Journal of the Egyptian Mathematical Society (2013) 21, 57–62

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

On weaker forms for concepts in theory of topological groupoids

Amin H. Saif, Adem Kılıçman *

Department of Mathematics, University Putra Malaysia, 43400 UPM, Serdang, Selangor, Malaysia

Received 27 April 2012; revised 19 July 2012; accepted 1 September 2012
Available online 2 November 2012

KEYWORDS

Topological groupoid;
α-Open;
α-Continuous;
α-Irresolute

Abstract In this paper, we investigate the topologically weak concepts of topological groupoids by giving the concepts of α-topological groupoid and α-topological subgroupoid. Furthermore, we show the role of the density condition to allow α-topological subgroupoid inherited properties from α-topological groupoid and the irresoluteness property for the structure maps in α-topological groupoid is studied. We also give some results about the fibers of α-topological groupoids.

AMS CLASSIFICATION: 22A22, 54C55

1. Introduction

Groupoids have been first introduced by Brandt in 1926 as algebraic structures generalizing groups, by allowing the group product to be partially defined. Groupoid can be usefully seen as the 'categorification' of equivalence relations: indeed, since every morphism is an iso, any two objects joined by at least one arrow are equivalent in "as many ways" as there are arrows between them. Moreover, equivalence relations can often be meaningfully represented as the orbit equivalence relations of some nontrivial groupoids over their domains. Topological groupoid was invented by Ehresmann, [1], around 1958–1970 in the field of differential geometry. The structure formulated was groupoid equipped with topology compatible with the map, domain, codomain, composite and inversion. These properties later translated in categorical language as a groupoid with all the structure maps, i.e. source, target, object, partial multiplication and inversion, are all continuous maps. For more details about topological groupoid see [2–4].

In topological spaces, Njastad, [5], 1965 introduced the notion of α-sets in topological space. In 1983, Mashhour et al. [6] introduced, with the help of α-sets, a weak form of continuity which they termed as α-continuity. Since then it has been widely investigated in the literature (see [7–10]). In 1980, Maheshwari and Thakur, [11] introduced the irresoluteness of α-functions in topological spaces. Recently, continuity and irresoluteness of functions in topological spaces have been researched by many mathematicians and quantum physicists (see [12–16]).

This paper is organized as follows: It consists of four sections. Section 2 is devoted to some preliminaries. In Section 3, we investigate some properties of α-continuous maps given to structure maps of a groupoid. In Section 4, we start by giving the notions of α-topological groupoid and α-topological subgroupoid. Next, we show the role of the density condition to allow α-topological subgroupoid inherited properties from α-topological groupoid and study the irresoluteness property

* Corresponding author.
E-mail address: akilicman@putra.upm.edu.my (A. Kılıçman).

Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

1110-256X © 2012 Egyptian Mathematical Society. Production and hosting by Elsevier B.V. Open access under CC BY-NC-ND license.
http://dx.doi.org/10.1016/j.joems.2012.09.001

for the structure maps in α-topological groupoid. Finally, we show the relation between a set of objects and a set of identities in α-topological groupoid. In Section 5, we give some results about the fibers of α-topological groupoids.

2. Preliminaries

Throughout this paper, by X we mean a topological space (X, τ), and $cl(A)$ will denote the closure of set a set A and $int(A)$ the interior of set A. Recall [5] that a subset A of topological space X is called *α-open* set if there exists open subset U of X such that $U \subset A \subset int(cl(U))$. The complement of α-open set is called *α-closed* set. Every open set is α-open but the converse need not be true; the intersection of two α-open sets is α-open and the arbitrary union of α-open sets is α-open set, that is, the collection of all α-open sets in X forms a topology on X. The set M is α-open in X if and only if there exists α-open set A in X such that $A \subset M \subset int(cl(A))$; if A is α-open set in X and $A \subset Y \subset X$, then A is α-open in Y; if $A \subset Y$ and Y is α-open subset of X, then A is α-open in Y if and only if A is α-open in X; for more details see [8–10,17].

Recall [6] that a map $f{:}X \to Y$ is called *α-continuous* if $f^{-1}(U)$ is α-open set in X for any open sets U in Y. Every continuous map is α-continuous but the converse not necessarily true; A map $f{:}X \to Y$ is α-continuous if and only if $f^{-1}(F)$ is α-closed set in X for any closed sets F in Y; the cartesian product of two α-continuous maps is α-continuous. A map $f{:}X \to Y$ is called *α-open* (resp. *α-closed*) map if the direct image of open set (resp. closed) in X is α-open (resp. α-closed) set in Y.

Definition 2.1. [18] A *groupoidG* is a small category consisting of two sets G and O_G, called respectively the set of elements (or arrows) and the set of objects (or vertices) of the groupoid, together with two maps μ, $\beta{:}G \to O_G$, called respectively the *source* and*target* maps of groupoid, the map $\varepsilon{:}O_G \to G$ which is defined by $\varepsilon(x) = 1_x$, where 1_x is called the identity element at x in O_G and ε is called the *object* map, and the *partial multiplication* map $\gamma{:}(G\times G)_{\mu=\beta} \to G$which is defined by $\gamma(g,h) = gh$, where $(G\times G)_{\mu=\beta} = \{(g,h) \in G\times G{:}\mu(g) = \beta(h)\}$. Theses terms must satisfy the following axioms:

G1.$\mu(gh) = \mu(h)$ and $\beta(gh) = \beta(g)$,
G2. $(gh)k = g(hk)$,
G3. $\mu(1_x) = \beta(1_x) = x$ for all $x \in O_G$,
G4. $g1_{\mu(g)} = g$ and $1_{\beta(g)}g = g$,
G5. $g^{-1}g = 1_{\mu(g)}$ and $gg^{-1} = 1_{\beta(g)}$
for all g, h, $k \in G$.

Remark 2.2. For a groupoid G:

1. σ: $G \to G$ is the *inversion* map of G defined by $\sigma(g) = g^{-1}$ which is a bijective,
2. δ: $(G\times G)_\mu \to G$ is the *difference* map of G defined by $\delta(h,g) = gh^{-1}$, where

$$(G\times G)_\mu = \{(g,h) \in G\times G : \mu(h) = \mu(g)\},$$

3. π: $G \to O_G \times O_G$ is a map, defined by $\pi(g) = (\mu(g), \beta(g))$.

For a groupoid G and $x, y \in O_G$, we denote the star of G at x by $st_G x$ of the fiber $\mu^{-1}(x) = \{g \in G{:}\mu(g) = x\}$, the co-star of G at y by $cost_G y$ of the fiber $\beta^{-1}(y) = \{g \in G{:}\ \beta(g) = y\}$ and $G(x,y) = st_G x \cap cost_G y$.

Definition 2.3. [3] Let G be a groupoid. A *subgroupoid* of G is a pair of subset $N \subset G$ and $O_N \subset O_G$ such that $\mu(N) \subset O_N$, $\beta(N) \subset O_N$, $1_x \in N$ for all $x \in O_N$, and N is closed under partial multiplication and inversion maps in G.

A subgroupoid N of G is *wide* if $O_G = O_N$.

Definition 2.4. [4] A topological groupoid G is a groupoid G together with topologies on G and O_G such that the structure maps of G are continuous, that is; the source map μ, the target map β, the object map ε, the inversion map σ, and the partial multiplication map γ are continuous.

3. Structure maps of a groupoid

In this section, we investigate some properties of α-continuous maps given to structure maps of a groupoid.

Theorem 3.1. *Let G be a groupoid in which G and O_G have topologies. If the inversion map σ is continuous, then the source map μ is α-continuous if and only if the target map β is α-continuous.*

Proof. Suppose μ is α-continuous and U be open subset of O_G. Then $\mu^{-1}(U)$ is α-open in G which implies that there exists open subset B of G such that $B \subset \mu^{-1}(U) \subset int(cl(B))$. Since σ is a continuous, we get from its definition that it is a homeomorphism. Hence

$$\sigma^{-1}(B) \subset \sigma^{-1}(\mu^{-1}(U)) \subset \sigma^{-1}[int(cl(B))] = int[cl(\sigma^{-1}(B))].$$

Then $\sigma^{-1}(\mu^{-1}(U))$ is α-open subset of G. Since G is groupoid, then $\beta = \mu \circ \sigma$. Hence $\beta^{-1}(U) = \sigma^{-1}(\mu^{-1}(U))$. So, $\beta^{-1}(U)$ is α-open subset of G. That is, β is α-continuous.

Conversely, suppose β is α-continuous and V open subset of O_G. Since σ is homeomorphism and $\mu = \beta \circ \sigma$, then $\mu^{-1}(V) = \sigma^{-1}(\beta^{-1}(V))$ is α-open subset of G. Therefore μ is α-continuous. □

Theorem 3.2. *Let G be a groupoid in which G and O_G have topologies. If the inversion map σ is continuous, then the partial multiplication map γ is α-continuous if and only if the difference map δ is α-continuous.*

Proof. Since the identity map I: Similar for$G \to G$ and the inversion map σ are homeomorphisms, then the map $I\times\sigma{:}G\times G \to G\times G$, defined by $(I\times\sigma)(g,h) = (g,h^{-1})$, is also homeomorphism. If $(g,h) \in (G\times G)_\mu$ and since G is groupoid, then $\mu(g) = \mu(h) = \beta(h^{-1})$, that is $(g,h^{-1}) \in (G\times G)_{\mu=\beta}$. Hence the restriction map $r{:}(G\times G)_\mu \to (G\times G)_{\mu=\beta}$ of $I\times\sigma$ on $(G\times G)_\mu$ is homeomorphism. Now suppose the partial multiplication map γ is α-continuous and let U be open subset of G. Then $\gamma^{-1}(U)$ is α-open subset of $(G\times G)_{\mu=\beta}$. This implies that there exists open subset B of $(G\times G)_{\mu=\beta}$ such that $B \subset \gamma^{-1}(U) \subset int(cl(B))$. Hence

$$r^{-1}(B) \subset r^{-1}(\gamma^{-1}(U)) \subset r^{-1}[int(cl(B))] = int[cl(r^{-1}(B))].$$

That is, $r^{-1}(\gamma^{-1}(U))$ is α-open subset of $(G\times G)_\mu$. Since G is a groupoid, then $\delta = \gamma \circ r$. Hence $\delta^{-1}(U) = r^{-1}(\gamma^{-1}(U))$ is α-open subset of $(G\times G)_\mu$, that is, the difference map δ is α-continuous.

Conversely, suppose the difference map δ is α-continuous and let V be open subset of G. If $(g,h) \in (G\times G)_{\mu=\beta}$ and since G is groupoid, then $\mu(g) = \beta(h) = \mu(h^{-1})$, that is $(g,h^{-1}) \in (G\times G)_{\mu}$. Hence the restriction map $k{:}(G\times G)_{\mu=\beta} \to (G\times G)_{\mu}$ of $I\times\sigma$ on $(G\times G)_{\mu=\beta}$ is homeomorphism. Since V is open subset of G, then $\delta^{-1}(V)$ is α-open subset of $(G\times G)_{\mu}$. Hence $k^{-1}(\delta^{-1}(V))$ is α-open subset of $(G\times G)_{\mu=\beta}$. Since G is groupoid, then $\gamma = \delta\circ k$. Hence $\gamma^{-1}(V) = k^{-1}(\delta^{-1}(V))$ is α-open subset of $(G\times G)_{\mu=\beta}$, that is, the partial multiplication map γ is α-continuous. □

In the following theorem, we show the relation between the partial multiplication map and the map $D{:}(G\times G)_{\beta} \to G$ which is defined by $D(g,h) = g^{-1}h$.

Theorem 3.3. *Let G be a groupoid in which G and O_G have topologies. If the inversion map σ is continuous, then the partial multiplication map γ is α-continuous if and only if the map D is α-continuous.*

Proof. Similar in the proof of theorem above that $I\times\sigma{:}G\times G \to G\times G$ is homeomorphism. If $(g,h) \in (G\times G)_{\beta}$ and since G is groupoid, then $\beta(g) = \beta(h) = \mu(h^{-1})$, that is $(g,h^{-1}) \in (G\times G)_{\mu=\beta}$. Hence the restriction map $r{:}(G\times G)_{\beta} \to (G\times G)_{\mu=\beta}$ of $I\times\sigma$ on $(G\times G)_{\beta}$ is homeomorphism. Now suppose the partial multiplication map γ is α-continuous and let U be open subset of G. Then it is clear that $r^{-1}(\gamma^{-1}(U)) = D^{-1}(U)$ is α-open subset of $(G\times G)_{\beta}$, that is, the map D is α-continuous.

Conversely, suppose the map D is α-continuous and let V be open subset of G. If $(g,h) \in (G\times G)_{\mu=\beta}$ and since G is groupoid, then $\beta(g^{-1}) = \mu(g) = \beta(h)$, that is $(g^{-1},h) \in (G\times G)_{\beta}$. Hence the restriction map $m{:}(G\times G)_{\mu=\beta} \to (G\times G)_{\beta}$ of $I\times\sigma$ on $(G\times G)_{\mu=\beta}$ is homeomorphism. Since V is open subset of G and G is a groupoid, then it easy to see that $\gamma^{-1}(V) = m^{-1}(D^{-1}(V))$ is α-open subset of $(G\times G)_{\mu=\beta}$, that is, the partial multiplication map γ is α-continuous. □

Corollary 3.4. *Let G be a groupoid in which G and O_G have topologies. If the inversion map σ is continuous, then the difference map δ is α-continuous if and only if the map D is α-continuous.*

Proof. From Theorems 3.2 and 3.3. □

4. α-Topological groupoids

In this section, we start by giving the notions of α-topological groupoid and α-topological subgroupoid. Next, we show the role of the density condition to allow α-topological subgroupoid inherited properties from α-topological groupoid and study the irresoluteness property for the structure maps in α-topological groupoid.

Definition 4.1. A α-topological groupoid G is a groupoid G together with topologies on G and O_G such that the inversion map σ is continuous and the remainder structure maps of G are α-continuous. That is; the source map μ, the target map β, the object map ε, and the partial multiplication map γ are α-continuous.

Note that every topological groupoid is α-topological groupoid but the converse need not be true since any α-continuous need not be continuous, [7].

Definition 4.2. Let G be α-topological groupoid. The α-topological subgroupoid of G is a subgroupoid B together with subspace topologies on B and O_B such that the restriction map $\sigma'{:}B \to B$ of the inversion map σ is continuous and the remainder structure maps of B are α-continuous, that is; the restriction maps μ', $\beta'{:}B \to O_B$, $\varepsilon'{:}O_B \to B$, and $\gamma' : (B\times B)_{\mu'=\beta'} \to B$ of the source (target) map μ, β, the object map ε, and the partial multiplication map γ, respectively, are α-continuous.

Recall [19] that U is open subset of X if and only if $cl[U\cap cl(A)] = cl[U\cap A]$, for every $A \subset X$. Moreover, $D\subset X$ is dense in X if $cl(D) = X$. This will be used in the following lemmas to describe hot to restrict α-continuous map.

Lemma 4.3. *If D is a dense in X, then $cl(A) = cl(A\cap D)$ for any α-open set A in X.*

Proof. Since A is α-open set in X, then there exits open subset U of X such that $U\subset A\subset int(cl(U))\subset cl(U)$. Hence since D is dense in X, we get

$$cl(A) = cl(U) = cl(U\cap X) = cl[U\cap cl(D)] = cl(U\cap D) \subset cl(A\cap D).$$

That is, $cl(A) = cl(A\cap D)$. □

Lemma 4.4. *Let $f{:}X \to Y$ be α-continuous map. If D is dense subset of a space X, then the restriction $f|_D{:}D \to Y$ is α-continuous.*

Proof. Let U be open subset of Y. Then $f^{-1}(U)$ is α-open subset of X, that is, there exists open subset O of X such that $O\subset f^{-1}(U)\subset int(cl(O))$. This implies

$$O\cap D\subset f^{-1}(U)\cap D\subset int(cl(O))\cap D.$$

Since D is dense in X, we get

$$int_D(cl_D(O\cap D)) = int(cl(O\cap D))\cap D = int[cl(O\cap cl(D))]\cap D = int(cl(O\cap X))\cap D = int(cl(O))\cap D.$$

That is, $f^{-1}(U)\cap D$ is α-open in D. Hence $f|D$ is α-continuous. □

Theorems 4.5 and 4.6 deal with the density condition which allows α-topological subgroupoid inherited properties from α-topological groupoid.

Theorem 4.5. *Let G be α-topological groupoid and B is a wide subgroupoid of G. If $(B\times B)_{\mu=\beta}$ is dense in $G\times G$, then B is α-topological subgroupoid.*

Proof. Since $(B\times B)_{\mu=\beta}$ is dense in $G\times G$, then we have

$$cl_{(G\times G)_{\mu=\beta}}[(B\times B)_{\mu=\beta}] = cl[(B\times B)_{\mu=\beta}]\cap(G\times G)_{\mu=\beta} = (G\times G)_{\mu=\beta}.$$

This implies that $(B \times B)_{\mu=\beta}$ is dense in $(G \times G)_{\mu=\beta}$. Since $(B \times B)_{\mu=\beta} \subset B \times B$, then $B \times B$ is dense in $G \times G$ and B is dense in G. Then by Lemma 4.4, the restriction maps $\gamma' : (B \times B)_{\mu'=\beta'} \to B$ and $\mu', \beta' : B \to O_B$ are α-continuous. Further, the restriction $\sigma' : B \to B$ is continuous. Finally, since $O_G = O_B$, then the restriction $\varepsilon' : O_B \to B$ is α-continuous. Hence B is α-topological subgroupoid. □

Theorem 4.6. *Let G be α-topological groupoid and B is a wide and dense subgroupoid of G. If $(G \times G)_{\mu=\beta}$ is α-open in $G \times G$, then B is α-topological subgroupoid.*

Proof. Since B is dense in G, then $B \times B$ is dense in $G \times G$. Moreover, since $(G \times G)_{\mu=\beta}$ is α-open subset of $G \times G$, then by Lemma 4.3,

$$\begin{aligned} cl_{(G\times G)_{\mu=\beta}}[(B \times B)_{\mu=\beta}] &= cl[(B \times B)_{\mu=\beta}] \cap (G \times G)_{\mu=\beta} \\ &= cl[(G \times G)_{\mu=\beta} \cap (B \times B)] \cap (G \times G)_{\mu=\beta} \\ &= cl[(G \times G)_{\mu=\beta}] \cap (G \times G)_{\mu=\beta} = (G \times G)_{\mu=\beta}. \end{aligned}$$

That is, $(B \times B)_{\mu=\beta}$ is dense in $(G \times G)_{\mu=\beta}$. Hence B is α-topological subgroupoid. □

Theorem 4.7. *Let G be α-topological groupoid. If B is open subgroupoid of G, then B is α-topological subgroupoid.*

Proof. Since B is open in G, then the restriction maps μ', $\beta' : B \to O_B$ of μ, β are α-continuous and $(B \times B)_{\mu=\beta} = (B \times B) \cap (G \times G)_{\mu=\beta}$ is open subset of $(G \times G)_{\mu=\beta}$. Hence the restriction $\gamma' : (B \times B)_{\mu'=\beta'} \to B$ of γ is α-continuous. Finally, we will show that the restriction $\varepsilon' : O_B \to B$ is α-continuous. Let U be open subset of B which implies that U is also open set in G. Since B is subgroupoid of G, then for $x \in \varepsilon^{-1}(U)$,

$$1_x \in U \Rightarrow \mu(1_x) \in \mu(U) \subset \mu(B) \subset O_B \Rightarrow x \in O_B.$$

Hence $(\varepsilon')^{-1}(U) = \varepsilon^{-1}(U) \cap O_B = \varepsilon^{-1}(U)$, that is $(\varepsilon')^{-1}(U)$ is α-open set in O_G. Hence B is α-topological subgroupoid. □

Here we show the relation between a set of objects and a set of identities in α-topological groupoid. This relation was studied in a topological subgroupoid as the set of objects is homeomorphic to the set of identities under the object map. However many authors such as Aof and Brown [18,4] identified the set of objects with the set of identities in groupoid. If we take this definition, then the set of objects becomes a subspace in the set of arrows G, and the object map will become an inclusion, which implies that the object map is continuous. Moreover, the set of identities is a wide subgroupoid of G. Actually, there is a relation between the set of objects and the set of identities as follows: $1_x = 1_y$ if and only if $\mu(1_x) = \mu(1_y)$ if and only if $\beta(1_x) = \beta(1_y)$ if and only if $x = y$, under object, source and target maps. Therefore, the set of identities is a wide subgroupoid. The following result shows that this situation in α-topological groupoid case.

Theorem 4.8. *Let G be α-topological groupoid. Then the object map ε is a bijective and α-continuous onto $A = \{1_x : x \in O_G\}$.*

Proof. Since G is α-topological groupoid, then the object map $\varepsilon : O_G \to G$ is α-continuous. Define the restriction map $r : O_G \to A$ by $r(x) = 1_x$, for all $x \in O_G$. Then for $x, y \in O_G$,

$$r(x) = r(y) \Rightarrow 1_x = 1_y \Rightarrow \mu(1_x) = \mu(1_y) \Rightarrow x = y,$$

that is, r is injective. For $1_x \in A$, there exists $x \in O_G$ such that $r(x) = 1_x$, that is, r is surjective. Hence r is bijective. Now suppose U is open set of A. Then $U = V \cap A$, where V is open set of G, but $r^{-1}(U) = \varepsilon^{-1}(V) \cap O_G = \varepsilon^{-1}(V)$. Therefore r is α-continuous. □

From Theorem above, if the object $\varepsilon : O_G \to G$ is α-open map, then the restriction r is α-open map. This follows from the fact that for any open subset U of O_G, then $\varepsilon(U)$ is α-open subset of G but since $\varepsilon(U) \subset A$ and $r(U) = \varepsilon(U) \cap A = \varepsilon(U)$, this implies that $r(U)$ is α-open set in G. Then, there exists an open subset O of G such that $O \subset r(U) \subset int(cl(O))$. This implies that $O \cap A \subset r(U) \cap A \subset int(cl(O)) \cap A$. Since $O \subset r(U) \subset A$, then $O \subset r(U) \subset int_A[cl_A(O)]$, that is, $r(U)$ is α-open subset of A. Therefore, r is α-open map.

Analogous to the pervious explanation, the following theorem deals with a new case of a set of identities in α-topological groupoids.

Theorem 4.9. *Let G be α-topological groupoid. If the set of identities A is open subgroupoid of G, then*

1. A is a α-topological subgroupoid;
2. the object $\varepsilon : O_G \to G$ is α-open map;
3. the bijective map $r : O_G \to A$, $r(x) = \varepsilon(x) = 1_x$ for all $x \in O_G$, is α-open map.

Proof.

1. Since A is open subgroupoid, then by Theorem 4.7, A is α-topological subgroupoid.
2. Let U be open subset of O_G. Since the source map μ is α-continuous, the $\mu^{-1}(U)$ is α-open subset of G. Since A is open subset of G, then $\mu^{-1}(U) \cap A$ is α-open subset of G and α-open subset of A. Now we prove that $\varepsilon(U) = \mu^{-1}(U) \cap A$. Suppose that $y \in \mu^{-1}(U) \cap A$. Since $y \in A$, then there exists $x \in O_G$, such that $y = 1_x$. Hence $\mu(1_x) = x \in \mu[\mu^{-1}(U)]$. That is,

$$x \in U \Rightarrow \varepsilon(x) = 1_x \in \varepsilon(U) \Rightarrow y \in \varepsilon(U).$$

   Conversely, suppose that $h \in \varepsilon(U)$. Since $\varepsilon(U) \subset A$, then there exists $z \in U$ such that $\varepsilon(z) = 1_z = h$, which implies that $z = \mu(1_z) = \mu(h)$. Then $h \in \mu^{-1}(z) \subset \mu^{-1}(U)$. Hence ε is α-open map.
3. Since $\varepsilon(U) \subset A$, then $r(U) = \varepsilon(U)$. But ε is α-open map, this implies that r is also α-open map. □

Theorem 4.10. *Let G be α-topological groupoid. Then the source map μ is α-closed (resp. α-open) map if and only if the target map β is α-closed (resp. α-open) map.*

Proof. Let the source μ be α-closed map and M be a closed subset of G. Then the set $\sigma(M)$ is also closed in G. Since the source map μ is α-closed, then $\mu(\sigma(M))$ is α-closed subset of O_G. Since G is a groupoid, then we get that $\beta(M) = \mu(\sigma(M))$. Hence β is α-closed. Conversely, For any closed subset F of G, the set $\beta(\sigma(F))$ is α-closed subset of O_G. Since $\mu(F) = \beta(\sigma(F))$, then μ is α-closed. A similar argument holds for α-open map. □

We illustrate further the relation between the difference map and the multiplication map in α-topological groupoid.

Theorem 4.11. *Let G be α-topological groupoid. Then the multiplication map is α-closed (resp. α-open) map if and only if the difference map is α-closed (resp. α-open) map.*

Proof. Suppose the multiplication map $\gamma:(G\times G)_{\mu=\beta}\to G$is α-open map. Since G is α-topological groupoid, then the restriction map $r:(G\times G)_{\mu}\to(G\times G)_{\mu=\beta}$ of $I\times\sigma$, which is defined by $r(g,h)=(g,h^{-1})$, is a homeomorphism. Then for α-open subset U of $(G\times G)_{\mu}$, the set $\gamma(r(U))$ is α-open subset of G. Further, since G is α-topological groupoid, then $\delta(U)=\gamma(r(U))$. Therefore the difference map δ is α-open. Conversely, since G is α-topological groupoid, then $\gamma=\delta\circ r^{-1}$. Then the proof of this part will be similar for the proof the first part. □

Moreover, recall from Caldas and Navalagi, [17], that a topological space X is*hyperconnected* if every nonempty open subset of X is dense in X.

Theorem 4.12. *Let G be α-topological groupoid such that the set of arrows G is a hyperconnected space and B is a wide subgroupoid of G. If $(B\times B)_{\mu=\beta}$ is α-open in $G\times G$, then B is α-topological groupoid.*

Proof. Since G is a hyperconnected, it is easy to see that $G\times G$ is also hyperconnected. Further, since every nonempty α-open set contains open set and $(B\times B)_{\mu=\beta}$ is α-open subset of $G\times G$, then $(B\times B)_{\mu=\beta}$ is dense in $G\times G$. Hence B is also dense in G. And since B is wide subgroupoid then by Theorem 4.6, B is α-topological subgroupoid. □

Recall from Maheshwari and Thakur, [11] that a map $f:X\to Y$ is called anα-irresolute if the inverse image of any α-open set in Y is α-open set in X. We say that a map $f:X\to Y$ is *α-pre-open* map if $f^{-1}[int(cl(U))]\subset int[cl(f^{-1}(U))]$ for any α-open subset U of Y.

Theorem 4.13. *Let G be α-topological groupoid. If the source map μ is α-pre-open, then:*

1. The source map μ is α-irresolute.
2. The target map β is α-irresolute.
3. The composite maps $\varepsilon\circ\mu$ and $\varepsilon\circ\beta$ are α-continuous.

Proof.

1. Suppose μ is α-pre-open map and U be α-open in O_G. Then there exists open subset O of O_G such that $O\subset U\subset int(cl(O))$, which implies

$$\mu^{-1}(O)\subset\mu^{-1}(U)\subset\mu^{-1}[int(cl(O))].$$

Since μ is α-pre-open, then $\mu^{-1}[int(cl(O))]\subset int[cl(\mu^{-1}(O))]$. Also, since μ is α-continuous, then $\mu^{-1}(U)$ is α-open which implies $\mu^{-1}(U)$ α-open subset of G. This implies μ is α-irresolute.

2. Since the inversion map σ is a homeomorphism, then from the part (1), $\sigma^{-1}[\mu^{-1}(U)]$ is α-open in G. Since G is α-topological groupoid, then $\beta^{-1}(U)=\sigma^{-1}[\mu^{-1}(U)]$ which implies $\beta^{-1}(U)$ is α-open set in G. That is, β is α-irresolute.
3. Since β and μ are α-irresolute maps, ε is α-continuous and from the parts (1) and (2), we get that the composite maps $\varepsilon\circ\mu$ and $\varepsilon\circ\beta$ are α-continuous. □

The above theorem also holds for α-irresolute target map.

Theorem 4.14. *Let G be α-topological groupoid. If the object map ε is α-pre-open, then:*

1. The target map ε is α-irresolute.
2. The composite maps $\varepsilon\circ\mu$ and $\varepsilon\circ\beta$ are α-continuous.

Proof.

1. The proof is similar for the proof of part (1) in Theorem 4.13.
2. Let U be open subset of O_G. Since μ is α-continuous, then $\mu^{-1}(U)$ is α-open subset of G. That is, there exists open subset B of G such that $B\subset\mu^{-1}(U)\subset int(cl(B))$. This implies

$$\varepsilon^{-1}(B)\subset\varepsilon^{-1}[\mu^{-1}(U)]\subset\varepsilon^{-1}[int(cl(B))].$$

Since ε is α-pre-open, then $\varepsilon^{-1}[cl(B)]\subset cl[\varepsilon^{-1}(B)]$. Hence

$$\varepsilon^{-1}(B)\subset\varepsilon^{-1}[\mu^{-1}(U)]\subset int[cl(\varepsilon^{-1}(B))].$$

And since ε is α-continuous, then $\varepsilon^{-1}(B)$ is α-open, which implies $\varepsilon^{-1}[\mu^{-1}(U)]$ is α-open set in O_G. That is, $\varepsilon\circ\mu$ is α-continuous. Similarly, $\varepsilon\circ\beta$ will be α-continuous. □

Theorem 4.15. *Let G be α-topological groupoid. If the multiplication map γ is α-pre-open, then:*

1. *The multiplication map γ is α-irresolute.*
2. *The difference map δ is α-irresolute.*
3. *The map $k:(G\times G)_{\beta}\to G$, which is defined by $k(g,h)=g^{-1}h$, is α-irresolute.*

Proof.

1. Suppose that the multiplication map γ is α-pre-open map. Let U be α-open in G. Then there exists open subset B of G such that $B\subset U\subset int(cl(B))$, which implies $\gamma^{-1}(B)\subset\gamma^{-1}(U)\subset\gamma^{-1}[int(cl(B))]$. Since γ is α-continuous, then $\gamma^{-1}(B)$ is α-open in $(G\times G)_{\mu=\beta}$. And since γ is α-pre-open, then $\gamma^{-1}[int(cl(B))]\subset int[cl(\gamma^{-1}(B))]$. Hence $\gamma^{-1}(U)$ is α-open subset of $(G\times G)_{\mu=\beta}$. This implies γ is α-irresolute.
2. Since G is α-topological groupoid, then the restriction map $r:(G\times G)_{\mu}\to(G\times G)_{\mu=\beta}$ of $I\times\sigma$, which is defined by $r(g,h)=(g,h^{-1})$, is a homeomorphism. Now let U be α-open subset of G. Then by the part (1), $\gamma^{-1}(U)$ is α-open in $(G\times G)_{\mu=\beta}$. Then there exists open subset O of $(G\times G)_{\mu=\beta}$ such that $O\subset\gamma^{-}(U)\subset int(cl(O))$, which implies $r^{-1}(O)\subset r^{-1}[\gamma^{-}(U)]\subset r^{-1}[int(cl(O))]$. Since r is homeomorphism, then $r^{-1}(O)$ is open set in $(G\times G)_{\mu}$ and $r^{-1}[int(cl(O))]=int[cl(r^{-1}(O))]$. Hence $r^{-1}[\gamma^{-}(U)]$ is α-open in $(G\times G)_{\mu}$. Further, since G is a groupoid, then $\delta=\gamma\circ r$. Hence δ is α-irresolute.
3. Since G is α-topological groupoid, then the restriction map $m:(G\times G)_{\beta}\to(G\times G)_{\mu=\beta}$ of $\sigma\times I$, which is defined by $m(g,h)=(g^{-1},h)$, is a homeomorphism. Now let U be α-

open subset of G. Since γ is α-pre-open and G is groupoid, then $m^{-1}[\gamma^{-1}(U)] = k^{-1}(U)$ is α-open subset of $(G\times G)_\beta$. That is, k is α-irresolute. □

5. Fibers of α-topological groupoid

In this section, we give some results about the fibers of α-topological groupoids.

Theorem 5.1. *Let G be α-topological groupoid. If the singleton $\{g\}$ is open subset of G such that $g \in G(x,y)$, then*

1. *The left translation L_g:$cost_G x \to cost_G y$, which is defined by $L_g(h) = gh$, is bijective, α-continuous and α-open.*
2. *The right translation R_g:$st_G y \to st_G x$, which is defined by $R_g(t) = tg$, is bijective, α-continuous and α-open.*

Proof.

1. Since G is α-topological groupoid, then the multiplication map $\gamma:(G\times G)_{\mu=\beta} \to G$ is α-continuous. So, the restriction $R:\{g\}\times cost_G x \to cost_G y$ of γ, which is defined by $R(g,t) = gt$, is bijective and α-continuous. This follows from: Let $(g,t) \in \{g\}\times cost_G x$. Then $\mu(g) = \beta(t) = x$. And since $\beta(gt) = \beta(g) = y$, then $gt \in cost_G y$. So $R(\{g\}\times cost_G x) \subset cost_G y$. If t_1, $t_2 \in cost_G x$ and $R(g,t_1) = R(g,t_2)$, then $gt_1 = gt_2$, implies, $g^{-1}gt_1 = g^{-1}gt_2$, implies, $1_{\mu(g)}t_1 = 1_{\mu(g)} t_2$. But $\mu(g)\beta(t)$, that is, $t_1 = t_2$. Hence R is injective. Also for $b \in cost_G y$ and $g \in G(x,y)$, $g^{-1}b \in cost_G x$. So, $R(g, g^{-1}b) = (gg^{-1})b = b$. That is, R is bijective. Now let V be open subset of $cost_G y$. That is, there exists open subset U of G such that $V = U \cap cost_G y$. Then $R^{-1}(V) = \gamma^{-1}(U) \cap (\{g\}\times cost_G x)$. Since $\{g\}$ is open in G and $\{g\}\times cost_G x = (\{g\}\times G) \cap (G\times G)_{\mu=\beta}$, then $\{g\}\times cost_G x$ is open subset of $(G\times G)_{\mu=\beta}$. Since γ is α-continuous, then $\gamma^{-1}(U)$ is α-open in $(G\times G)_{\mu=\beta}$, that is, $R^{-1}(V)$ is α-open in $\{g\}\times cost_G x$. Hence R is α-continuous. Therefore, its easy to see that the map $M:cost_G x \to \{g\}\times cost_G x$, which is defined by $M(t) = (g,t)$, is a homeomorphism and

$$L_g = R \circ M : cost_G x \xrightarrow{M} \{g\} \times cost_G x \xrightarrow{R} cost_G y,$$

is a bijective and α-continuous.
To prove that L_g is α-open, it is enough to prove its inverse $L_g^{-1} : cost_G y \to cost_G x$, which is defined by $L_g^{-1}(h) = g^{-1}h$, is α-continuous. Similarly, we can write L_g^{-1} as a composite:

$$L_g^{-1} = m \circ q : cost_G y \xrightarrow{q(t)=(g^{-1},t)} \{g^{-1}\} \times cost_G y \xrightarrow{m(g^{-1},t)=g^{-1}t} cost_G x.$$

2. Similar for the Part (1). □

Remark 5.2. The results of Theorem 5.1 are hold if we replace the word 'dense' for the singleton instead of 'open'. It is also useful to remark that $\{g\}$ dense in G for $g \in G(x,y)$ implies that $G(x,y)$ is dense in G.

Theorem 5.3. *Let G be α-topological groupoid. If the singleton $\{g\}$ is dense of G such that $g \in G(x,y)$, then*

1. The restriction R_1:$G(y,x) \to G(y,y)$ of the left translation L_g, which is defined by $R_1(h) = gh$, is α-continuous.
2. The restriction R_2:$G(x,y) \to G(y,y)$ of the right translation $R_{g^{-1}}$, which is defined by $R_2(h) = hg^{-1}$, is α-continuous.
3. The restriction R_3:$G(x,y)\times G(y,x) \to G(y,y)$ of γ, which is defined by $R_3(h,g) = hg$, is α-continuous.

Proof. We observe that the map R_1, R_2 and R_3 are the restriction maps of the left translation L_g, the right translation $R_{g^{-1}}$ and the partial multiplication map γ to the sets $G(y,x)$, $G(y,x)$ and $G(x,y)\times G(y,x)$, respectively. Since $G(x,y)$ is dense in G and by Lemma 4.4 and Remark 5.2, then R_1, R_2 and R_3 are α-continuous. □

Acknowledgement

The authors would like to express their sincere thanks and gratitude to the reviewer(s) for their details comments and valuable suggestions that improved the manuscript very much.

References

[1] C. Ehresmann, Catgories topologiques et catgories diffrentiables, Colloque Gom. Diff. Globale Bruxelles (1958) 137–150.
[2] G. Danesh-Naruie, Theory of Topological Groupoids, PhD. Thesis, The University of Southampton, England, 1970.
[3] K. Anthony, On the continuity of Haar measure on topological groupoids, Proceedings of the American Mathematical Society 96 (1986) 115–120.
[4] M. Aof, R. Brown, The holonomy groupoid of locally topological groupoid, Topology and its Application 47 (1992) 97–113.
[5] O. Njastsd, On some classes of nearly open sets, Pacific Journal of Mathematics 15 (1965) 961–970.
[6] A. Mashhour, S. Hasanein, S. Deeb, α-continuity and α-open mappings, Acta Mathematica Hungarica 41 (1983) 213–218.
[7] I. Reilly, M. Vamanamurthy, On α-continuity in topological spaces, Acta Mathematica Hungarica 45 (1985) 27–32.
[8] H. Maki, R. Devi, K. Balachandran, Associated topologies on generalized α-closed sets and α-generalized closed sets, Memoirs of the Faculty of Science Kochi University Series A. Mathematics 17 (1994) 51–63.
[9] D. Jankovic, I. Reilly, On semi-separation properties, Indian Journal of Pure and Applied Mathematics 16 (1985) 957–964.
[10] A.H. Kocaman, S. Yuksel, A. Acikgoz, On some strongly functions defined by α-open, Chaos, Solitons and Fractals 39 (2009) 1346–1355.
[11] S.N. Maheshwari, S.S. Thakur, On α-irresolute mappings, Tamkang Journal of Mathematics 11 (1980) 209–214.
[12] Y. Beceren, T. Noiri, Some functions defined by α-open and preopen sets, Chaos, Solitons and Fractals 37 (2008) 1097–1103.
[13] M.S. Naschie, Quantum gravity, clifford algebras, fuzzy set theory and the fundamental constants of nature, Chaos, Solitons and Fractals 20 (2004) 437–450.
[14] T. Noiri, Mildly normal spaces and some functions, Kyungpook Mathematical Journal 36 (1996) 183–190.
[15] J.K. Park, J.H. Park, Mildly generalized closed sets, almost normal and mildly normal spaces, Chaos, Solitons and Fractals 20 (2004) 1103–1111.
[16] J.K. Park, J.H. Park, B. Lee, On locally d-generalized closed sets and LdGC-continuous functions, Chaos, Solitons and Fractals 19 (2004) 995–1002.
[17] M. Caldas, G. Navalagi, On weak forms of preopen and preclosed functions, Tomus 40 (2004) 128–199.
[18] M. Aof, Topological Aspects of Holonomy Groupoid, Ph.D thesis, University of Wales, Bagor, 1989.
[19] N. Bourbaki, General Topology, Herman, Baris, 1966.