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ORIGINAL ARTICLE

Eigenvalues for the Steklov problem via Ljusternic–Schnirelman principle

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Abstract This paper deals with the existence of nondecreasing sequence of nonnegative eigenvalues for the systems

$$\begin{cases} div(a(x)|\nabla u|^{p-2}\nabla u) = b(x)|u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = \lambda c(x)|u|^{p-2}u & \text{on } \partial\Omega, \end{cases}$$

by using the Ljusternic–Schnirelman principle, where Ω is a bounded domain in $R^N (N \geqslant 2)$.

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1. Introduction

Eigenvalue problems for the p-Laplacian operator on a bounded domain have been studied extensively and many interesting results have been obtained, see e.g. [13] and [14].

Beside being of mathematical interest, the study of the p-Laplacian operator is also of interest in the theory of Non-Newtonian fluids both for the case $p \geqslant 2$ (dilatant fluids) and the case $1 < p < 2$ (pseudo-plastic fluids), see [3].

In this work we study the existence of nondecreasing sequence of nonnegative eigenvalues for the systems

$$\begin{cases} div(a(x)|\nabla u|^{p-2}\nabla u) = b(x)|u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = \lambda c(x)|u|^{p-2}u & \text{on } \partial\Omega, \end{cases} \quad (1)$$

by using the Ljusternic–Schnirelman principle, where Ω is a bounded domain in $R^N (N \geqslant 2)$ and $1 < p \leqslant N$. We assume that

$a(x), b(x)$ is positive a.e. in Ω,

$$a \in L^1_{loc}(\Omega), a^{-s} \in L^1(\Omega),\ s \in \left(\frac{N}{p}, \infty\right) \cap \left[\frac{1}{p-1}, \infty\right). \quad (2)$$

We define

$$p_s = \frac{ps}{s+1}, p_s^* = \frac{Np_s}{N-p_s} = \frac{Nps}{N(s+1)-ps}, \quad (3)$$

In addition we assume

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$$\text{meas}\ \{x \in \partial\Omega : c(x) > 0\} > 0,$$
$$c \in L^{\frac{q}{q-p}}(\partial\Omega), \text{ for some } p \leqslant q < p_s^*. \qquad (4)$$

Many results have been obtained on the structure of the spectrum of the Dirichlet problem

$$\begin{cases} div(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

(e.g. see [4,7,9]). It is shown in [5] that there exists a nondecreasing sequence of positive eigenvalues λ_n tending to ∞ as $n \to \infty$, also in [12], the author establish the results on existence of such sequence and some properties of the spectral of above problem. The existence of such a sequence of eigenvalues can be proved using the theory of Ljusternic–Schnirelman (e.g. see [6,8]). For that reason we call this sequence *the* $L - S$ *sequence* $\{\lambda_n\}$. Motivated by above-mentioned papers and the results in [15], we deal with the existence of L-S sequence and simplicity of the principal eigenvalue of problem (1).

Let $X := W^{1,p}(a, \Omega)$, the weighted Sobolev is defined to the set of all real valued measurable functions u for which

$$\|u\|_{1,p,a} = \left(\int_\Omega a|\nabla u|^p dx + \int_\Omega |u|^p dx\right)^{\frac{1}{p}}. \qquad (5)$$

Then X equipped with the norm $\|\cdot\|_{1,p,a}$ is a uniformly convex Banach space, thus, by Milman's Theorem (see [10]) is a reflexive Banach space. Moreover we have these continuous embedding

$$X \hookrightarrow W^{1,p_s}(\Omega) \hookrightarrow L^{p_s^*}(\Omega)$$

with $p_s = \frac{ps}{s+1}$ and $p_s^* = \frac{Np_s}{N-p_s}$.

Notice that the compact embedding

$$X \hookrightarrow L^r(\partial\Omega) \qquad (6)$$

holds provided that $1 \leqslant r < p_s^*$, see [1] and [2]. It follows from the weighted Friedrichs inequality (see [7] (formula (1.28))) that the norm

$$\|u\| = \left(\int_\Omega a|\nabla u|^p dx\right)^{\frac{1}{p}}$$

on the space X is equivalent to the norm $\|\cdot\|_{1,p,a}$ defined in (5).

Definition 1. We say $\lambda > 0$ is a positive eigenvalue of (1), if there exists a nontrivial function $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega a|\nabla u|^{p-2}\nabla u \nabla v dx + \int_\Omega b|u|^{p-2}uv dx = \lambda \int_{\partial\Omega} c(t)|u|^{p-2}uv dt \qquad (7)$$

holds for any $v \in X$. Then u is called an eigenfunction corresponding to the eigenvalue λ. The pair (u, λ) is called an eigenpair.

2. The Ljusternic–Schnirelman principle

Let X be a real Banach space and F, G be two functionals on X. For fixed $\alpha > 0$, we consider the eigenvalue problem

$$F'(u) = \mu G'(u), \qquad u \in N_\alpha, \quad \lambda \in R \qquad (8)$$

with the level set

$$N_\alpha := \{u \in X; G(u) = \alpha\}.$$

We assume that:

(H$_1$)F, $G: X \to R$ are even functionals such that F, $G \in C^1(X, R)$ and $F(0) = G(0) = 0$. In particular, it follows from this that F' and G' are odd potential operators.

(H$_2$) The operator F' is strongly continuous (i.e. $u_n \rightharpoonup u \Rightarrow F'(u_n) \to F'(u)$) and $F(u) \neq 0, u \in \overline{coN_\alpha}$ implies $F'(u) \neq 0$, where $\overline{coN_\alpha}$ is the closed convex hull of N_α.

(H$_3$) The operator G' is uniformly continuous on bounded sets and satisfies (S_0), i.e. as $n \to \infty$,

$$u_n \rightharpoonup u, G'(u_n) \rightharpoonup v, \langle G'(u_n), u_n\rangle \to \langle v, u\rangle \text{ implies } u_n \to u.$$

(H$_4$) The level set N_α is bounded and

$$u \neq 0 \text{ implies } \langle G'(u), u\rangle > 0, \lim_{t\to\infty} G(tu) = +\infty,$$

and

$$\inf_{u \in N_\alpha} \langle G'(u), u\rangle > 0.$$

It is known that u is a solution of (8) if and only if u is a critical point of F with respect to N_α (see Zeidler [8, Proposition 43.21]).

For any positive integer n, denote by A_n the class of all compact, symmetric subsets K of N_α such that $F(u) > 0$ on K and $\gamma(K) \geqslant 0$, where $\gamma(K)$ denote the genus of K, i.e., $\gamma(K) := \inf\{k \in N; \exists h: K \to R^k \setminus \{0\}$ such that h is continuous and odd$\}$.

We define:

$$a_n = \begin{cases} \sup_{H \in A_n} \inf_{u \in H} F(u) & \text{if } A_n \neq \emptyset \\ 0 & \text{if } A_n = \emptyset. \end{cases} \qquad (9)$$

Also let

$$\chi = \begin{cases} \sup\{n \in N; a_n > 0\} & \text{if } a_1 > 0, \\ 0 & \text{if } a_1 = 0. \end{cases}$$

Now, we state the L–S principle.

Theorem 1. *Under assumptions (H$_1$)–(H$_4$), the following assertions hold:*

[1] (*Existence of an eigenvalue*) If $a_n > 0$, then (1) possesses a pair $\pm u_n$ of eigenvectors and an eigenvalue $\mu_n \neq 0$; furthermore $F(u_n) = a_n$.
[2] (*Multiplicity*) If $\chi = \infty$, (8) has infinitely many pairs $\pm u_n$ of eigenvectors corresponding to nonzero eigenvalues.
[3] (*Critical levels*) $\infty > a_1 \geqslant a_2 \geqslant \cdots \geqslant 0$ and $a_n \to 0$ as $n \to \infty$.
[4] (*Infinitely many eigenvalues*) If $\chi = \infty$ and $F(u) = 0, u \in \overline{coN_\alpha}$ implies $\langle F'(u), u\rangle = 0$, then there exists an infinite sequence $\{\mu_n\}$ of distinct eigenvalues of (8) such that $\mu_n \to 0$ as $n \to \infty$.
[5] (*Weak convergence of eigenvectors*) Assume that $F(u) = 0, u \in \overline{coN_\alpha}$ implies $u = 0$, Then $\chi = \infty$ and there exists a sequence of eigenpairs $\{(u_n, \mu_n)\}$ of (8) such that $u_n \rightharpoonup 0, \mu_n \to 0$ as $n \to \infty$ and $\mu_n \neq 0$ for all n.

Proof. We refer to [6] or [8] for the proof. □

Define on X the functionals

$$F(u) = \int_{\partial\Omega} c(t)|u(t)|^p dt, \qquad (10)$$

$$G(u) = \int_\Omega a|\nabla u|^p dx + \int_\Omega b|u|^p dx. \qquad (11)$$

It is easy to see that F and G are differentiable with $A = \frac{1}{p}F'$ and $B = \frac{1}{p}G'$ given by

$$\langle Au, v\rangle = \int_{\partial\Omega} c(t)|u(t)|^{p-2}uvdt, \tag{12}$$

$$\langle Bu, v\rangle = \int_{\Omega} a|\nabla u|^{p-2}\nabla u\nabla vdx + \int_{\Omega} b|u|^{p-2}uvdx. \tag{13}$$

Then (8) becomes $Au = \mu Bu$, where $G(u) = 1$.

We claim that F and G satisfy $(\mathbf{H_1})$–$(\mathbf{H_4})$. It is clear that F and G are even and $(\mathbf{H_4})$ holds. It remain to verify $(\mathbf{H_2})$ and $(\mathbf{H_3})$.

Lemma 1. *Let Ω be a domain in R^N and let $\phi: R^+ \to R^+$ be a Young function which satisfies a Δ_2-condition, i.e., there is $c > 0$ such that $\phi(2t) \leqslant c\phi(t)$ for all $t \geqslant 0$. If $\{u_n\}$ is a sequence of integrable functions in Ω such that*

$$u(x) = \lim_{n\to\infty} u_n(x), \text{ a.e. } x \in \Omega \quad \text{and} \quad \int_{\Omega}\phi(|u|)dx = \lim_{n\to\infty}\int_{\Omega}\phi(|u_n|)dx,$$

then

$$\lim_{n\to\infty}\phi(|u_n - u|)dx = 0.$$

Proof. See [11, Theorem 12], for the proof. □

Proposition 1. *The functional F given by (10) satisfies $(\mathbf{H_2})$.*

Proof. It is sufficient to show that A is strongly continuous. Let $u_n \rightharpoonup u$ in X, we show that $Au_n \to Au$ in X^*.

For any $v \in X$, by Holder's inequality and compact embedding $X \hookrightarrow L^p(\partial\Omega)$, it follows that

$$\begin{aligned}|\langle Au_n - Au, v\rangle| &= \left|\int_{\partial\Omega} c(|u_n|^{p-2}u_n - |u|^{p-2}u)vds\right| \\ &\leqslant \|c\|_{L^\alpha(\partial\Omega)}\||u_n|^{p-2}u_n - |u|^{p-2}u\|_{L^{\frac{\beta}{p-1}}(\partial\Omega)}\|v\|_{L^{p_s^*}(\partial\Omega)} \\ &\leqslant k\|c\|_{L^\alpha(\partial\Omega)}\||u_n|^{p-2}u_n - |u|^{p-2}u\|_{L^{\frac{\beta}{p-1}}(\partial\Omega)}\|v\|,\end{aligned}$$

where α, β are such that $\frac{1}{\alpha} + \frac{p-1}{\beta} + \frac{1}{p_s^*} = 1$. We observe that

$$\frac{p_s^* - p}{p_s^*} + \frac{p-1}{p_s^*} + \frac{1}{p_s^*} = 1. \tag{14}$$

Since c is in $L^{\frac{q}{q-p}}(\partial\Omega)$ and $\frac{p_s^*}{p_s^*-p} < \frac{q}{q-p}$, whenever $p < q < p_s^*$, we can choose α such that $\frac{p_s^*}{p_s^*-p} < \alpha < \frac{q}{q-p}$. With this choice of α, it follows from (14) that $1 < \beta < p_s^*$. We next show that $|u_n|^{p-2}u_n \to |u|^{p-2}u$ in $L^{\frac{\beta}{p-1}}(\partial\Omega)$. To see this, let $w_n = |u_n|^{p-2}u_n$ and $w = |u|^{p-2}u$. Since $u_n \rightharpoonup u$ in X, $u_n \to u$ in $L^\beta(\partial\Omega)$ by (6), it follows that

$$w_n(x) \to w(x), a.e. \text{ on } \partial\Omega \quad \text{and} \quad \int_{\partial\Omega}|w_n|^{\frac{\beta}{p-1}}ds \to \int_{\partial\Omega}|w|^{\frac{\beta}{p-1}}ds.$$

Using Lemma 1, we conclude that $w_n \to w$ in $L^{\frac{\beta}{p-1}}(\partial\Omega)$. Therefore $Au_n \to Au$ in X^*. □

Lemma 2. *Let B be defined in (13), then for any u, $v \in X$ one has*

$$\langle Bu - Bv, u - v\rangle \geqslant (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|).$$

Furthermore, $\langle Bu - Bv, u - v\rangle = 0$ if and only if $u = v$ a.e. in Ω.

Proof. Straightforward computation gives us for any u, v in X

$$\begin{aligned}|\langle Bu - Bv, u - v\rangle| = &\int_{\Omega}[a|\nabla u|^p + a|\nabla v|^p]dx \\ &- \int_{\Omega} a|\nabla u|^{p-2}\nabla u\nabla vdx - a|\nabla v|^{p-2}\nabla v\nabla udx \\ &+ \int_{\Omega}[b|u|^p + b|v|^p]dx - \int_{\Omega} b|u|^{p-2}uvdx \\ &- \int_{\Omega} b|v|^{p-2}vudx.\end{aligned}$$

Also, we have

$$\begin{aligned}&\int_{\Omega} b(|u|^p + |v|^p - |u|^{p-2}uv - |v|^{p-2}uv)dx \\ &\geqslant \int_{\Omega} b(|u|^p + |v|^p - |u|^{p-1}|v| - |v|^{p-1}|u|)dx \\ &= \int_{\Omega} b(|u|^{p-1} - |v|^{p-1})(|u| - |v|)dx \geqslant 0,\end{aligned}$$

where the last inequality follows from the fact that $t \to |t|^{p-1}$ is strictly increasing. As the function a is positive, it follows from Holder's inequality that

$$\int_{\Omega} a|\nabla u|^{p-2}\nabla u\nabla vdx \leqslant \left(\int_{\Omega} a|\nabla u|^p\right)^{\frac{p-1}{p}}\left(\int_{\Omega} a|\nabla v|^p\right)^{\frac{1}{p}} = \|u\|^{p-1}\|v\|. \tag{15}$$

Similarly we have

$$\int_{\Omega} a|\nabla v|^{p-2}\nabla v\nabla udx \leqslant \|v\|^{p-1}\|u\|.$$

Therefore,

$$\begin{aligned}\langle Bu - Bv, u - v\rangle &\geqslant \|u\|^p + \|v\|^p - \|u\|^{p-1}\|v\| - \|v\|^{p-1}\|u\| \\ &= (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|).\end{aligned}$$

Now let u and v be such that $\langle B\,u - B\,v, u - v\rangle = 0$. Then we have

$$\langle Bu - Bv, u - v\rangle = (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|) = 0.$$

It follows that $\|u\| = \|v\|$ and that the equality holds in (15). As equality in Holder's inequality is characterized, we obtain that $u = kv$ a.e. in Ω, for some constant $k \geqslant 0$, which implies $\|u\| = k\|v\|$. Therefore, $k = 1$ and $u = v$ a.e. in Ω. □

Proposition 2. *Let G be defined in (11), then G' satisfies $(\mathbf{H_3})$.*

Proof. As $B = \frac{G'}{p}$, it suffices to show this for B. It is easy to see that B is bounded. Using Holder's inequality and Sobolev embedding theorem we have

$$\begin{aligned}\langle Bu_n - Bu, v\rangle = &\left|\int_{\Omega} a(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla vdx + \int_{\Omega} b(|u_n|^{p-2}u_n\right. \\ &\left. -|u|^{p-2}u)vdx\right| \leqslant \left(\int_{\Omega}\left|a^{\frac{p-1}{p}}\right|\nabla u_n|^{p-2}\nabla u_n - a^{\frac{p-1}{p}}|\nabla u|^{p-2}\nabla u|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\|v\| \\ &+ c\left(\int_{\Omega}\left|b^{\frac{p-1}{p}}\right|u_n|^{p-2}u_n - b^{\frac{p-1}{p}}|u|^{p-2}u|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\|v\|\end{aligned}$$

Since $u_n \to u$ in X, $u_n \to u$ in $L^p(\Omega)$. Let $w_n = a^{\frac{p-1}{p}}|\nabla u_n|^{p-2}\nabla u_n$ and $w = a^{\frac{p-1}{p}}|\nabla u|^{p-2}\nabla u$ then

$$w_n(x) \to w(x), \text{ a.e.in } \Omega \quad \text{and} \quad \int_\Omega |w_n|^{\frac{p}{p-1}}dx \to \int_\Omega |w|^{\frac{p}{p-1}}dx.$$

Thus, $w_n \to w$ in $L^{\frac{p}{p-1}}(\Omega)$ and similarly we can prove that $b^{\frac{p-1}{p}}|u_n|^{p-2}u_n \to b^{\frac{p-1}{p}}|u|^{p-2}u$ in $L^{\frac{p}{p-1}}(\Omega)$.

It remains to show that B satisfies condition S_0. That means if $\{u_n\}$ is a sequence in X such that

$$u_n \rightharpoonup u, Bu_n \rightharpoonup v \quad \text{and} \quad \langle Bu_n, u_n\rangle \to \langle v, u\rangle$$

for some $v \in X^*$ and $u \in X$, then $u_n \to u$ in X.

Since X is a uniformly convex Banach space, Weak convergence and norm convergence imply (strong) convergence. Thus to showing $u_n \to u$, we need to show $\|u_n\| \to \|u\|$. To do this, we first observe that

$$\lim_{n\to\infty}\langle Bu_n - Bu, u_n - u\rangle = \lim_{n\to\infty}(\langle Bu_n, u_n\rangle - \langle Bu_n - u\rangle - \langle Bu, u_n - u\rangle) = 0.$$

On the other hand, it follows from Lemma 2 that

$$\langle Bu_n - Bu, u_n - u\rangle \geqslant (\|u_n\|^{p-1} - \|u\|^{p-1})(\|u_n\| - \|u\|).$$

Thus $\|u_n\| \to \|u\|$ as $n \to \infty$. Therefore B satisfies condition S_0. □

Theorem 2 (*Existence of L–S sequence*). *Let F and G be the two functionals defined in (10) and (11). Then there exists a nondecreasing sequence of positive eigenvalues $\{\mu_n\}$ obtained from the Ljusternik–Schnirelman principle such that $\mu_n \to 0$ as $n \to \infty$,*
where

$$\mu_n = \sup_{H\in A_n}\inf_{u\in H}F(u) \tag{16}$$

and each μ_n is an eigenvalue of $F'(u) = \mu\, G'(u)$.

Proof. It is easy to see that N_α contains compact subsets of arbitrary large genus. Thus A_n is nonempty for any n. Given a set H in A_n, since H is compact and F is positive on H, $\inf_{u\in H}F(u) > 0$. It follows that a_n is the critical value defined by (9). The existence of such a sequence μ_n follows from Theorem 1 – [1], [2] and [3]. Also

$$\mu_n = \mu G(u_n) = \mu\langle Bu_n, u_n\rangle = \langle Au_n, u_n\rangle = F(u_) = a_n.$$

Combining this with (9), we obtain (16). □

3. Simplicity of the first eigenvalue

In this section we will show that the first element λ_1 of the L–S sequence of eigenvalue is simple.

Lemma 3.

(i) Let $p \geqslant 2$ then for all $x, y \in R^N$

$$|y|^p \geqslant |x|^p + p|x|^{p-2}x\cdot(y-x) + C(p)|x-y|^p. \tag{17}$$

(ii) Let $1 < p < 2$, then for all $x, y \in R^N$,

$$|y|^p \geqslant |x|^p + p|x|^{p-2}x\cdot(y-x) + C(p)\frac{|x-y|^2}{(|x|+|y|)^{2-p}}. \tag{18}$$

(iii) For any $x \neq y$, $p > 1$

$$|y|^p \geqslant |x|^p + p|x|^{p-2}x\cdot(y-x). \tag{19}$$

In the above $C(p)$ is a constant depending only on p.

Proof. We refer to Lindqvist [4] for the proof. □

Lemma 4. *If u_1 is an eigenfunction associated with λ_1, then either $u_1 > 0$ or $u_1 < 0$ in $\overline{\Omega}$.*

Proof. We refer to [12] for the proof. □

Theorem 3. *The principal eigenvalue λ_1 is simple, i.e., if u and v are two eigenfunctions associated with λ_1, then there exists a constant c such that $u = cv$.*

Proof. By Lemma 4 we can assume u and v are positive in $\overline{\Omega}$. Let

$$\eta_1 = \frac{(u^p - v^p)}{u^{p-1}} \quad \text{and} \quad \eta_2 = \frac{(v^p - u^p)}{v^{p-1}},$$

then take them as test functions, we get

$$\int_\Omega a|\nabla u|^{p-2}\nabla u.\nabla\left(\frac{u^p - v^p}{u^{p-1}}\right)dx = \lambda_1\int_{\partial\Omega}c|u|^{p-2}u\left(\frac{u^p - v^p}{u^{p-1}}\right)ds - \int_\Omega b|u|^{p-2}u\left(\frac{u^p - v^p}{u^{p-1}}\right)dx,$$

and

$$\int_\Omega a|\nabla v|^{p-2}\nabla v\cdot\nabla\left(\frac{v^p - u^p}{v^{p-1}}\right)dx = \lambda_1\int_{\partial\Omega}c|v|^{p-2}v\left(\frac{v^p - u^p}{v^{p-1}}\right)ds - \int_\Omega b|v|^{p-2}v\left(\frac{v^p - u^p}{v^{p-1}}\right)dx.$$

Summing up, we obtain

$$0 = \int_\Omega a|\nabla u|^{p-2}\nabla u\cdot\nabla\left(\frac{u^p - v^p}{u^{p-1}}\right)dx + \int_\Omega a|\nabla v|^{p-2}\nabla v \cdot\nabla\left(\frac{v^p - u^p}{v^{p-1}}\right)dx. \tag{20}$$

We know that

$$\nabla\left(\frac{u^p - v^p}{u^{p-1}}\right) = \nabla u - p\frac{v^{p-1}}{u^{p-1}}\nabla v + (p-1)\frac{v^p}{u^p}\nabla u.$$

Using this, the first term of (20) becomes

$$\int_\Omega a|\nabla u|^p dx - p\int_\Omega a\frac{v^{p-1}}{u^{p-1}}|\nabla u|^{p-2}\nabla v\nabla u dx + (p-1)\int_\Omega a \times \frac{v^p}{u^p}|\nabla u|^p dx$$
$$= \int_\Omega a|\nabla \ln u|^p u^p dx - p\int_\Omega a v^p|\nabla \ln u|^{p-2}\nabla\ \ln\ u\cdot\nabla \ln v dx + (p-1)\int_\Omega a|\nabla \ln u|^p v^p dx$$
$$= \int_\Omega a|\nabla \ln u|^p(u^p - v^p)dx - p\int_\Omega a v^p|\nabla \ln u|^{p-2}\nabla \ln u\nabla \ln v + p\int_\Omega a|\nabla \ln u|^p v^p dx$$

and for the second term of (20) we have

$$\int_\Omega a|\nabla v|^p dx - p\int_\Omega a\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla u\nabla v dx + (p-1)$$

$$\int_\Omega a\frac{u^p}{v^p}|\nabla v|^p dx$$

$$= \int_\Omega a|\nabla \ln v|^p v^p dx - p\int_\Omega a u^p|\nabla \ln v|^{p-2}\nabla \ \ln \ v\cdot\nabla \ln u dx$$

$$+ (p-1)\int_\Omega a|\nabla \ln v|^p u^p dx$$

$$= \int_\Omega a|\nabla \ln v|^p (v^p - u^p)dx - p\int_\Omega a u^p|\nabla \ln v|^{p-2}\nabla \ln v\nabla \ln u$$

$$+ p\int_\Omega a|\nabla \ln v|^p u^p dx.$$

Thus (20) becomes

$$0 = \int_\Omega a(u^p - v^p)(|\nabla \ln u|^p - |\nabla \ln v|^p)dx - p$$

$$\times\int_\Omega a v^p|\nabla \ln u|^{p-2}\nabla \ln u\cdot(\nabla \ln v - \nabla \ln u)dx - p$$

$$\times\int_\Omega a u^p|\nabla \ln v|^{p-2}\nabla \ln v\cdot(\nabla \ln u - \nabla \ln v)dx.$$

For $p \geqslant 2$, taking $x = \$\ln u$, $y = \$\ln v$ and vice versa, it follows from inequality (17) in Lemma 3 that

$$0 \geqslant C(p)\int_\Omega a|\nabla \ln u - \nabla \ln v|^p (u^p + v^p)dx.$$

Therefore,

$$0 = |\nabla \ln u - \nabla \ln v|$$

This implies that $u = kv$. For $p < 2$ we use inequality (18) in Lemma 3 to obtain the same results. □

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