended generalized ϕ-recurrent Sasakian manifold, which is neither ϕ-recurrent nor generalized ϕ-recurrent. □

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