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ORIGINAL ARTICLE

Inclusion properties for some subclasses of analytic functions associated with generalized integral operator

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Abstract In the present paper, we introduce several subclasses of analytic functions, which are defined by means of generalized integral operator and investigate various inclusion properties of these subclasses. Some interesting applications involving these and other families of integral operators are also considered.

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1. Introduction

Let $\mathcal{A}$ denote the class of function $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. If f and g are analytic in U, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function ω, analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$, such that $f(z) = g(\omega(z))$ $(z \in U)$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

For two functions $f(z)$ given by (1.1) and $g(z)$ given by:

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.2}$$

their Hadamard product (or convolution) is defined by:

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.3}$$

For $0 \leqslant \alpha < 1$, we denote by $S^*(\alpha)$ and $C(\alpha)$ the subclasses of $\mathcal{A}$ consisting of all analytic functions which are, respectively, starlike of order α and convex of order α in U.

Let $\mathcal{P}_k(\alpha)$ be the class of functions $p(z)$ analytic in U satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re\{p(z)\} - \alpha}{1 - \alpha} \right| d\theta \leqslant k\pi, \tag{1.4}$$

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where $z = re^{i\theta}$, $k \geqslant 2$ and $0 \leqslant \alpha < 1$. This class was introduced by Padmanabhan and Parvatham [4]. For $\alpha = 0$, the class $\mathcal{P}_k(0) = \mathcal{P}_k$ was introduced by Pinchuk [3]. Also we note that $\mathcal{P}_2(\alpha) = \mathcal{P}(\alpha)$, where $\mathcal{P}(\alpha)$ is the class of functions with positive real part greater than α and $\mathcal{P}_2(0) = \mathcal{P}$, where $\mathcal{P}$ is the class of functions with positive real part. From (1.4), we have $p(z) \in \mathcal{P}_k(\alpha)$ if and only if there exists $p_1, p_2 \in \mathcal{P}(\alpha)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z) \quad (z \in U). \tag{1.5}$$

It is known that [6]the class $\mathcal{P}_k(\alpha)$ is a convex set.

Making use of the class $\mathcal{P}_k(\alpha)$, we introduce the subclasses $S_k^*(\alpha)$ and $C_k(\alpha)$, $0 \leqslant \alpha < 1$, of the class $\mathcal{A}$ which are defined by:

$$S_k^*(\alpha) = \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \in \mathcal{P}_k(\alpha) \text{ in } U\right\}, \tag{1.6}$$

$$C_k(\alpha) = \left\{f \in \mathcal{A} : \frac{(zf'(z))'}{f'(z)} \in \mathcal{P}_k(\alpha) \text{ in } U\right\}. \tag{1.7}$$

We note that

$$S_2^*(\alpha) = S^*(\alpha) \quad \text{and} \quad C_2(\alpha) = C(\alpha).$$

For complex parameters $a_1, \ldots, a_q; b_1, \ldots, b_s$ ($b_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}$; $j = 1, \ldots, s$), the generalized hypergeometric function ${}_qF_s(a_1, \ldots, a_q;\ b_1, \ldots, b_s;\ z)$ is given by:

$$ {}_qF_s(a_1, \ldots, a_q;\ b_1, \cdots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_q)_n}{(b_1)_n \ldots (b_s)_n} \frac{z^n}{n!} \quad (q \leqslant s+1; q,\ s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \mathbb{N} = \{1, 2, \ldots\}; z \in U), \tag{1.8}$$

where $(x)_n$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & (n = 0), \\ x(x+1)\ldots(x+n-1) & (n \in \mathbb{N}). \end{cases} \tag{1.9}$$

Corresponding to a function $h(a_1, \ldots, a_q;\ b_1, \ldots, b_s;\ z)$ defined by:

$$h(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = z\,{}_qF_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z). \tag{1.10}$$

Dziok and Srivastava [8] considered a linear operator $H(a_1, \ldots, a_q; b_1, \ldots, b_s) : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned} &H(a_1, \ldots, a_q; b_1, \ldots, b_s)\, f(z) \\ &\quad = h(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \ * \ f(z) \\ &\quad = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} \cdots (a_q)_{n-1}}{(b_1)_{n-1} \cdots (b_s)_{n-1}(n-1)!} a_n z^n. \end{aligned} \tag{1.11}$$

We note that many subclasses of analytic functions, associated with the Dziok–Srivastava operator and many special cases, were investigated recently by Aghalary and Azadi [17], Dziok and Srivastava [13,14], Liu [15], Liu and Srivastava [16] and others.

Corresponding to the function $h(a_1, \ldots, a_q;\ b_1, \ldots, b_s;\ z)$, defined by (1.10), we introduce a function $h_\mu(a_1, \ldots, a_q;\ b_1, \ldots, b_s;\ z)$ given by:

$$\begin{aligned} &h(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \ * \ h_\mu(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \\ &\quad = \frac{z}{(1-z)^\mu} \quad (\mu > 0). \end{aligned} \tag{1.12}$$

Analogous to $H(a_1, \ldots, a_q;\ b_1, \ldots, b_s)$, Kwon and Cho [18] introduced the linear operator

$$H^\mu(a_1, \ldots, a_q; b_1, \ldots, b_s) : \mathcal{A} \to \mathcal{A}$$

as follows:

$$\begin{aligned} &H^\mu(a_1, \ldots, a_q; b_1, \ldots, b_s)\, f(z) = h_\mu(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \ * \ f(z) \\ &\quad = z + \sum_{n=2}^{\infty} \frac{(\mu)_{n-1}(b_1)_{n-1} \cdots (b_s)_{n-1}}{(a_1)_{n-1} \cdots (a_q)_{n-1}} a_n z^n \\ &(a_i, b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-, i = 1, \ldots, s, j = 1, \ldots, q; \mu > 0; f \in \mathcal{A}; z \in U). \end{aligned} \tag{1.13}$$

For $q = s + 1$ and $a_2 = b_1, \ldots, a_q = b_s$, we note that

$$H^\mu(\mu, \ldots, a_q; b_1, \ldots, b_s)\, f(z) = f(z) \quad \text{and}$$
$$H^2(1, \ldots, a_q; b_1, \ldots, b_s)\, f(z) = zf'(z).$$

For convenience, we write

$$H^\mu_{q,s}(a_1) = H^\mu_p(a_1, \ldots, a_q; b_1, \ldots, b_s).$$

It is easily verified from the definition (1.13) that

$$z\left(H^\mu_{q,s}(a_1)f(z)\right)' = \mu H^{\mu+1}_{q,s}(a_1)f(z) - (\mu - 1)H^\mu_{q,s}(a_1)f(z) \tag{1.14}$$

and

$$z(H^\mu_{q,s}(a_1 + 1)f(z))' = a_1 H^\mu_{q,s}(a_1)f(z) - (a_1 - 1)H^\mu_{q,s}(a_1 + 1)f(z). \tag{1.15}$$

In particular, the operator $H^\mu_{2,1}(\lambda + 1, 1; 1)$ $(\mu > 0; \lambda > -1)$ was introduced by Choi et al. [12], who investigated (among other things) several inclusion properties involving various subclasses of analytic and univalent functions. For $\lambda = n$ $(n \in \mathbb{N}_0)$ and $\mu = 2$, we also note that the Choi–Sago–Srivastava operator $H^\mu_{2,1}(\lambda + 1, 1; 1)$ is the Noor integral operator of n-th order of $f(z)$ studied by Liu [11] and Noor [9]and Noor and Noor [10].

Next, by using the operator $H^\mu_{q,s}(a_1)$, we introduce the following classes of analytic functions for $k \geqslant 2$, $\mu > 0$, $a_i \in \mathbb{C}(i = 1, \ldots, q), b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-(j = 1, \ldots, s)$ and $0 \leqslant \alpha < 1$ as follows:

$$S^*_{k,q,s}(\mu, a_1; \alpha) = \left\{f \in \mathcal{A} : H^\mu_{q,s}(a_1)f(z) \in S_k^*(\alpha)\right\}, \tag{1.16}$$

$$C_{k,q,s}(\mu, a_1; \alpha) = \left\{f \in \mathcal{A} : H^\mu_{q,s}(a_1)f(z) \in C_k(\alpha)\right\}. \tag{1.17}$$

We also note that

$$f(z) \in C_{k,q,s}(\mu, a_1; \alpha) \Longleftrightarrow zf'(z) \in S^*_{k,q,s}(\mu, a_1; \alpha). \tag{1.18}$$

In particular, we set

$$S^*_{2,q,s}(\mu, a_1; \alpha) = S^*_{q,s}(\mu, a_1; \alpha) \quad \text{and} \quad C_{2,q,s}(\mu, a_1; \alpha) = C_{q,s}(\mu, a_1; \alpha).$$

In this paper, we investigate several inclusion properties of the classes $S^*_{k,q,s}(\mu, a_1; \alpha)$ and $C_{k,q,s}(\mu, a_1; \alpha)$ associated with the generalized integral operator $H^{\mu}_{q,s}(a_1)$. Some applications involving integral operators are also considered.

2. Main results

The following results will be required in our investigation.

Lemma 1 5. *Let $\phi(z)$ be convex univalent in U with $\phi(0) = 1$ and $\Re\{\beta\phi(z) + \gamma\} > 0$ $(\beta, \gamma \in \mathbb{C})$. If $p(z)$ is analytic in U with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \phi(z) \tag{2.1}$$

implies

$$p(z) \prec \phi(z). \tag{2.2}$$

Theorem 1. *Let $k \geqslant 2$ and a_1, $\mu > 1$. Then,*

$$S^*_{k,q,s}(\mu+1, a_1; \alpha) \subset S^*_{k,q,s}(\mu, a_1; \alpha) \subset S^*_{k,q,s}(\mu, a_1+1; \alpha) \quad (0 \leqslant \alpha < 1).$$

Proof. First of all, we will show that

$$S^*_{k,q,s}(\mu+1, a_1; \alpha) \subset S^*_{k,q,s}(\mu, a_1; \alpha).$$

Let $f \in S^*_{k,q,s}(\mu+1, a_1; \alpha)$ and set

$$p(z) = \frac{z\left(H^{\mu}_{p,q,s}(a_1)f(z)\right)'}{H^{\mu}_{p,q,s}(a_1)f(z)} \quad (z \in U), \tag{2.3}$$

where $p(z)$ is analytic in U with $p(0) = 1$. Using (1.14) and (2.3), we have

$$\frac{z\left(H^{\mu+1}_{p}(a_1)f(z)\right)'}{H^{\mu+1}_{p}(a_1)f(z)} = p(z) + \frac{zp'(z)}{p(z)+\mu-1} \in \mathcal{P}_k(\alpha) \quad (k \geqslant 2; 0 \leqslant \alpha < 1; z \in U). \tag{2.4}$$

Our aim is to show that $p(z) \in \mathcal{P}_k(\alpha)$. If $p(z) + \frac{zp'(z)}{p(z)+\mu-1} \in \mathcal{P}_k(\alpha)$, then there exist two functions $h_1(z)$, $h_2(z) \in \mathcal{P}(\alpha)$ such that

$$p(z) + \frac{zp'(z)}{p(z)+\mu-1} = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z).$$

Now let

$$h_i(z) = p_i(z) + \frac{zp_i'(z)}{p_i(z)+\mu-1} \prec \frac{1+(1-2\alpha)z}{1-z} \quad (i=1,2;\ 0 \leqslant \alpha < 1), \tag{2.5}$$

since $\mu > 1$, we see that

$$\Re\left\{\frac{1+(1-2\alpha)z}{1-z} + \mu - 1\right\} > 0 \quad (0 \leqslant \alpha < 1; z \in U).$$

Applying Lemma 1 to (2.5), it follows that

$$p_i(z) \prec \frac{1+(1-2\alpha)z}{1-z} \quad (i = 1,2;\ 0 \leqslant \alpha < 1).$$

This means that $\Re(p_i(z)) > \alpha$, $i = 1,2$. Now, if

$$p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z),$$

then $p(z) \in \mathcal{P}_k(\alpha)$, that is, $f \in S^*_{k,q,s}(\mu, a_1; \alpha)$.

To prove the second part, let $f \in S^*_{k,q,s}(\mu, a_1; \alpha)$ and put

$$q(z) = \frac{z\left(H^{\mu}_{q,s}(a_1+1)f(z)\right)'}{H^{\mu}_{q,s}(a_1+1)f(z)} \quad (z \in U), \tag{2.6}$$

where $q(z)$ is analytic function with $q(0) = 1$. Then, by using the arguments similar to those detailed above with (1.15), it follows that $q \in \mathcal{P}_k(\alpha)$ in U, which implies that $f \in S^*_{k,q,s}(\mu, a_1+1; \alpha)$. This complete the proof of Theorem 1. □

Theorem 2. *Let $k \geqslant 2$ and a_1, $\mu > 1$. Then,*

$$C_{k,q,s}(\mu+1, a_1; \alpha) \subset C_{k,q,s}(\mu, a_1; \alpha) \subset C_{k,q,s}(\mu, a_1+1; \alpha) \quad (0 \leqslant \alpha < 1).$$

Proof. Applying (1.18) and Theorem 1, we observe that

$$\begin{aligned} f(z) \in C_{k,q,s}(\mu+1, a_1; \alpha) &\Longleftrightarrow H^{\mu+1}_{q,s}(a_1)\, f(z) \\ &\in C_k(\alpha) \Longleftrightarrow z\left(H^{\mu+1}_{q,s}(a_1)f(z)\right)' \\ &\in S^*_k(\alpha) \Longleftrightarrow H^{\mu+1}_{q,s}(a_1)(zf'(z)) \in S^*_k(\alpha) \Longleftrightarrow zf'(z) \\ &\in S^*_{k,q,s}(\mu+1, a_1; \alpha) \Rightarrow zf'(z) \\ &\in S^*_{k,q,s}(\mu, a_1; \alpha) \quad \text{(by Theorem 1)} \Longleftrightarrow f(z) \\ &\in C_{k,q,s}(\mu, a_1; \alpha) \quad \text{(by Eq.(2.7))}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} f(z) \in C_{k,q,s}(\mu, a_1; \alpha) &\Longleftrightarrow zf'(z) \in S^*_{k,q,s}(\mu, a_1; \alpha) \quad \text{(by Eq.(2.7))} \\ &\Rightarrow zf'(z) \in S^*_{k,q,s}(\mu, a_1+1; \alpha) \quad \text{(by Theorem 1)} \Longleftrightarrow f(z) \\ &\in C_{k,q,s}(\mu, a_1+1; \alpha), \end{aligned}$$

which evidently proves Theorem 2. □

Remark 1. Taking $k = 2$ in Theorems 1 and 2, respectively, we obtain the results obtained by Kwon and Cho [18, Theorems 2.3 and 2.4, respectively].

3. Inclusion properties involving the integral operator

In this section, we consider the generalized Libera integral operator F_c(cf. [1,2,7]) defined by

$$F_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (f \in A; c > -1). \tag{3.1}$$

We first prove the following theorem.

Theorem 3. *If $f \in S^*_{k,q,s}(\mu, a_1; \alpha)$, then $F_c(f)(z) \in S^*_{k,q,s}(\mu, a_1; \alpha)$ $(k \geqslant 2, c \geqslant 0)$.*

Proof. Let $f \in S^*_{k,q,s}(\mu, a_1; \alpha)$ and set

$$p(z) = \frac{z\left(H^{\mu}_{q,s}(a_1)F_c(f)(z)\right)'}{H^{\mu}_{q,s}(a_1)F_c(f)(z)} \quad (z \in U), \tag{3.2}$$

where $p(z)$ is analytic in U with $p(0) = 1$. From (3.1), we have

$$z\Big(H^{\mu}_{q,s}(a_1)F_c(f)(z)\Big)' = (c+1)H^{\mu}_{q,s}(a_1)f(z) - cH^{\mu}_{q,s}(a_1)F_c(f)(z). \tag{3.3}$$

Then, by using (3.2) and (3.3), we obtain

$$(c+1)\frac{H^{\mu}_{q,s}(a_1)f(z)}{H^{\mu}_{q,s}(a_1)F_c(f)(z)} = p(z) + c. \tag{3.4}$$

Taking the logarithmic differentiation on both sides of (3.4) and multiplying by z, we have

$$p(z) + \frac{zp'(z)}{p(z)+c} = \frac{z\Big(H^{\mu}_{q,s}(a_1)f(z)\Big)'}{H^{\mu}_{q,s}(a_1)f(z)} \in \mathcal{P}_k(\alpha)$$
$$(k \geqslant 2; 0 \leqslant \alpha < 1; z \in U). \tag{3.5}$$

Our aim is to show that $p(z) \in \mathcal{P}_k(\alpha)$. If $p(z) + \frac{zp'(z)}{p(z)+c} \in \mathcal{P}_k(\alpha)$, then there exist two functions $h_1(z), h_2(z) \in \mathcal{P}(\alpha)$ such that

$$p(z) + \frac{zp'(z)}{p(z)+c} = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z).$$

Now let

$$h_i(z) = p_i(z) + \frac{zp_i'(z)}{p_i(z)+c} \prec \frac{1+(1-2\alpha)z}{1-z} \quad (i = 1,2;\ 0 \leqslant \alpha < 1), \tag{3.6}$$

since $c \geqslant 0$, we see that

$$\Re\left\{\frac{1+(1-2\alpha)z}{1-z} + c\right\} > 0 \quad (0 \leqslant \alpha < 1; z \in U).$$

Applying Lemma 1 to (3.6), it follows that

$$p_i(z) \prec \frac{1+(1-2\alpha)z}{1-z} \quad (i = 1,2;\ 0 \leqslant \alpha < 1; z \in U).$$

This means that $\Re(p_i(z)) > \alpha$, $i = 1, 2$. Now, if

$$p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)\ p_2(z),$$

then $p(z) \in \mathcal{P}_k(\alpha)$, that is, $F_c(f)(z) \in S^*_{k,q,s}(\mu, a_1; \alpha)$. □

Next, we derive an inclusion property involving $F_c(f)(z)$, which is given by the following theorem.

Theorem 4. *If $f \in C_{k,q,s}(\mu, a_1; \alpha)$, then $F_c(f)(z) \in C_{k,q,s}(\mu, a_1; \alpha)$ $(k \geqslant 2,\ c \geqslant 0)$.*

Proof. By applying Theorem 3, it follows that

$$f(z) \in C_{k,q,s}(\mu, a_1; \alpha) \Longleftrightarrow zf'(z) \in S^*_{k,q,s}(\mu, a_1; \alpha) \Rightarrow F_c(zf'(z)) \in S^*_{k,q,s}(\mu, a_1; \alpha) \quad \text{(by Theorem 3)} \Longleftrightarrow z(F_c(f)(z))' \in S^*_{k,q,s}(\mu, a_1; \alpha) \Longleftrightarrow F_c(f)(z) \in C_{k,q,s}(\mu, a_1; \alpha),$$

which proves Theorem 4. □

Remark 2. Taking $k = 2$ in Theorems 3 and 4, respectively, we obtain the results obtained by Kwon and Cho [18, Theorems 3.1 and 3.2, respectively].

Theorem 5. *The function $f(z)$ belongs to the class $S^*_{k,q,s}(\mu, a_1; \alpha)$ (or $C_{k,q,s}(\mu, a_1; \alpha)$) if and only if the function $g(z)$ defined by:*

$$g(z) = \frac{\mu}{z^{\mu-1}} \int_0^z t^{\mu-2} f(t)dt \quad (\mu > 0) \tag{3.7}$$

belongs to the class $S^*_{k,q,s}(\mu+1, a_1; \alpha)$ (or $C_{k,q,s}(\mu+1, a_1; \alpha)$).

Proof. From (3.7), we have

$$\mu f(z) = (\mu-1)g(z) + zg'(z). \tag{3.8}$$

Using (1.14) and (3.8), we can write

$$\mu H^{\mu}_{q,s}(a_1)f(z) = (\mu-1)H^{\mu}_{q,s}(a_1)g(z) + z\Big(H^{\mu}_{q,s}(a_1)g(z)\Big)' = \mu H^{\mu+1}_{q,s}(a_1)g(z).$$

Therefore

$$H^{\mu}_{q,s}(a_1)f(z) = H^{\mu+1}_{q,s}(a_1)g(z)$$

and this proves our result. □

Theorem 6. *The function $f(z)$ belongs to the class $S^*_{k,q,s}(\mu, a_1+1; \alpha)$ (or $C_{k,q,s}(\mu, a_1+1; \alpha)$) if and only if the function $r(z)$ defined by:*

$$r(z) = \frac{a_1}{z^{a_1-1}} \int_0^z t^{a_1-2} f(t)dt \quad (a_1 > 0) \tag{3.9}$$

belongs to the class $S^*_{k,q,s}(\mu, a_1; \alpha)$ (or $C_{k,q,s}(\mu, a_1; \alpha)$).

Proof. From (3.9), we have

$$a_1 f(z) = (a_1-1)r(z) + zr'(z). \tag{3.10}$$

Using (1.15) and (3.10), we can write

$$a_1 H^{\mu}_{q,s}(a_1+1)f(z) = (a_1-1)H^{\mu}_{q,s}(a_1+1)r(z) + z\Big(H^{\mu}_{q,s}(a_1+1)r(z)\Big)' = a_1 H^{\mu}_{q,s}(a_1+1)r(z).$$

Therefore

$$H^{\mu}_{q,s}(a_1+1)f(z) = H^{\mu}_{q,s}(a_1)r(z),$$

which proves Theorem 6. □

References

[1] R.J. Libera, Some classes of regular univalent functions, Proc. Am. Math. Soc. 16 (1965) 658–755.

[2] A.E. Livingston, On the radius of univalence of certain analytic functions, Proc. Am. Math. Soc. 17 (1966) 352–357.

[3] B. Pinchuk, Functions with bounded boundary rotation, Isr. J. Math. 10 (1971) 7–16.

[4] K.S. Padmanabhan, R. Parvatham, Properties of a class of functions with bounded boundary rotation, Ann. Polon. Math. 31 (1975) 311–323.

[5] P. Eenigenburg, S.S. Miller, P.T. Mocanu, M.O. Reade, On a Briot–Bouquet differential subordination, in: General Inequalities, 3 (Oberwolfach, 1981), Internationale Schriftenreihe zur Numerischen Mathematik, vol. 64, Birkhäuser, Basel, Switzerland, 1983, pp. 339–348.

[6] K.I. Noor, On subclasses of close-to-convex functions of higher order, Int. J. Math. Math. Sci. 15 (1992) 279–290.
[7] S.D. Bernardi, Convex and univalent functions, Trans. Am. Math. Soc. 135 (1996) 429–446.
[8] J. Dziok, H.M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, Appl. Math. Comput. 103 (1) (1999) 1–13.
[9] K.I. Noor, On new classes of integral operators, J. Nat. Geom. 16 (1–2) (1999) 71–80.
[10] K.I. Noor, M.A. Noor, On integral operators, J. Math. Anal. Appl. 238 (2) (1999) 341–352.
[11] J.-L. Liu, The Noor integral and strongly starlike functions, J. Math. Anal. Appl. 261 (2) (2001) 441–447.
[12] J.H. Choi, M. Saigo, H.M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl. 276 (1) (2002) 432–445.
[13] J. Dziok, H.M. Srivastava, Some subclasses of analytic functions with fixed argument of coefficients associated with the generalized hypergeometric function, Adv. Stud. Contemp. Math. 5 (2) (2002) 115–125.
[14] J. Dziok, H.M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Integ. Transf. Spec. Funct. 14 (1) (2003) 7–18.
[15] J.- L. Liu, Strongly starlike functions associated with the Dziok–Srivastava operator, Tamkang J. Math. 35 (2004) 37–42.
[16] J.-L. Liu, H.M. Srivastava, Certain properties of the Dziok–Srivastava operator, Appl. Math. Comput. 159 (2) (2004) 485–493.
[17] R. Aghalary, G.H. Azadi, The Dziok–Srivastava operator and k-uniformly starlike functions, J. Ineq. Pure Appl. Math. 6 (2) (2005) 1–7. Art. 52.
[18] O.S. Kwon, N.E. Cho, Inclusion properties for certain subclasses of analytic functions associated with the Dziok–Srivastava operator, J. Inequal. Appl. 2007 (2007) 1–10. Art. ID 51079.