Journal of the Egyptian Mathematical Society (2013) **21**, 21–24

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

On a class of entire functions represented by Dirichlet series

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Received 11 August 2012; revised 22 October 2012; accepted 31 October 2012
Available online 14 December 2012

KEYWORDS

Dirichlet series;
Banach algebra;
Topological zero divisor;
Division algebra;
Continuous linear functional

Abstract The present paper deals with the study on a class of entire functions represented by Dirichlet series whose coefficients belong to a commutative Banach algebra with identity. We consider a class of such series which satisfy certain conditions and establish some results.

MATHEMATICS SUBJECT CLASSIFICATION: 30B50, 46J15, 17A35

1. Introduction

Let

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad s = \sigma + it, \quad (\sigma, t \in R) \tag{1.1}$$

and E be a commutative Banach algebra with identity element e_n such that $\|e_n\| = 1$. If $a_n's$ belong to E and $\lambda_n's \in R$ which satisfy the condition $0 < \lambda_1 < \lambda_2 < \lambda_3 \cdots < \lambda_n \cdots ; \lambda_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\log \|a_n\|}{\lambda_n} = -\infty \tag{1.2}$$

$$\limsup_{n\to\infty} \frac{\log\ n}{\lambda_n} = K < \infty \tag{1.3}$$

Then from [1] the Dirichlet series (1.1) represents an entire function. Recently some properties of such type of series were discussed by Srivastava and Sharma in [2–5]. In this paper let F be the set of series (1.1) for which $\lambda_n^{c_1\lambda_n}\ e^{\{c_2 n - c_1\}\lambda_n}\|a_n\|$ is bounded where $c_1,\ c_2 \geqslant 0$ and $c_1,\ c_2$ are simultaneously not zero. Then every element of F represents an entire function. If

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \quad \text{and} \quad g(s) = \sum_{n=1}^{\infty} b_n e^{\lambda_n s}$$

Define binary operation i.e. addition and scalar multiplication in F as

$$f(s) + g(s) = \sum_{n=1}^{\infty} (a_n + b_n) e^{\lambda_n s},$$

$$\alpha \cdot f(s) = \sum_{n=1}^{\infty} (\alpha \cdot a_n) e^{\lambda_n s},$$

$$f(s) \cdot g(s) = \sum_{n=1}^{\infty} [\lambda_n^{c_1\lambda_n}\ e^{\{c_2 n - c_1\}\lambda_n} a_n b_n]\ e^{\lambda_n s}.$$

So far many authors considered set of entire functions with weighted norms and studied results on it. In the present paper we generalize the weighted norm by taking the conditions of papers [6,7] and prove some results which would further be useful in the study of the spaces like FK-space, Frechet space, and Montel space. Using the results of this paper the spectrum of the set can also be determined.

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2012.10.008

The norm in F is defined as follows:

$$\|f\| = \sum_{n=1}^{\infty} \lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} \|a_n\|. \tag{1.4}$$

If $a_n's$ belong to the set of complex numbers, $c_1 = 0$, $c_2 = 1$ gives the norm as defined in [6]. Again if $c_1 = 1$, $c_2 = 0$ we get the condition as defined in [7].

2. Main results

In this section we prove our main results. For the definitions of terms used we refer to [8,9].

Theorem 1. *F is a commutative Banach algebra with identity.*

Proof. In order to prove the above theorem we first show that F is complete under the norm defined by (1.4). For this let $\{f_{k_1}\}$ be a cauchy sequence in F. For given $\epsilon > 0$ we can find k such that

$$\|f_{k_1} - f_{k_2}\| < \epsilon \quad \text{where} \quad k_1, k_2 \geqslant k$$

This implies that

$$\sum_{n=1}^{\infty} \lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} \|a_{k_1 n} - a_{k_2 n}\| < \epsilon \quad \text{where} \quad k_1, k_2 \geqslant k.$$

This shows that $\{a_{k_1 n}\}$ forms a cauchy sequence in a Banach space E for every value of $n \geqslant 1$ hence converges to a_n. Therefore $f_{k_1} \to f$. Also

$$\sum_{n=1}^{\infty} \lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} \|a_n\| \leqslant \sum_{n=1}^{\infty} \lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} \|a_{k_1 n} - a_n\| + \sum_{n=1}^{\infty} \lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} \|a_{k_1 n}\|.$$

Hence $f \in F$.

If f, $g \in F$ then

$$\|f.g\| = \sum_{n=1}^{\infty} \lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} \|a_n \; b_n \; \lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n}\| \leqslant \sum_{n=1}^{\infty} \lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} \|a_n\| \cdot \lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} \|b_n\| = \|f\| \cdot \|g\|$$

The identity element in F is

$$e(s) = \sum_{n=1}^{\infty} e_n \; \lambda_n^{-c_1\lambda_n} \; e^{\{c_1 - c_2 n\}\lambda_n} \; e^{\lambda_n s}.$$

This completes the proof of the theorem. □

Theorem 2. *The function* $f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}$ *is invertible in F if and only if*

$$\{\|d_n \; \lambda_n^{-c_1\lambda_n} \; e^{\{c_1 - c_2 n\}\lambda_n}\|\}$$

is a bounded sequence where d_n *is the inverse of* a_n.

Proof. Let $f(s)$ be invertible and $g(s) = \sum_{n=1}^{\infty} b_n e^{\lambda_n s}$ be its inverse then $f(s) \cdot g(s) = e(s)$. This implies that

$$\lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} a_n b_n = e_n \; \lambda_n^{-c_1\lambda_n} \; e^{\{c_1 - c_2 n\}\lambda_n}$$

or equivalently one can write

$$\lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} b_n = e_n \; \{\lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} a_n\}^{-1}$$
$$\lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} \|b_n\| = \|e_n \; \{\lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} a_n\}^{-1}\|$$
$$\lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} \|b_n\| = \|e_n \; a_n^{-1} \; \lambda_n^{-c_1\lambda_n} \; e^{\{c_1 - c_2 n\}\lambda_n}\|$$
$$\lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} \|b_n\| = \|d_n \; \lambda_n^{-c_1\lambda_n} \; e^{\{c_1 - c_2 n\}\lambda_n}\|$$

Since $g(s) \in F$ hence $\{\|d_n \; \lambda_n^{-c_1\lambda_n} \; e^{\{c_1 - c_2 n\}\lambda_n}\|\}$ is a bounded sequence.

Conversely suppose $\{\|d_n \; \lambda_n^{-c_1\lambda_n} \; e^{\{c_1 - c_2 n\}\lambda_n}\|\}$ be a bounded sequence. Define $g(s)$ such that

$$g(s) = \sum_{n=1}^{\infty} e_n \; \lambda_n^{-2c_1\lambda_n} \; e^{\{2c_1 - 2c_2 n\}\lambda_n} a_n^{-1} \; e^{\lambda_n s}$$

Obviously $g(s) \in F$. Moreover

$$f(s) \cdot g(s) = \sum_{n=1}^{\infty} \{(a_n \; e_n \; \lambda_n^{-2c_1\lambda_n} \; e^{\{2c_1 - 2c_2 n\}\lambda_n} \; a_n^{-1}) \; \lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n}\} \; e^{\lambda_n s} = e(s)$$

Hence the theorem. □

Theorem 3. *A necessary and a sufficient condition that an element f(s) of F be a topological zero divisor is that*

$$\lim_{n\to\infty} \lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} \|a_n\| = 0$$

Proof. Let the given condition holds. We need to prove that $f(s)$ is a topological zero divisor. Construct a sequence $\{g_n\}$ such that

$$g_n(s) = \sum_{n=1}^{\infty} \; \lambda_n^{-c_1\lambda_n} \; e^{\{c_1 - c_2 n\}\lambda_n} \; e^{\lambda_n s}$$

Thus for all $n \geqslant 1$, $g_n \in F$ and $\|g_n\| = 1$.

Now

$$g_n(s) \cdot f(s) = f(s) \cdot g_n(s) = \sum_{n=1}^{\infty} [\lambda_n^{-c_1\lambda_n} \; e^{\{c_1 - c_2 n\}\lambda_n} \; a_n \; \lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n}] e^{\lambda_n s} = \sum_{n=1}^{\infty} a_n \; e^{\lambda_n s}$$

Therefore

$$\|g_n \cdot f\| = \|f \cdot g_n\| = \sum_{n=1}^{\infty} \; \lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} \|a_n\|$$

As $n \to \infty$

$$\|g_n \cdot f\| = \|f \cdot g_n\| \to 0$$

Thus $f(s)$ is a topological zero divisor.

Conversely suppose if possible the given condition is not true that is

$$\lim_{n\to\infty} \lambda_n^{c_1\lambda_n} \; e^{\{c_2 n - c_1\}\lambda_n} \|a_n\| = \alpha > 0$$

Then given η with $0 < \eta < \alpha$ we can find an integer $n_0 \geqslant 1$ such that for all $n \geqslant n_0$

$$\lambda_n^{c_1\lambda_n} \ e^{\{c_2 n - c_1\}\lambda_n}\|a_n\| > \alpha - \eta$$

hold true. Also since $f(s)$ is a topological zero divisor therefore there exists an arbitrary sequence $\{g_t\}$ of elements in F with unit norm such that for all $t \geqslant 1$ one has

$$g_t(s) = \sum_{t=1}^{\infty} b_t e^{\lambda_t s}, \Rightarrow \sum_{t=1}^{\infty} \lambda_t^{c_1\lambda_t} \ e^{\{c_2 t - c_1\}\lambda_t}\|b_t\| = 1$$

Next for given ϵ satisfying $0 < \epsilon < 1$ there exists an integer N_t and a subsequence $\{n_i\}$ of the sequence of indices $\{n\}$ such that

$$\lambda_n^{c_1\lambda_n} \ e^{\{c_2 n - c_1\}\lambda_n}\|b_{nt}\| > 1 - \epsilon \quad \text{for all } n = n_i \geqslant N_t.$$

Hence we have

$$\lambda_n^{c_1\lambda_n} \ e^{\{c_2 n - c_1\}\lambda_n} \left\{\lambda_n^{c_1\lambda_n} \ e^{\{c_2 n - c_1\}\lambda_n}\|a_n \cdot b_{nt}\|\right\} > c > 0 \quad \text{for all } n_i \geqslant N_t.$$

Therefore

$$\|f(s) \cdot g_t(s)\| \nrightarrow 0$$

which is a contradiction to the fact that $f(s)$ is a topological zero divisor. Hence our initial supposition is not true. This completes the proof of the theorem. □

Theorem 4. *F is not a Division Algebra.*

Proof. Let

$$p(s) = \sum_{n=1}^{\infty} \{n^{-1}\lambda_n^{-c_1\lambda_n} \ e^{\{c_1 - c_2 n\}\lambda_n}\} \ e^{\lambda_n s}$$

$p(s) \in F$ and does not possess inverse in F.

Let if possible

$$q(s) = \sum_{n=1}^{\infty} d_n e^{\lambda_n s}$$

be its inverse. Hence

$$\begin{aligned} p(s) \cdot q(s) &= e(s) \\ &\Rightarrow \sum_{n=1}^{\infty} \{n^{-1}\lambda_n^{-c_1\lambda_n} \ e^{\{c_1 - c_2 n\}\lambda_n} \ d_n \ \lambda_n^{c_1\lambda_n} \ e^{\{c_2 n - c_1\}\lambda_n}\} e^{\lambda_n s} \\ &= \sum_{n=1}^{\infty} e_n \ \lambda_n^{-c_1\lambda_n} \ e^{\{c_1 - c_2 n\}\lambda_n} e^{\lambda_n s} \Rightarrow d_n \\ &= n \ e_n \ \lambda_n^{-c_1\lambda_n} \ e^{\{c_1 - c_2 n\}\lambda_n} \text{ does not belong to } F. \end{aligned}$$

Hence the theorem. □

Theorem 5. *Every continuous linear functional $\theta: F \to E$ is of the form*

$$\theta(f) = \sum_{n=1}^{\infty} a_n \ d_n \ \lambda_n^{c_1\lambda_n} \ e^{\{c_2 n - c_1\}\lambda_n}$$

where

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}$$

and $\{d_n\}$ is a bounded sequence in E.

Proof. Let us first assume that $\theta: F \to E$ be a continuous linear functional.

Then since θ is continuous

$$\theta(f) = \theta(\lim_{N\to\infty} f^{(N)})$$

where

$$f^{(N)}(s) = \sum_{n=1}^{N} a_n \ e^{\lambda_n s} \Rightarrow \theta(f) = \theta(\lim_{N\to\infty} \sum_{n=1}^{N} \ a_n e^{\lambda_n s})$$

Now let us define a sequence $\{f_n\} \subseteq F$ as

$$f_n(s) = \lambda_n^{-c_1\lambda_n} \ e^{\{c_1 - c_2 n\}\lambda_n} \ e^{\lambda_n s}$$

$$\begin{aligned} \theta(f) &= \theta(\lim_{N\to\infty} \sum_{n=1}^{N} \ a_n \ \lambda_n^{c_1\lambda_n} \ e^{\{c_2 n - c_1\}\lambda_n} \ f_n) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \ a_n \ \lambda_n^{c_1\lambda_n} \ e^{\{c_2 n - c_1\}\lambda_n} \ \theta(f_n) \end{aligned}$$

Since θ is a linear functional therefore

$$\theta(f_n) = d_n \Rightarrow \theta(f) = \sum_{n=1}^{\infty} \ a_n \ d_n \ \lambda_n^{c_1\lambda_n} \ e^{\{c_2 n - c_1\}\lambda_n}.$$

Now we show $\{d_n\}$ is a bounded sequence in E.

$$\|d_n\| = \|\theta(f_n)\| \leqslant M\|f_n\|$$

and $\|f_n\| = 1$

$$\Rightarrow \|d_n\| \leqslant M$$

Thus $\{d_n\}$ is a bounded sequence.

Conversely let $\{d_n\}$ is a bounded sequence in E satisfying

$$\theta(f) = \sum_{n=1}^{\infty} \ a_n \ d_n \ \lambda_n^{c_1\lambda_n} \ e^{\{c_2 n - c_1\}\lambda_n}$$

Then θ is well defined and linear. Now

$$\begin{aligned} \|\theta(f)\| &= \sum_{n=1}^{\infty} \ \|a_n d_n\| \ \lambda_n^{c_1\lambda_n} \ e^{\{c_2 n - c_1\}\lambda_n} \\ &\leqslant \sum_{n=1}^{\infty} \ \|a_n\| \ \|d_n\| \ \lambda_n^{c_1\lambda_n} \ e^{\{c_2 n - c_1\}\lambda_n} \leqslant M\|f\| \end{aligned}$$

Thus θ is a continuous linear functional. □

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