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ORIGINAL ARTICLE

Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem

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Abstract In this paper, we introduce an iterative method to approximate a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem for a nonexpansive mapping in real Hilbert spaces. We prove that the sequences generated by the iterative scheme converge strongly to a common solution of the split equilibrium problem, the variational inequality problem and the fixed point problem for a nonexpansive mapping. The results presented in this paper extend and generalize many previously known results in this research area.

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1. Introduction

Throughout the paper unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $\{x_n\}$ be a sequence in H_1, then $x_n \to x$ (respectively, $x_n \rightharpoonup x$) denotes strong (respectively, weak) convergence of the sequence $\{x_n\}$ to a point $x \in H_1$.

A mapping S: $C \to C$ is called *nonexpansive*, if

$$\|Sx - Sy\| \leqslant \|x - y\|, \quad \forall x, y \in C.$$

The *fixed point problem* (in short, FPP) for the mapping S: $C \to C$ is to find $x \in C$ such that

$$Sx = x. \tag{1.1}$$

The solution set of FPP (1.1) is denoted by Fix(S).

The *variational inequality problem* (in short, VIP) is to find $x \in C$ such that

$$\langle Dx, y - x\rangle \geqslant 0, \quad \forall y \in C, \tag{1.2}$$

where D: $C \to H_1$ be a nonlinear mapping. The solution set of VIP (1.2) is denoted by Γ.

For solving the VIP in a finite-dimensional Euclidean space $\mathbb{R}^n$, Korpelevich [1] introduced an iterative method so-called extragradient method. Further motivated by the idea of Korpelevich extragradient method, Nadezhkina and Takahashi [2] introduced an iterative method for finding the common element of the set $\mathrm{Fix}(S) \cap \Gamma$ and proved the strong convergence theorem. For related works, we refer to see [3,4].

The *equilibrium problem* (in short, EP) is to find $x \in C$ such that

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$$F(x,y) \geqslant 0, \quad \forall y \in C, \tag{1.3}$$

which has been introduced studied by Blum and Oettli [5]. The solution set of EP (1.3) is denoted by EP(F).

Recently, Combettes and Hirstoaga [6] introduced and studied an iterative method for finding the best approximation to the initial data when EP(F) $\neq \emptyset$ and proved a strong convergence theorem. Subsequently, Takahashi and Takahashi [12] introduced another iterative scheme for finding the common element of the set EP(F) $\cap$ Fix(S). Using the idea of Takahashi and Takahashi [7], Plubtieng and Punpaeng [8] introduced the general iterative method for finding the common element of the set EP(F) $\cap$ Fix(S) $\cap$ Γ. Recently Liu et al. [4] introduced and studied an iterative method, an extention of the viscosity approximation method, for finding the common element of the set $\bigcap_{i=1}^{\infty} \text{Fix}(S_i) \cap \text{EP}(F) \cap \Gamma$. For further related works, we refer to see [3,9–11].

Recently, Censor and Segal [12] introduced and studied the following split common fixed point problem which is a generalization of split feasibility problem and convex feasibility problem:

Let A be a real $m \times n$ matrix and let $U : \mathbb{R}^n \to \mathbb{R}^n$ and $T : \mathbb{R}^m \to \mathbb{R}^m$ be operators with nonempty $\text{Fix}\,U = C$ and $\text{Fix}\,T = Q$. The problem is to:

$$\text{find} \quad x^* \in C \quad \text{such that} \quad Ax^* \in Q.$$

Later Moudafi [13] studied the split common fixed point problem in Hilbert spaces.

Recently, Censor et al. [14] introduced and studied some iterative methods for the following *split variational inequality problem* (in short, SVIP): Find $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geqslant 0, \quad \forall x \in C, \tag{1.4}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves } \langle g(y^*), y - y^* \rangle \geqslant 0, \quad \forall y \in Q, \tag{1.5}$$

where f: $H_1 \to H_1$ and g: $H_2 \to H_2$ are nonlinear mappings and A: $H_1 \to H_2$ is a bounded linear operator.

The special cases of SVIP (1.4) and (1.5) is split zero problem and split feasibility problem which has already been studied and used in practice as a model in intensity-modulated radiation therapy treatment planning, see [15,16].

Very recently, Moudafi [17] introduced an iterative method, an extension of a method given by Censor et al. [14] for the following split monotone variational inclusions:

$$\text{Find } x^* \in H_1 \text{ such that } f(x^*) + B_1(x^*) \ni 0$$
$$\text{and such that } y^* = Ax^* \in H_2 \text{ solves } g(y^*) + B_2(y^*) \ni 0,$$

where $B_i : H_i \to 2^{H_i}$ is a set-valued mapping for $i = 1, 2$. Later on Byrne et al. [18] generalize and extend the work of Censor et al. [14] and Moudafi [17].

In this paper we consider the following split equilibrium problem (in short, SEP) [17]:

Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and A: $H_1 \to H_2$ be a bounded linear operator, then the *split equilibrium problem* (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geqslant 0, \quad \forall x \in C, \tag{1.6}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) \geqslant 0, \quad \forall y \in Q. \tag{1.7}$$

When looked separately, (1.6) is the classical equilibrium problem EP and we denoted its solution set by $EP(F_1)$. The SEP(1.6) and (1.7) constitutes a pair of equilibrium problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator A, of the solution x^* of the EP (1.6) in H_1 is the solution of another EP (1.7) in another space H_2, we denote the solution set of EP (1.7) by $EP(F_2)$.

The solution set of SEP (1.6) and (1.7) is denoted by $\Omega = \{p \in \text{EP}(F_1) : Ap \in \text{EP}(F_2)\}$.

Motivated by the work of Censor et al. [12,14], Moudafi [17], Byrne et al. [18], Plubtieng et al. [8], Liu et al. [4] and by the ongoing research in this direction, we suggest and analyze an iterative method for approximating a common solution of SEP(1.6) and (1.7), VIP (1.2)–FPP(1.1) for a nonexpansive mapping in real Hilbert spaces. Furthermore, we prove that the sequences generated by the iterative scheme converge strongly to a common solution of SEP(1.6) and (1.7), VIP(1.2) and FPP(1.1). The results presented in this paper extend and generalize many previously known results in this research area, for instance, see [4].

2. Preliminaries

We recall some concepts and results which are needed in sequel.

Definition 2.1. Let D: $C \to H_1$ be a nonlinear mapping. Then D is called:

(*i*) *monotone*, if

$$\langle Dx - Dy, x - y \rangle \geqslant 0, \quad \forall x, y \in C;$$

(*ii*) *α-strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Dx - Dy, x - y \rangle \geqslant \alpha \|x - y\|^2, \quad \forall x, y \in C;$$

(*iii*) *β-inverse strongly monotone*, if there exists a constant $\beta > 0$ such that

$$\langle Dx - Dy, x - y \rangle \geqslant \beta \|Dx - Dy\|^2, \quad \forall x, y \in C;$$

(*iv*) *k-Lipschitz continuous*, if there exists a constant $k > 0$ such that

$$\|Dx - Dy\| \leqslant k \|x - y\|, \quad \forall x, y \in C.$$

It is easy to observe that every α-inverse strongly monotone mapping D is monotone and Lipschitz continuous.

A mapping P_C is said to be *metric projection* of H_1 onto C if for every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leqslant \|x - y\|, \quad \forall y \in C.$$

It is well known that P_C is nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geqslant \|P_C x - P_C y\|^2, \quad \forall x, y \in H_1. \tag{2.1}$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x \rangle \leqslant 0, \tag{2.2}$$

and

$$\|x - y\|^2 \geqslant \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H_1, y \in C. \tag{2.3}$$

It is well known that every nonexpansive operator T: $H_1 \to H_1$ satisfies, for all $(x,y) \in H_1 \times H_1$, the inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x) \rangle \leqslant (1/2)\|(T(x) - x) - (T(y) - y)\|^2 \tag{2.4}$$

and therefore, we get, for all $(x,y) \in H_1 \times \mathrm{Fix}(T)$,

$$\langle x - T(x), y - T(x) \rangle \leqslant (1/2)\|T(x) - x\|^2, \tag{2.5}$$

see e.g., [19], Theorem 3 and [20], Theorem 1.

It is also known that H_1 satisfies Opial's condition [21], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.6}$$

holds for every $y \in H_1$ with $y \neq x$.

Further, It is easy to see that the following is true:

$$x \in \Gamma \Longleftrightarrow x = P_C(x - \lambda Dx), \quad \lambda > 0. \tag{2.7}$$

A set valued mapping $B : H_1 \to 2^{H_1}$ is called *monotone* if for all $x, y \in H_1$, $u \in Bx$ and $v \in By$ imply $\langle x - y, u - v \rangle \geqslant 0$. A monotone mapping $B : H_1 \to 2^{H_1}$ is *maximal* if the graph $G(B)$ of B is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping B is maximal if and only if for $(x,u) \in H_1 \times H_1$, $\langle x - y, u - v \rangle \geqslant 0$, for every $(y,v) \in G(B)$ implies $u \in Bx$. Let D: $C \to H_1$ be an inverse-strongly monotone mapping and let $N_C x$ be the normal cone to C at $x \in C$, i.e., $N_C x := \{z \in H_1 : \langle y - x, z \rangle \geqslant 0, \forall y \in C\}$. Define

$$Bx = \begin{cases} Dx + N_C x, & \forall x \in C, \\ \emptyset, & \forall x \notin C. \end{cases}$$

Then B is maximal monotone and $0 \in Bx$ if and only if $x \in \Gamma$, see [2].

Assumption 2.1 (5). Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

(i) $F(x,x) = 0$, $\forall x \in C$;
(ii) F is monotone, i.e., $F(x,y) + F(y,x) \leqslant 0$, $\forall x \in C$;
(iii) For each $x, y, z \in C$, $\limsup_{t\to 0} F(tz + (1-t)x, y) \leqslant F(x,y)$;
(iv) For each $x \in C$, $y \to F(x,y)$ is convex and lower semicontinuous.
(v) Fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H_1 and $x \in C \cap K$ such that

$$F(y,x) + \frac{1}{r}\langle y - x, x - z \rangle < 0, \quad \forall y \in C \setminus K.$$

Lemma 2.1 (6). *Assume that $F_1 : C \times C \to \mathbb{R}$ satisfying Assumption 2.1. For $r > 0$ and for all $x \in H_1$, define a mapping $J_r^{F_1} : H_1 \to C$ as follows:*

$$J_r^{F_1} x = \left\{ z \in C : F_1(z,y) + \frac{1}{r}\langle y - z, z - x \rangle \geqslant 0, \quad \forall y \in C \right\}.$$

Then the following hold:

(i) *$J_r^{F_1}$ is nonempty and single-valued;*
(ii) *$J_r^{F_1}$ is firmly nonexpansive, i.e.,*

$$\left\|J_r^{F_1} x - J_r^{F_1} y\right\|^2 \leqslant \langle J_r^{F_1} x - J_r^{F_1} y, x - y \rangle, \quad \forall x, y \in H_1;$$

(iii) *$Fix\left(J_r^{F_1}\right) = EP(F_1)$;*
(iv) *$EP(F_1)$ is closed and convex.*

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ satisfying Assumption 2.1. For $s > 0$ and for all $w \in H_2$, define a mapping $J_s^{F_2} : H_2 \to Q$ as follows:

$$J_s^{F_2}(w) = \left\{ d \in Q : F_2(d,e) + \frac{1}{s}\langle e - d, d - w \rangle \geqslant 0, \quad \forall e \in Q \right\}.$$

Then, we easily observe that $J_s^{F_2}$ is nonempty, single-valued and firmly nonexpansive, $EP(F_2, Q)$ is closed and convex and $\mathrm{Fix}\left(J_s^{F_2}\right) = EP(F_2, Q)$, where $EP(F_2, Q)$ is the solution set of the following equilibrium problem:

Find $y^* \in Q$ such that $F_2(y^*, y) \geqslant 0$, $\forall y \in Q$.

We observe that $EP(F_2) \subset EP(F_2, Q)$. Further, it is easy to prove that Γ is closed and convex set.

Lemma 2.2 22. *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1 hold and let $J_r^{F_1}$ be defined as in Lemma 2.1 for $r > 0$. Let $x, y \in H_1$* and *$r_1, r_2 > 0$. Then:*

$$\left\|J_{r_2}^{F_1} y - J_{r_1}^{F_1} x\right\| \leqslant \|y - x\| + \left|\frac{r_2 - r_1}{r_2}\right| \left\|J_{r_2}^{F_1} y - y\right\|.$$

Lemma 2.3 23. *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n$, for all integers $n \geqslant 0$ and $\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leqslant 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.4 24. *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space, then for all $x, y \in X$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 2.5 25. *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leqslant (1 - \alpha_n) a_n + \delta_n, \quad n \geqslant 0,$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leqslant 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main result

In this section, we prove a strong convergence theorem based on the proposed iterative method for computing the common approximate solution of SEP(1.6)–(1.7), VIP(1.2) and FPP(1.1) for a nonexpansive mapping in real Hilbert spaces.

We assume that $\Omega \neq \emptyset$.

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let A: $H_1 \to H_2$ be a bounded linear operator. Let D: $C \to H_1$ be a τ-inverse strongly monotone mapping. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption 2.1 and F_2 is upper semicontinuous in first argument. Let S: $C \to C$ be a nonexpansive mapping such that $\Theta := Fix(S) \cap \Omega \cap \Gamma \neq \emptyset$. For a given $x_0 = v \in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{aligned} u_n &= J_{r_n}^{F_1}\left(x_n + \gamma A^*\left(J_{r_n}^{F_2} - I\right)Ax_n\right);\\ y_n &= P_C(u_n - \lambda_n Du_n);\\ x_{n+1} &= \alpha_n v + \beta_n x_n + \gamma_n S y_n, \end{aligned} \tag{3.1}$$

where $r_n \subset (0,\infty)$, $\lambda_n \in (0,2\tau)$ and $\gamma \in (0,1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0,1)$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty}\beta_n \leqslant \limsup_{n\to\infty}\beta_n < 1$;
(iv) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < +\infty$;
(v) $\lim_{n\to\infty}\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) = 0$;
(vi) $0 < \liminf_{n\to\infty}\lambda_n \leqslant \limsup_{n\to\infty}\lambda_n < 2\alpha$ and $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Theta$, where $z = P_\Theta v$.

Proof. For any $x, y \in C$, we have

$$\begin{aligned} \|(I-\lambda_n D)x-(I-\lambda_n D)y\|^2 &= \|(x-y)-\lambda_n(Dx-Dy)\|^2\\ &\leqslant \|x-y\|^2 - 2\lambda_n\langle x-y, Dx-Dy\rangle\\ &\quad + \lambda_n^2\|Dx-Dy\|^2\\ &\leqslant \|x-y\|^2 - \lambda_n(2\tau-\lambda_n)\|Dx-Dy\|^2\\ &\leqslant \|x-y\|^2. \end{aligned} \tag{3.2}$$

This shows that the mapping $(I - \lambda_n D)$ is nonexpansive.

Let $p \in \Theta := \mathrm{Fix}(S) \cap \Omega \cap \Gamma$, i.e., $p \in \Omega$, we have $p = J_{r_n}^{F_1}p$ and $Ap = J_{r_n}^{F_2}Ap$.

We estimate

$$\begin{aligned} \|u_n - p\|^2 &= \left\|J_{r_n}^{F_1}\left(x_n + \gamma A^*\left(J_{r_n}^{F_2} - I\right)Ax_n\right) - p\right\|^2\\ &= \left\|J_{r_n}^{F_1}\left(x_n + \gamma A^*\left(J_{r_n}^{F_2} - I\right)Ax_n\right) - J_{r_n}^{F_1}p\right\|^2\\ &\leqslant \left\|x_n + \gamma A^*\left(J_{r_n}^{F_2} - I\right)Ax_n - p\right\|^2\\ &\leqslant \|x_n - p\|^2 + \gamma^2\left\|A^*\left(J_{r_n}^{F_2} - I\right)Ax_n\right\|^2\\ &\quad + 2\gamma\left\langle x_n - p, A^*\left(J_{r_n}^{F_2} - I\right)Ax_n\right\rangle. \end{aligned} \tag{3.3}$$

Thus, we have

$$\begin{aligned} \|u_n - p\|^2 &\leqslant \|x_n - p\|^2\\ &\quad + \gamma^2\left\langle\left(J_{r_n}^{F_2} - I\right)Ax_n, AA^*\left(J_{r_n}^{F_2} - I\right)Ax_n\right\rangle\\ &\quad + 2\gamma\left\langle x_n - p, A^*\left(J_{r_n}^{F_2} - I\right)Ax_n\right\rangle. \end{aligned} \tag{3.4}$$

Now, we have

$$\begin{aligned} &\gamma^2\left\langle\left(J_{r_n}^{F_2} - I\right)Ax_n, AA^*\left(J_{r_n}^{F_2} - I\right)Ax_n\right\rangle\\ &\quad \leqslant L\gamma^2\left\langle\left(J_{r_n}^{F_2} - I\right)Ax_n, \left(J_{r_n}^{F_2} - I\right)Ax_n\right\rangle\\ &\quad = L\gamma^2\left\|\left(J_{r_n}^{F_2} - I\right)Ax_n\right\|^2. \end{aligned} \tag{3.5}$$

Denoting $\Lambda = 2\gamma\left\langle x_n - p, A^*\left(J_{r_n}^{F_2} - I\right)Ax_n\right\rangle$ and using (2.5), we have

$$\begin{aligned} \Lambda &= 2\gamma\left\langle x_n - p, A^*\left(J_{r_n}^{F_2} - I\right)Ax_n\right\rangle\\ &= 2\gamma\left\langle A(x_n - p), \left(J_{r_n}^{F_2} - I\right)Ax_n\right\rangle\\ &= 2\gamma\left\langle A(x_n - p) + \left(J_{r_n}^{F_2} - I\right)Ax_n - \left(J_{r_n}^{F_2} - I\right)Ax_n, \left(J_{r_n}^{F_2} - I\right)Ax_n\right\rangle\\ &= 2\gamma\left\{\left\langle J_{r_n}^{F_2}Ax_n - Ap, \left(J_{r_n}^{F_2} - I\right)Ax_n\right\rangle - \left\|\left(J_{r_n}^{F_2} - I\right)Ax_n\right\|^2\right\}\\ &\leqslant 2\gamma\left\{\frac{1}{2}\left\|\left(J_{r_n}^{F_2} - I\right)Ax_n\right\|^2 - \left\|\left(J_{r_n}^{F_2} - I\right)Ax_n\right\|^2\right\}\\ &\leqslant -\gamma\left\|\left(J_{r_n}^{F_2} - I\right)Ax_n\right\|^2. \end{aligned} \tag{3.6}$$

Using (3.4), (3.5) and (3.6), we obtain

$$\|u_n - p\|^2 \leqslant \|x_n - p\|^2 + \gamma(L\gamma - 1)\left\|\left(J_{r_n}^{F_2} - I\right)Ax_n\right\|^2. \tag{3.7}$$

From the definition of γ, we obtain

$$\|u_n - p\|^2 \leqslant \|x_n - p\|^2. \tag{3.8}$$

Now, we estimate

$$\begin{aligned} \|y_n - p\|^2 &= \|P_C(u_n - \lambda_n Du_n) - P_C(p - \lambda_n Dp)\|^2\\ &\leqslant \|(u_n - \lambda_n Du_n) - (p - \lambda_n Dp)\|^2\\ &\leqslant \|u_n - p\|^2 - \lambda_n(2\tau - \lambda_n)\|Du_n - Dp\|^2\\ &\leqslant \|u_n - p\|^2\\ &\leqslant \|x_n - p\|^2. \end{aligned} \tag{3.9}$$

Further, we estimate

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n v + \beta_n x_n + \gamma_n S y_n - p\|\\ &\leqslant \alpha_n\|v - p\| + \beta_n\|x_n - p\| + \gamma_n\|Sy_n - p\|\\ &\leqslant \alpha_n\|v - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\|\\ &\leqslant \alpha_n\|v - p\| + \beta_n\|x_n - p\| + \gamma_n\|x_n - p\|\\ &\leqslant \alpha_n\|v - p\| + (1 - \alpha_n)\|x_n - p\|\\ &\leqslant \max\{\|v - p\|, \|x_0 - p\|\} = \|v - p\|. \end{aligned} \tag{3.10}$$

Hence $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}$, $\{y_n\}$ and $\{Sy_n\}$ are bounded. On the other hand, from the nonexpansivity of the mapping $(I - \lambda_n D)$, we have

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|P_C(u_{n+1} - \lambda_{n+1}Du_{n+1}) - P_C(u_n - \lambda_n Du_n)\|\\ &\leqslant \|(u_{n+1} - \lambda_{n+1}Du_{n+1}) - (u_n - \lambda_n Du_n)\|\\ &= \|(u_{n+1} - u_n) - \lambda_{n+1}(Du_{n+1} - Du_n)\\ &\quad + (\lambda_{n+1} - \lambda_n)Du_n\|\\ &\leqslant \|(u_{n+1} - u_n) - \lambda_{n+1}(Du_{n+1} - Du_n)\|\\ &\quad + |\lambda_{n+1} - \lambda_n|\|Du_n\|\\ &\leqslant \|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n|\|Du_n\|. \end{aligned} \tag{3.11}$$

Since $u_n = J_{r_n}^{F_1}\left(x_n + \gamma A^*\left(J_{r_n}^{F_2} - I\right)Ax_n\right)$ and $u_{n+1} = J_{r_{n+1}}^{F_1}\left(x_{n+1} + \gamma A^*\left(J_{r_{n+1}}^{F_2} - I\right)Ax_{n+1}\right)$. It follows from Lemma 2.2 that

$$\begin{aligned}\|u_{n+1}-u_n\| &\leqslant \Big\|x_{n+1}-x_n+\gamma\Big[A^*\Big(J^{F_2}_{r_{n+1}}-I\Big)Ax_{n+1}\\&\quad -A^*\big(J^{F_2}_{r_n}-I\big)Ax_n\Big]\Big\|\\&\quad+\Big|1-\frac{r_n}{r_{n+1}}\Big|\Big\|J^{F_1}_{r_{n+1}}\Big(x_{n+1}+\gamma A^*\Big(J^{F_2}_{r_{n+1}}-I\Big)Ax_{n+1}\Big)\\&\quad-\Big(x_{n+1}+\gamma A^*\Big(J^{F_2}_{r_{n+1}}-I\Big)Ax_{n+1}\Big)\Big\|\\&\leqslant \|x_{n+1}-x_n+\gamma A^*A(x_{n+1}-x_n)\|\\&\quad+\gamma\|A\|\Big\|J^{F_2}_{r_{n+1}}Ax_{n+1}-J^{F_2}_{r_n}Ax_n\Big\|+\delta_n,\\&\leqslant\Big\{\|x_{n+1}-x_n\|^2-2\gamma\|Ax_{n+1}\\&\quad-Ax_n\|^2+\gamma^2\|A\|^4\|x_{n+1}-x_n\|^2\Big\}^{\frac12}\\&\quad+\gamma\|A\|\Big\{\|Ax_{n+1}-Ax_n\|+\Big|1-\frac{r_n}{r_{n+1}}\Big|\|J^{F_2}_{r_{n+1}}Ax_{n+1}-Ax_{n+1}\|\Big\}+\delta_n\\&\leqslant(1-2\gamma\|A\|^2+\gamma^2\|A\|^4)^{\frac12}\|x_{n+1}-x_n\|+\gamma\|A\|^2\|x_{n+1}-x_n\|\\&\quad+\gamma\|A\|\sigma_n+\delta_n\\&=(1-\gamma\|A\|^2)\|x_{n+1}-x_n\|+\gamma\|A\|^2\|x_{n+1}-x_n\|+\gamma\|A\|\sigma_n+\delta_n\\&=\|x_{n+1}-x_n\|+\gamma\|A\|\sigma_n+\delta_n,\end{aligned}\tag{3.12}$$

where

$$\sigma_n=\Big|1-\frac{r_n}{r_{n+1}}\Big|\Big\|J^{F_2}_{r_{n+1}}Ax_{n+1}-Ax_{n+1}\Big\|$$

and

$$\delta_n=\Big|1-\frac{r_n}{r_{n+1}}\Big|\Big\|J^{F_1}_{r_{n+1}}\Big(x_{n+1}+\gamma A^*\Big(J^{F_2}_{r_{n+1}}-I\Big)Ax_{n+1}\Big)-\Big(x_{n+1}+\gamma A^*\Big(J^{F_2}_{r_{n+1}}-I\Big)Ax_{n+1}\Big)\Big\|.$$

Using (3.11) and (3.12), we obtain

$$\|y_{n+1}-y_n\|\leqslant\|x_{n+1}-x_n\|+\gamma\|A\|\sigma_n+\delta_n+|\lambda_{n+1}-\lambda_n|\|Du_n\|.\tag{3.13}$$

Setting $x_{n+1}=\beta_nx_n+(1-\beta_n)e_n$, which implies from (3.1) that

$$e_n=\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}=\frac{\alpha_nv+\gamma_nSy_n}{1-\beta_n}.$$

Further, it follows that

$$\begin{aligned}e_{n+1}-e_n&=\frac{\alpha_{n+1}v+\gamma_{n+1}Sy_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_nv+\gamma_nSy_n}{1-\beta_n}\\&=\Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\Big)v+\frac{\gamma_{n+1}(Sy_{n+1}-Sy_n)}{1-\beta_{n+1}}\\&\quad+\Big(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\Big)Sy_n.\end{aligned}$$

Using (3.13), we have

$$\begin{aligned}\|e_{n+1}-e_n\|&\leqslant\Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\Big|\|v\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|y_{n+1}-y_n\|\\&\quad+\Big|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\Big|\|Sy_n\|\\&\leqslant\Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\Big|\|v\|\\&\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}[\|x_{n+1}-x_n\|+\gamma\|A\|\sigma_n+\delta_n\\&\quad+|\lambda_{n+1}-\lambda_n|\|Du_n\|]+\Big|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\Big|\|Sy_n\|\\&\leqslant\Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\Big|\|v\|+(1-\alpha_{n+1})[\|x_{n+1}-x_n\|\\&\quad+\gamma\|A\|\sigma_n+\delta_n\\&\quad+|\lambda_{n+1}-\lambda_n|\|Du_n\|]+\Big|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\Big|\|Sy_n\|\\&\leqslant\Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\Big|\|v\|+\|x_{n+1}-x_n\|+\gamma\|A\|\sigma_n+\delta_n\\&\quad+|\lambda_{n+1}-\lambda_n|\|Du_n\|+\Big|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\Big|\|Sy_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|e_{n+1}-e_n\|\leqslant&\Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\Big|\|v\|+\|x_{n+1}-x_n\|\\&+\gamma\|A\|\sigma_n+\delta_n+|\lambda_{n+1}-\lambda_n|\|Du_n\|\\&+\Big|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\Big|\|Sy_n\|,\end{aligned}$$

which implies that

$$\begin{aligned}\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\|\leqslant&\Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\Big|\|v\|+\gamma\|A\|\sigma_n+\delta_n\\&+|\lambda_{n+1}-\lambda_n|\|Du_n\|\\&+\Big|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\Big|\|Sy_n\|.\end{aligned}$$

Hence it follows by conditions (ii)–(vi) that

$$\limsup_{n\to\infty}[\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\|]\leqslant0.\tag{3.14}$$

From Lemma 2.3, we get $\lim_{n\to\infty}\|e_n-x_n\|=0$ and

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|e_n-x_n\|=0.\tag{3.15}$$

Now,

$$\begin{aligned}x_{n+1}-x_n&=\alpha_nv+\beta_nx_n+\gamma_nSy_n-x_n\\&=\alpha_n(v-x_n)+\gamma_n(Sy_n-x_n).\end{aligned}$$

Since $\|x_{n+1}-x_n\|\to0$ and $\alpha_n\to0$ as $n\to\infty$, we obtain $\|Sy_n-x_n\|\to0$ as $n\to\infty$.

It follows from (3.7) and Lemma 2.4 that

$$\begin{aligned}\|x_{n+1}-p\|^2&\leqslant\alpha_n\|v-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|Sy_n-p\|^2\\&\leqslant\alpha_n\|v-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2\\&\leqslant\alpha_n\|v-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|u_n-p\|^2\\&\leqslant\alpha_n\|v-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\Big[\|x_n-p\|^2\\&\quad+\gamma(L\gamma-1)\big\|\big(J^{F_2}_{r_n}-I\big)Ax_n\big\|^2\Big]\\&\leqslant\alpha_n\|v-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\&\quad+\gamma(L\gamma-1)\big\|\big(J^{F_2}_{r_n}-I\big)Ax_n\big\|^2\\&\leqslant\alpha_n\|v-p\|^2+\|x_n-p\|^2+\gamma(L\gamma-1)\big\|\big(J^{F_2}_{r_n}-I\big)Ax_n\big\|^2.\end{aligned}\tag{3.16}$$

Therefore,

$$\begin{aligned}\gamma(1-L\gamma)\big\|\big(J^{F_2}_{r_n}-I\big)Ax_n\big\|^2&\leqslant\alpha_n\|v-p\|^2\\&\quad+\big(\|x_n-p\|^2-\|x_{n+1}-p\|^2\big)\\&\leqslant\alpha_n\|v-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|.\end{aligned}$$

Since $\gamma(1-L\gamma)>0$, $\alpha_n \to 0$, and $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty}\left\|\left(J_{r_n}^{F_2}-I\right)Ax_n\right\|=0. \tag{3.17}$$

Next, we show that $\|x_n - u_n\| \to 0$ as $n \to \infty$. Since $p \in \Theta$, we obtain

$$\begin{aligned}\|u_n-p\|^2 &= \left\|J_{r_n}^{F_1}\left(x_n+\gamma A^*\left(J_{r_n}^{F_2}-I\right)Ax_n\right)-p\right\|^2\\ &= \left\|J_{r_n}^{F_1}\left(x_n+\gamma A^*\left(J_{r_n}^{F_2}-I\right)Ax_n\right)-J_{r_n}^{F_1}p\right\|^2\\ &\leqslant \left\langle u_n-p, x_n+\gamma A^*\left(J_{r_n}^{F_2}-I\right)Ax_n-p\right\rangle\\ &= \frac{1}{2}\Big\{\|u_n-p\|^2+\|x_n+\gamma A^*\left(J_{r_n}^{F_2}-I\right)Ax_n-p\|^2\\ &\quad -\left\|(u_n-p)-\left[x_n+\gamma A^*\left(J_{r_n}^{F_2}-I\right)Ax_n-p\right]\right\|^2\Big\}\\ &= \frac{1}{2}\Big\{\|u_n-p\|^2+\|x_n-p\|^2\\ &\quad -\left\|u_n-x_n-\gamma A^*\left(J_{r_n}^{F_2}-I\right)Ax_n\right\|^2\Big\}\\ &= \frac{1}{2}\Big\{\|u_n-p\|^2+\|x_n-p\|^2-\Big[\|u_n-x_n\|^2\\ &\quad +\gamma^2\|A^*\left(J_{r_n}^{F_2}-I\right)Ax_n\|^2-2\gamma\left\langle u_n-x_n, A^*\left(J_{r_n}^{F_2}-I\right)Ax_n\right\rangle\Big]\Big\}.\end{aligned}$$

Hence, we obtain

$$\|u_n-p\|^2 \leqslant \|x_n-p\|^2-\|u_n-x_n\|^2+2\gamma\|A(u_n - x_n)\|\left\|\left(J_{r_n}^{F_2}-I\right)Ax_n\right\|. \tag{3.18}$$

It follows from (3.16) and (3.17) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leqslant \alpha_n\|v-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|u_n-p\|^2\\ &\leqslant \alpha_n\|v-p\|^2+\beta_n\|x_n-p\|^2\\ &\quad +\gamma_n\Big[\|x_n-p\|^2-\|u_n-x_n\|^2\\ &\quad +2\gamma\|A(u_n-x_n)\|\left\|\left(J_{r_n}^{F_2}-I\right)Ax_n\right\|\Big]\\ &\leqslant \alpha_n\|v-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\gamma_n\|u_n-x_n\|^2\\ &\quad +2\gamma_n\gamma\|A(u_n-x_n)\|\left\|\left(J_{r_n}^{F_2}-I\right)Ax_n\right\|\\ &\leqslant \alpha_n\|v-p\|^2+\|x_n-p\|^2-\gamma_n\|u_n-x_n\|^2\\ &\quad +2\gamma\|A(u_n-x_n)\|\left\|\left(J_{r_n}^{F_2}-I\right)Ax_n\right\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma_n\|u_n-x_n\|^2 &\leqslant \alpha_n\|v-p\|^2+\left(\|x_n-p\|^2-\|x_{n+1}-p\|^2\right)\\ &\quad +2\gamma\|A(u_n-x_n)\|\left\|\left(J_{r_n}^{F_2}-I\right)Ax_n\right\|\\ &\leqslant \alpha_n\|v-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\ &\quad +2\gamma\|A(u_n-x_n)\|\left\|\left(J_{r_n}^{F_2}-I\right)Ax_n\right\|.\end{aligned}$$

Since $\alpha_n \to 0$, $\left\|\left(J_{r_n}^{F_2}-I\right)Ax_n\right\| \to 0$ and $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.19}$$

Next, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leqslant \alpha_n\|v-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|Sy_n-p\|^2\\ &\leqslant \alpha_n\|v-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2\\ &\leqslant \alpha_n\|v-p\|^2+\beta_n\|x_n-p\|^2\\ &\quad +\gamma_n\{\|P_C(u_n-\lambda_nDu_n)-P_C(p-\lambda_nDp)\|^2\}\\ &\leqslant \alpha_n\|v-p\|^2+\beta_n\|x_n-p\|^2\\ &\quad +\gamma_n\{\|u_n-p\|^2+\lambda_n(\lambda_n-2\tau)\|Du_n-Dp\|^2\}\\ &\leqslant \alpha_n\|v-p\|^2+\beta_n\|x_n-p\|^2\\ &\quad +\gamma_n\{\|x_n-p\|^2+\lambda_n(\lambda_n-2\tau)\|Du_n-Dp\|^2\}\\ &\leqslant \alpha_n\|v-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\ &\quad +\gamma_n\{\lambda_n(\lambda_n-2\tau)\|Du_n-Dp\|^2\}\\ &\leqslant \alpha_n\|v-p\|^2+\|x_n-p\|^2\\ &\quad +\gamma_n\lambda_n(\lambda_n-2\tau)\|Du_n-Dp\|^2,\end{aligned}$$

which yields

$$\begin{aligned}\gamma_n\lambda_n(\lambda_n-2\tau)\|Du_n-Dp\|^2 &\leqslant \alpha_n\|v-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\leqslant \alpha_n\|v-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\\ &\quad \|x_n-x_{n+1}\|.\end{aligned}$$

Since $\|x_{n+1}-x_n\| \to 0$, $\alpha_n \to 0$ as $n \to \infty$, we obtain $\lim_{n\to\infty}\|Du_n - Dp\| = 0$.

Furthermore, we observe that

$$\begin{aligned}\|y_n-p\|^2 &= \|P_C(u_n-\lambda_nDu_n)-P_C(p-\lambda_nDp)\|^2\\ &\leqslant \langle y_n-p,(u_n-\lambda_nDu_n)-(p-\lambda_nDp)\rangle\\ &\leqslant \frac{1}{2}\{\|y_n-p\|^2+\|(u_n-\lambda_nDu_n)-(p-\lambda_nDp)\|^2\\ &\quad -\|(y_n-u_n)+\lambda_n(Du_n-Dp)\|^2\}\\ &\leqslant \frac{1}{2}\{\|y_n-p\|^2+\|u_n-p\|^2-\|y_n-u_n+\lambda_n(Du_n-Dp)\|^2\}.\end{aligned}$$

Hence,

$$\begin{aligned}\|y_n-p\|^2 &\leqslant \|u_n-p\|^2-\|y_n-u_n\|^2-\lambda_n^2\|Du_n-Dp\|^2\\ &\quad +2\lambda_n\langle y_n-u_n, Du_n-Dp\rangle\\ &\leqslant \|u_n-p\|^2-\|y_n-u_n\|^2+2\lambda_n\|y_n-u_n\|\|Du_n-Dp\|\\ &\leqslant \|x_n-p\|^2-\|y_n-u_n\|^2+2\lambda_n\|y_n-u_n\|\|Du_n-Dp\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leqslant \alpha_n\|v-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|Sy_n-p\|^2\\ &\leqslant \alpha_n\|v-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2\\ &\leqslant \alpha_n\|v-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n[\|x_n-p\|^2-\|y_n-u_n\|^2\\ &\quad +2\lambda_n\|y_n-u_n\|\|Du_n-Dp\|]\\ &\leqslant \alpha_n\|v-p\|^2+(1-\alpha_n)\|x_n-p\|^2-\gamma_n\|y_n-u_n\|^2\\ &\quad +2\gamma_n\lambda_n\|y_n-u_n\|\|Du_n-Dp\|]\\ &\leqslant \alpha_n\|v-p\|^2+\|x_n-p\|^2-\gamma_n\|y_n-u_n\|^2\\ &\quad +2\gamma_n\lambda_n\|y_n-u_n\|\|Du_n-Dp\|.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\gamma_n\|y_n-u_n\|^2 &\leqslant \alpha_n\|v-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad +2\gamma_n\lambda_n\|y_n-u_n\|\|Du_n-Dp\|\\ &\leqslant \alpha_n\|v-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\ &\quad +2\gamma_n\lambda_n\|y_n-u_n\|\|Du_n-Dp\|.\end{aligned}$$

Since $\|x_{n+1} - x_n\| \to 0$, $\alpha_n \to 0$ as $n \to \infty$ and $\lim_{n\to\infty}\|Du_n - Dp\| = 0$, we obtain

$$\lim_{n\to\infty}\|y_n - u_n\| = 0. \tag{3.20}$$

Since, we can write

$$\|Sy_n - y_n\| \leqslant \|Sy_n - x_n\| + \|x_n - u_n\| + \|u_n - y_n\| \to 0 \text{ as } n \to \infty.$$

Next, we show that $\limsup_{n\to\infty}\langle v - z, x_n - z\rangle \leqslant 0$, where $z = P_{\mathrm{Fix}(S)\cap\Omega\cap\Gamma}v$. To show this inequality, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle v - z, Sy_n - z\rangle = \lim_{i\to\infty}\langle v - z, Sy_{n_i} - z\rangle.$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to some $w \in C$. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup w$. Further, from $\|Sy_n - y_n\| \to 0$, we obtain $Sy_{n_i} \rightharpoonup w$ as $i \to \infty$.

Now, we prove that $w \in \mathrm{Fix}(S) \cap \Omega \cap \Gamma$. Let us first show that $w \in \mathrm{Fix}(S)$. Assume that $w \notin \mathrm{Fix}(S)$. Since $y_{n_i} \rightharpoonup w$ and $Sw \neq w$. Form Opial's condition (2.6), we have

$$\begin{aligned}\liminf_{i\to\infty}\|y_{n_i} - w\| &< \liminf_{i\to\infty}\|y_{n_i} - Sw\| \\ &\leqslant \liminf_{i\to\infty}\{\|y_{n_i} - Sy_{n_i}\| + \|Sy_{n_i} - Sw\|\} \\ &\leqslant \liminf_{i\to\infty}\|y_{n_i} - w\|,\end{aligned}$$

which is a contradiction. Thus, we obtain $w \in \mathrm{Fix}(S)$.

Next, we show that $w \in EP(F_1)$. Since $u_n = J_{r_n}^{F_1}x_n$, we have

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0, \quad \forall y \in C.$$

It follows from monotonicity of F_1 that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant F_1(y, u_n)$$

and hence

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_n}\right\rangle \geqslant F_1(y, u_{n_i}).$$

Since $\|u_n - x_n\| \to 0$, $\|Sy_n - x_n\| \to 0$ and $\|Sy_n - y_n\| \to 0$, we get $u_{n_i} \rightharpoonup w$ and $\frac{u_{n_i} - x_{n_i}}{r_n} \to 0$. It follows by Assumption 2.1(*iv*) that $0 \geqslant F_1(y, w)$, $\forall w \in C$. For t with $0 < t \leqslant 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$, $w \in C$, we get $y_t \in C$ and hence $F_1(y_t, w) \leqslant 0$. So from Assumption 2.1(*i*) and (*iv*) we have

$$0 = F_1(y_t, y_t) \leqslant tF_1(y_t, y) + (1-t)F_1(y_t, w) \leqslant tF_1(y_t, y).$$

Therefore $0 \leqslant F_1(y_t, y)$. From Assumption 2.1(*iii*), we have $0 \leqslant F_1(w, y)$. This implies that $w \in EP(F_1)$.

Next, we show that $Aw \in EP(F_2)$. Since $\|u_n - x_n\| \to 0$, $u_n \rightharpoonup w$ as $n \to \infty$ and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w$ and since A is a bounded linear operator so that $Ax_{n_k} \rightharpoonup Aw$.

Now setting $v_{n_k} = Ax_{n_k} - J_{r_{n_k}}^{F_2}Ax_{n_k}$. It follows that from (3.17) that $\lim_{k\to\infty}v_{n_k} = 0$ and $Ax_{n_k} - v_{n_k} = J_{r_{n_k}}^{F_2}Ax_{n_k}$.

Therefore from Lemma 2.1, we have

$$F_2(Ax_{n_k} - v_{n_k}, z) + \frac{1}{r_{n_k}}\langle z - (Ax_{n_k} - v_{n_k}), (Ax_{n_k} - v_{n_k}) - Ax_{n_k}\rangle \geqslant 0, \quad \forall z \in Q.$$

Since F_2 is upper semicontinuous in first argument, taking lim sup to above inequality as $k \to \infty$ and using condition (*iv*), we obtain

$$F_2(Aw, z) \geqslant 0, \quad \forall z \in Q,$$

which means that $Aw \in EP(F_2)$ and hence $w \in \Omega$.

Finally, by using the arguments as in the proof of Theorem 3.1 [2], we can show that $w \in \Gamma$.

Next, we claim that $\limsup_{n\to\infty}\langle v - z, x_n - z\rangle \leqslant 0$, where $z = P_\Theta v$. Now from (2.2), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle v - z, x_n - z\rangle &= \limsup_{n\to\infty}\langle v - z, Sy_n - z\rangle \\ &= \limsup_{i\to\infty}\langle v - z, Sy_{n_i} - z\rangle \\ &= \langle v - z, w - z\rangle \\ &\leqslant 0.\end{aligned} \tag{3.21}$$

Finally, we show that $x_n \to z$.

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \langle \alpha_n v + \beta_n x_n + \gamma_n Sy_n - z, x_{n+1} - z\rangle \\ &= \alpha_n\langle v - z, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle \\ &\quad + \gamma_n\langle Sy_n - z, x_{n+1} - z\rangle \\ &\leqslant \frac{\beta_n}{2}\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\} \\ &\quad + \frac{\gamma_n}{2}\{\|Sy_n - z\|^2 + \|x_{n+1} - z\|^2\} \\ &\quad + \alpha_n\langle v - z, x_{n+1} - z\rangle \\ &\leqslant \frac{\beta_n}{2}\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\} \\ &\quad + \frac{\gamma_n}{2}\{\|y_n - z\|^2 + \|x_{n+1} - z\|^2\} \\ &\quad + \alpha_n\langle v - z, x_{n+1} - z\rangle \\ &\leqslant \frac{\beta_n}{2}\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\} \\ &\quad + \frac{\gamma_n}{2}\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\} \\ &\quad + \alpha_n\langle v - z, x_{n+1} - z\rangle \\ &\leqslant \frac{(1-\alpha_n)}{2}\left\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right\} \\ &\quad + \alpha_n\langle v - z, x_{n+1} - z\rangle \\ &\leqslant \frac{1}{2}\left\{(1-\alpha_n)\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right\} \\ &\quad + \alpha_n\langle v - z, x_{n+1} - z\rangle.\end{aligned}$$

This implies that

$$\|x_{n+1} - z\|^2 \leqslant (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle v - z, x_{n+1} - z\rangle.$$

Finally, by using (3.21) and Lemma 2.5, we deduce that $x_n \to z$. This completes the proof.

We have following consequence which is a strong convergence theorem for computing the common approximate solution of EP(1.3), VIP(1.2) and FPP(1.1) for a nonexpansive mapping in real Hilbert space. □

Corollary 3.1. *Let H_1 be a real Hilbert space and $C \subseteq H_1$ be nonempty closed convex subset of Hilbert space H_1. Let D: $C \to H_1$ be a τ-inverse strongly monotone mapping. Assume that $F_1 : C \times C \to \mathbb{R}$ is a bifunction satisfying Assumption* 2.1. *Let S: $C \to C$ be a nonexpansive mapping such that $\Theta := \mathrm{Fix}(S) \cap EP(F_1) \cap \Gamma \neq \emptyset$. For a given $x_0 = v \in C$ arbitrarily, let the iterative sequences $\{u_n\}$, $\{x_n\}$ and $\{y_n\}$ be generated by*

$$u_n = J_{r_n}^{F_1} x_n;$$
$$y_n = P_C(u_n - \lambda_n D u_n);$$
$$x_{n+1} = \alpha_n v + \beta_n x_n + \gamma_n S y_n,$$

where $r_n \subset (0, \infty)$, $\lambda_n \in (0, 2\tau)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0,1)$ satisfying the conditions (i)–(vi) of Theorem 3.1. *Then the sequence $\{x_n\}$ converges strongly to $z \in Fix(S) \cap EP(F_1) \cap \Gamma$, where $z = P_{\mathrm{Fix}(S) \cap EP(F_1) \cap \Gamma} v$.*

Remark 3.1.

1. The algorithm considered in Theorem 3.1 is different from those considered in [12–14,17,18] in the sense that variable sequence $\{r_n\}$ has been taken in place of fixed r. Further the approach presented in this paper is also different.
2. The use of iterative method presented in this paper for the split monotone variational inclusions considered in Moudafi [17] and Byrne et al. [18] needs further research effort.

References

[1] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, Matecon 12 (1976) 747–756.

[2] N. Nadezhkina, W. Takahashi, Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings, SIAM J. Optim. 16 (2006) 1230–1241.

[3] W. Kumam, C. Jaiboon, A new hybrid iterative method for mixed equilibrium problems and variational inequality problems for relaxed cocoercive mappings with application to optimization problems, Nonlinear Anal. Hybrid Syst. 3 (2009) 510–530.

[4] Q. Liu, W.Y. Zeng, N.J. Huang, An iterative method for generalized equilibrium problem and fixed point problems and variational inequality problems, Fixed Point Theory and Applications, 2009, 20pp. Article ID 531308.

[5] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994) 123–145.

[6] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117–136.

[7] S. Takahashi, W. Takahashi, Viscosity approximation method for equilibrium problems and fixed point problems in Hilbert space, J. Math. Anal. Appl. 331 (2007) 506–515.

[8] S. Plubtieng, R. Punpaeng, A general iterative method for equilibrium problems and fixed point problems in Hilbert space, J. Math. Anal. Appl. 336 (2007) 455–469.

[9] K.R. Kazmi, F.A. Khan, Existence and iterative approximation of solutions of generalized mixed equilibrium problems, Comput. Math. Appl. 56 (5) (2008) 1314–1321.

[10] P. Kumam, N. Petrot, R. Wangkeeree, A general system of variational inequality problems and mixed equilibrium problems, Appl. Math. Mech. 29 (5) (2008) 571–581.

[11] Y. Yao, Y.C. Liou, J.C. Yao, Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings, Fixed Point Theory and Applications, 2007, 12pp. Article ID 64363.

[12] Y. Censor, A. Segal, The split common fixed point problem for directed operators, J. Convex Anal. 16 (2009) 587–600.

[13] A. Moudafi, The split common fixed point problem for demicontractive mappings, Inverse Problems 26 (2010) 055007, 6pp.

[14] Y. Censor, A. Gibali, S. Reich, Algorithms for the split variational inequality problem, Numerical Algorithms, in press. http://dx.doi.org/10.1007/s11075-011-9490-5. See also <http://arxiv.org/abs/1009.3780>.

[15] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problems in intensity modulated radiation therapy, Phys. Med. Biol. 51 (2006) 2353–2365.

[16] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in product space, Numer. Algorithms 8 (1994) 221–239.

[17] A. Moudafi, Split monotone variational inclusions, J. Optim. Theory Appl. 150 (2011) 275–283.

[18] C. Byrne, Y. Censor, A. Gibali, S. Reich, Weak and strong convergence of algorithms for the split common null point problem. <http://arxiv.org/abs/1108.5953>.

[19] G. Crombez, A hierarchical presentation of operators with fixed points on Hilbert spaces, Numer. Funct. Anal. Optim. 27 (2006) 259–277.

[20] G. Crombez, A geometrical look at iterative methods for operators with fixed points, Numer. Funct. Anal. Optim. 26 (2005) 157–175.

[21] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Am. Math. Soc. 73 (4) (1967) 595–597.

[22] F. Cianciaruso, G. Marino, L. Muglia, Y. Yao, A hybrid projection algorithm for finding solutions of mixed equilibrium problem and variational inequality problem, Fixed Point Theory Applications, 2010, 19pp. Article ID 383740.

[23] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one parameter nonexpansive semigroups without Bochner integrals, J. Math. Anal. Appl. 305 (2005) 227–239.

[24] M.O. Osilike, D.I. Igbokwe, Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations, Comput. Math. Appl. 40 (2000) 559–567.

[25] H.K. Xu, Viscosity approximation method for nonexpansive mappings, J. Math. Anal. Appl. 298 (2004) 279–291.