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ORIGINAL ARTICLE

Position vectors of slant helices in Euclidean 3-space

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Abstract In this paper, position vector of a slant helix with respect to standard frame in Euclidean space $\mathbf{E}^3$ is studied in terms of Frenet equations. First, a vector differential equation of third order is constructed to determine a position vector of an arbitrary slant helix. In terms of solution, we determine the parametric representation of the slant helices from the intrinsic equations. Thereafter, we apply this method to find the parametric representation of a Salkowski curve, anti-Salkowski curve and a curve of constant precession, as examples of a slant helices, by means of intrinsic equations.

1. Introduction

In the local differential geometry, we think of curves as a geometric set of points, or locus. Intuitively, we are thinking of a curve as the path traced out by a particle moving in $\mathbf{E}^3$. So, the investigating position vectors of the curves in a classical aim to determine behavior of the particle (curve).

Helix is one of the most fascinating curves in science and nature. Scientist have long held a fascinating, sometimes bordering on mystical obsession, for helical structures in nature. Helices arise in nano-springs, carbon nano-tubes, α-helices, DNA double and collagen triple helix, lipid bilayers, bacterial flagella in salmonella and escherichia coli, aerial hyphae in actinomycetes, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells [5,16,24]. Also we can see the helix curve or helical structures in fractal geometry, for instance hyperhelices [22]. In the field of computer aided geometric design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or the design of highways, etc. [25]. From the view of differential geometry, a helix is a geometric curve with non-vanishing constant curvature κ and non-vanishing constant torsion τ [3]. The helix may be called a *circular helix* or *W-curve* [10,18].

Its known that straight line ($\kappa(s) = 0$) and circle ($\tau(s) = 0$) is degenerate-helix [12]. In fact, circular helix is the simplest three-dimensional spirals. One of the most interesting spiral example is k-Fibonacci spirals. These curves appear naturally from studying the k-Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ and the related hyperbolic k-Fibonacci function [7]. Three-dimensional k-Fibonacci spirals was studied from a geometric point of view in [8].

Indeed a helix is a special case of the *general helix*. A curve of constant slope or general helix in Euclidean 3-space $\mathbf{E}^3$ is

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defined by the property that the tangent makes a constant angle with a fixed straight line called the axis of the general helix. A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (see [21] for details) says that: *A necessary and sufficient condition that a curve be a general helix is that the ratio*

$$\frac{\kappa}{\tau}$$

is constant along the curve, where κ *and* τ *denote the curvature and the torsion, respectively*. A general helices or *inclined curves* are well known curves in classical differential geometry of space curves [1,3,17,20,23].

Izumiya and Takeuchi [11] have introduced the concept of slant helix by saying that the normal lines make a constant angle with a fixed straight line. They characterize a slant helix if and only if the geodesic curvature

$$\kappa_g = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$$

of the principal image of the principal normal indicatrix is a constant function. They also called a curve is canonical geodesic if

$$\left(\frac{\tau}{\kappa}\right)'$$

is a constant function. Kula and Yayli [13] have studied spherical images of tangent indicatrix and binormal indicatrix of a slant helix and they showed that the spherical images are spherical helices. Recently, Kula et al. [14] investigated the relations between a general helix and a slant helix. Moreover, they obtained some differential equations which they are characterizations for a space curve to be a slant helix.

Many important results in the theory of the curves in $\mathbf{E}^3$ were initiated by Monge and Darboux pioneered the moving frame idea. Thereafter, Frenet defined his moving frame and his special equations which play important role in mechanics and kinematics as well as in differential geometry [4].

In this work, we use vector differential equations established by means of Frenet equations in Euclidean space $\mathbf{E}^3$ to determine position vectors of the arbitrary curves according to standard frame in $\mathbf{E}^3$. We obtain position vector of a slant helix from intrinsic equations in $\mathbf{E}^3$. Besides, we present some new characterizations of a slant helix and some of examples are illustrated.

2. Preliminaries

In Euclidean space $\mathbf{E}^3$, it is well known that to each unit speed curve with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ which are called respectively, the tangent, the principal normal and the binormal vector fields [9].

Let $\psi : I \subset \mathbb{R} \to \mathbf{E}^3, \psi = \psi(s)$, be an arbitrary curve in $\mathbf{E}^3$. The curve ψ is said to be of unit speed (or parameterized by the arc-length) if $\langle \psi'(s), \psi'(s)\rangle = 1$ for any $s \in I$. In particular, if $\psi(s) \neq 0$ for any s, then it is possible to re-parameterize ψ, that is, $\alpha = \psi(\phi(s))$ so that α is parameterized by the arc-length. Thus, we will assume throughout this work that ψ is a unit speed curve, where $\langle , \rangle$ is Euclidean inner product.

Let $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ be the Frenet moving frame along ψ. The Frenet equations for ψ are given by [21]:

$$\begin{bmatrix}\mathbf{T}'(s)\\ \mathbf{N}'(s)\\ \mathbf{B}'(s)\end{bmatrix} = \begin{bmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}(s)\\ \mathbf{N}(s)\\ \mathbf{B}(s)\end{bmatrix}. \tag{1}$$

If $\tau(s) = 0$ for any $s \in I$, then $\mathbf{B}(s)$ is a constant vector V and the curve ψ lies in a 2-dimensional affine subspace orthogonal to V, which is isometric to the Euclidean 2-space $\mathbf{E}^2$.

3. Position vectors of a slant helices

The problem of the determination of parametric representation of the position vector of an arbitrary space curve according to the intrinsic equations is still open in the Euclidean space $\mathbf{E}^3$ [6,15]. This problem is not easy to solve in general case. However, this problem is solved in three special cases only, Firstly, in the case of a plane curve ($\tau = 0$). Secondly, in the case of a helix (κ and τ are both non-vanishing constant). Recently, Ali [2] adapted fundamental existence and uniqueness theorem for space curves in Euclidean space $\mathbf{E}^3$ and constructed a vector differential equation to solve this problem in the case of a general helix ($\frac{\tau}{\kappa}$ is constant). However, this problem is not solved in other cases of the space curve.

In the light of our main problem, first we give:

Theorem 3.1. *Let* $\psi = \psi(s)$ *be a unit speed curve. Suppose* $\psi = \psi(\theta)$ *is another parametric representation of this curve by the parameter* $\theta = \int \kappa(s)ds$. *Then, the principal normal vector* $\mathbf{N}$ *satisfies a vector differential equation of third order as follows:*

$$\frac{1}{f(\theta)}\left[\frac{1}{f'(\theta)}\left(\mathbf{N}''(\theta) + (1+f^2(\theta))\mathbf{N}(\theta)\right)\right]' + \mathbf{N}(\theta) = 0, \tag{2}$$

where $f(\theta) = \frac{\tau(\theta)}{\kappa(\theta)}$.

Proof. Let $\psi = \psi(s)$ be an unit speed curve. If we write this curve in the another parametric representation $\psi = \psi(\theta)$, where $\theta = \int \kappa(s)ds$, we have new Frenet equations as follows:

$$\begin{bmatrix}\mathbf{T}'(\theta)\\ \mathbf{N}'(\theta)\\ \mathbf{B}'(\theta)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ -1 & 0 & f(\theta)\\ 0 & -f(\theta) & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}(\theta)\\ \mathbf{N}(\theta)\\ \mathbf{B}(\theta)\end{bmatrix}, \tag{3}$$

where $f(\theta) = \frac{\tau(\theta)}{\kappa(\theta)}$. If we differentiate the second equation of the new Frenet Eq. (3) and using the first and the third equations, we have

$$\mathbf{B}(\theta) = \frac{1}{f'(\theta)}[\mathbf{N}''(\theta) + (1+f^2(\theta))\mathbf{N}(\theta)]. \tag{4}$$

Differentiating the above equation and using the last equation from (3), we obtain a vector differential equation of third order (2) as desired. □

The Eq. (2) is not easy to solve in general case. If one solves this equation, the natural representation of the position vector of an arbitrary space curve can be determined as follows:

$$\psi(s) = \int\left(\int \kappa(s)\mathbf{N}(s)ds\right)ds + C, \tag{5}$$

or in parametric representation

$$\psi(\theta) = \int \frac{1}{\kappa(\theta)}\left(\int \mathbf{N}(\theta)d\theta\right)d\theta + C, \tag{6}$$

where $\theta = \int \kappa(s)ds$.

We can solve the Eq. (2) in the case of a slant helix. The following proposition are *new characterizations* for a slant helices in $\mathbf{E}^3$:

Lemma 3.2. *Let $\psi : I \to \mathbf{E}^3$ be a curve that is parameterized by arclength with intrinsic equations $\kappa = \kappa(s)$ and $\tau = \tau(s)$. The curve is a slant helix (its normal vectors make a constant angle, $\phi = \pm \arccos[n]$, with a fixed straight line in the space) if and only if*

$$\tau(s) = \pm \frac{m\ \kappa(s) \int \kappa(s)ds}{\sqrt{1 - m^2\left(\int \kappa(s)ds\right)^2}}, \tag{7}$$

where $m = \frac{n}{\sqrt{1-n^2}}$.

Proof. ($\Rightarrow$) Let $\mathbf{d}$ be a unit fixed vector makes a constant angle, $\phi = \pm\arccos[n]$, with the normal vector $\mathbf{N}$. Therefore

$$\langle \mathbf{N}, \mathbf{d} \rangle = n. \tag{8}$$

Differentiating the Eq. (8) with respect to the variable $\theta = \int \kappa(s)ds$ and using the new Frenet Eq. (3), we get

$$\langle -\mathbf{T}(\theta) + f(\theta)\mathbf{B}(\theta), \mathbf{d} \rangle = 0. \tag{9}$$

Therefore,

$$\langle \mathbf{T}, \mathbf{d} \rangle = f \langle \mathbf{B}, \mathbf{d} \rangle.$$

If we put $\langle \mathbf{B},\mathbf{d} \rangle = b$, we can write

$$\mathbf{d} = f\ b\ \mathbf{T} + n\ \mathbf{N} + b\ \mathbf{B}.$$

From the unitary of the vector $\mathbf{d}$ we get $b = \pm\sqrt{\frac{1-n^2}{1+f^2}}$. Therefore, the vector $\mathbf{d}$ can be written as

$$\mathbf{d} = \pm f \sqrt{\frac{1-n^2}{1+f^2}}\ \mathbf{T} + n\ \mathbf{N} \pm \sqrt{\frac{1-n^2}{1+f^2}}\ \mathbf{B}. \tag{10}$$

If we differentiate Eq. (9) again, we obtain

$$\langle f'\ \mathbf{B} - (1+f^2)\mathbf{N}, \mathbf{d} \rangle = 0. \tag{11}$$

Eqs. (10) and (11) lead to the following differential equation

$$\frac{f'}{(1+f^2)^{3/2}} = \pm\ m,$$

where $m = \frac{n}{\sqrt{1-n^2}}$. Integration the above equation, we get

$$\frac{f}{\sqrt{1+f^2}} = \pm\ m(\theta + c_1). \tag{12}$$

where c_1 is an integration constant. The integration constant can disappear with a parameter change $\theta \to \theta - c_1$. Solving the Eq. (12) with f as unknown we have

$$f(\theta) = \pm \frac{m\theta}{\sqrt{1-m^2\theta^2}}. \tag{13}$$

Finally, $\tau(s) = \kappa(s)f(s)$, we express the desired result.

($\Leftarrow$) Suppose that $\tau(s) = \pm \frac{m\kappa(s)\int \kappa(s)ds}{\sqrt{1-m^2\left(\int \kappa(s)ds\right)^2}}$. The function f can be written as $f(\theta) = \pm \frac{m\theta}{\sqrt{1-m^2\theta^2}}$ and let us consider the vector

$$\mathbf{d} = n\left(\theta\ \mathbf{T} + \mathbf{N} \pm \frac{1}{m}\sqrt{1-m^2\theta^2}\ \mathbf{B}\right).$$

We will prove that the vector $\mathbf{d}$ is a constant vector. Indeed, applying Frenet formula (3)

$$\mathbf{d}' = n\left(\mathbf{T} + \theta\ \mathbf{N} - \mathbf{T} + f\mathbf{B} \mp \frac{m\ \theta}{\sqrt{1-m^2\theta^2}}\mathbf{B} \mp \frac{f}{m}\sqrt{1-m^2\theta^2}\mathbf{N}\right) = 0$$

Therefore, the vector $\mathbf{d}$ is constant and $\langle \mathbf{N},\mathbf{d} \rangle = n$. This concludes the proof of Lemma 3.2. $\square$

Theorem 3.3. *The position vector $\psi = (\psi_1,\psi_2,\psi_3)$ of a slant helix is computed in the natural representation form:*

$$\begin{cases} \psi_1(s) = \frac{n}{m} \int \left[\int \kappa(s) \cos\left[\frac{1}{n} \arcsin\left(m \int \kappa(s)ds\right)\right] ds\right] ds, \\ \psi_2(s) = \frac{n}{m} \int \left[\int \kappa(s) \sin\left[\frac{1}{n} \arcsin\left(m \int \kappa(s)ds\right)\right] ds\right] ds, \\ \psi_3(s) = n \int \left[\int \kappa(s)ds\right] ds, \end{cases} \tag{14}$$

or in the parametric form

$$\begin{cases} \psi_1(\theta) = \frac{n}{m} \int \frac{1}{\kappa(\theta)} \left[\int \cos\left[\frac{1}{n} \arcsin(m\theta)\right] d\theta\right] d\theta, \\ \psi_2(\theta) = \frac{n}{m} \int \frac{1}{\kappa(\theta)} \left[\int \sin\left[\frac{1}{n} \arcsin(m\theta)\right] d\theta\right] d\theta, \\ \psi_3(\theta) = n \int \frac{\theta}{\kappa(\theta)} d\theta, \end{cases} \tag{15}$$

or in the useful parametric form:

$$\begin{cases} \psi_1(t) = \frac{n^3}{m^3} \int \frac{\cos[nt]}{\kappa(t)} \left[\int \cos[t]\cos[nt]dt\right] dt, \\ \psi_2(t) = \frac{n^3}{m^3} \int \frac{\cos[nt]}{\kappa(t)} \left[\int \sin[t]\cos[nt]dt\right] dt, \\ \psi_3(t) = \frac{n^2}{m^2} \int \frac{\sin[nt]\cos[nt]}{\kappa(t)} dt, \end{cases} \tag{16}$$

where $\theta = \int \kappa(s)ds$, $t = \frac{1}{n}\arcsin(m\theta)$, $m = \frac{n}{\sqrt{1-n^2}}$, $n = \cos[\phi]$ *and ϕ is the angle between the fixed straight line* (*axis* of a slant helix) *and the principal normal vector of the curve*.

Proof. If ψ is a slant helix whose principal normal vector $\mathbf{N}$ makes an angle $\phi = \pm\arccos[n]$ with a straight line U, then we can write $f(\theta) = \pm \frac{m\ \theta}{\sqrt{1-m^2\ \theta^2}}$, where $f = \frac{\tau}{\kappa}$, $\theta = \int \kappa(s)ds$ and $m = \frac{n}{\sqrt{1-n^2}}$. Therefore the Eq. (2) becomes

$$(1 - m^2\theta^2)\mathbf{N}'''(\theta) - 3m^2\theta\mathbf{N}''(\theta) + \mathbf{N}'(\theta) = 0. \tag{17}$$

If we write the principal normal vector as the following:

$$\mathbf{N} = N_1(\theta)\mathbf{e}_1 + N_2(\theta)\mathbf{e}_2 + N_3(\theta)\mathbf{e}_3. \tag{18}$$

Now, the curve ψ is a slant helix, i.e. the principal normal vector $\mathbf{N}$ makes a constant angle, ϕ, with the constant vector called the axis of the slant helix. So, without loss of generality, we take the axis of a slant helix parallel to $\mathbf{e}_3$. Then

$$N_3 = \langle \mathbf{N}, \mathbf{e}_3 \rangle = n. \tag{19}$$

On other hand the principal normal vector $\mathbf{N}$ is a unit vector, so the following condition is satisfied

$$N_1^2(\theta) + N_2^2(\theta) = 1 - n^2 = \frac{n^2}{m^2}. \tag{20}$$

The general solution of Eq. (20) can be written in the following form:

$$N_1 = \frac{n}{m}\cos[t(\theta)], \quad N_2 = \frac{n}{m}\sin[t(\theta)], \tag{21}$$

where t is an arbitrary function of θ. Every component of the vector $\mathbf{N}$ is satisfied the Eq. (17). So, substituting the components $N_1(\theta)$ and $N_2(\theta)$ in the Eq. (17), we have the following differential equations of the function $t(\theta)$

$$3t'\left(m^2\theta\ t' - (1-m^2\theta^2)t''\right)\sin[t] \\ -\left(t' - 3m^2\theta\ t'' - (1-m^2\theta^2)(t'^3 - t''')\right)\cos[t] = 0, \quad (22)$$

$$3t'\left(m^2\theta\ t' - (1-m^2\theta^2)t''\right)\cos[t] \\ +\left(t' - 3m^2\theta\ t'' - (1-m^2\theta^2)(t'^3 - t''')\right)\sin[t] = 0. \quad (23)$$

It is easy to prove that the above two equations lead to the following two equations:

$$m^2\theta\ t' - (1-m^2\theta^2)t'' = 0, \quad (24)$$

$$t' - 3m^2\theta\ t'' - (1-m^2\theta^2)(t'^3 - t''') = 0. \quad (25)$$

The general solution of Eq. (24) is

$$t(\theta) = c_2 + c_1\ \arcsin(m\ \theta), \quad (26)$$

or

$$t(\theta) = c_2 + c_1\ \arccos(m\ \theta), \quad (27)$$

where c_1 and c_2 are constants of integration. The constant c_2 can be disappear if we change the parameter $t \to t + c_2$. Substituting the solution (26) or (27) in the Eq. (25), we obtain the following condition:

$$m\ c_1\ (1 + m^2(1-c_1)) = 0$$

which leads to $c_1 = \frac{\sqrt{1+m^2}}{m} = \frac{1}{n}$, where $m \neq 0$ and $c_1 \neq 0$.

Now, the principal normal vector take the following form:

$$\mathbf{N}(\theta) = \left(\frac{n}{m}\cos\left[\frac{1}{n}\arcsin(m\ \theta)\right], \frac{n}{m}\sin\left[\frac{1}{n}\arcsin(m\ \theta)\right], n\right). \quad (28)$$

or

$$\mathbf{N}(\theta) = \left(\frac{n}{m}\cos\left[\frac{1}{n}\arccos(m\ \theta)\right], \frac{n}{m}\sin\left[\frac{1}{n}\arccos(m\ \theta)\right], n\right). \quad (29)$$

If we substitute the Eq. (28) in the two Eqs. (5) and (6), we have the two Eqs. (14) and (15). It is easy to arrive the Eq. (16), if we take the new parameter $t = \frac{1}{n}\arcsin(m\theta)$, which completes the proof.

On other hand if we used Eq. (29), we have the following theorem:

Theorem 3.4. *The position vector* $\psi = (\psi_1,\psi_2,\psi_3)$ *of a slant helix is given in the natural representation form:*

$$\begin{cases} \psi_1(s) = \frac{n}{m}\ \int\left[\int \kappa(s)\cos\left[\frac{1}{n}\arccos\left(m\int\kappa(s)ds\right)\right]ds\right]ds, \\ \psi_2(s) = \frac{n}{m}\ \int\left[\int \kappa(s)\sin\left[\frac{1}{n}\arccos\left(m\int\kappa(s)ds\right)\right]ds\right]ds, \\ \psi_3(s) = n\ \int\left[\int\kappa(s)ds\right]ds, \end{cases} \quad (30)$$

or in the parametric form

$$\begin{cases} \psi_1(\theta) = \frac{n}{m}\ \int\frac{1}{\kappa(\theta)}\left[\int\cos\left[\frac{1}{n}\arccos(m\theta)\right]d\theta\right]d\theta, \\ \psi_2(\theta) = \frac{n}{m}\ \int\frac{1}{\kappa(\theta)}\left[\int\sin\left[\frac{1}{n}\arccos(m\theta)\right]d\theta\right]d\theta, \\ \psi_3(\theta) = n\ \int\frac{\theta}{\kappa(\theta)}d\theta, \end{cases} \quad (31)$$

or in the useful parametric form:

$$\begin{cases} \psi_1(t) = \frac{n^3}{m^3}\ \int\frac{\sin[nt]}{\kappa(t)}\left[\int\cos[t]\sin[nt]dt\right]dt, \\ \psi_2(t) = \frac{n^3}{m^3}\ \int\frac{\sin[nt]}{\kappa(t)}\left[\int\sin[t]\sin[nt]dt\right]dt, \\ \psi_3(t) = -\frac{n^2}{m^2}\ \int\frac{\sin[nt]\cos[nt]}{\kappa(t)}dt, \end{cases} \quad (32)$$

where $\theta = \int\kappa(s)ds, t = \frac{1}{n}\arccos(m\theta), m = \frac{n}{\sqrt{1-n^2}}, n = \cos[\phi]$ *and* ϕ *is the angle between the fixed straight line* (*axis* of a slant helix) *and the principal normal vector of the curve.*

4. Examples

In this section, we take several choices for the curvature κ and torsion τ, and next, we apply Theorem 3.3.

Example 4.1. The case of a slant helix with

$$\kappa = 1, \quad \tau = \frac{m\ s}{\sqrt{1-m^2\ s^2}}, \quad (33)$$

which are the intrinsic equations of a Salkowski curve [18]. Substituting $\kappa(t) = 1$ in the Eq. (16) we have the explicit parametric representation of such curve as follows:

$$\begin{cases} \psi_1(t) = \frac{n}{4m}\left[\frac{n-1}{2n+1}\cos[(2n+1)t] + \frac{n+1}{2n-1}\cos[(2n-1)t] - 2\cos[t]\right], \\ \psi_2(t) = \frac{n}{4m}\left[\frac{n-1}{2n+1}\sin[(2n+1)t] - \frac{n+1}{2n-1}\sin[(2n-1)t] - 2\sin[t]\right], \\ \psi_3(t) = -\frac{n}{4m^2}\cos[2nt], \end{cases} \quad (34)$$

where $t = \frac{1}{n}\arcsin(ms)$. One can see a special examples of such curves in the Fig. 1.

Example 4.2. The case of a slant helix with

$$\kappa = \frac{m\ s}{\sqrt{1-m^2\ s^2}}, \quad \tau = 1. \quad (35)$$

which are the intrinsic equations of an anti-Salkowski curve [18]. Substituting

$$\kappa = \frac{m\ s}{\sqrt{1-m^2\ s^2}} = \frac{\sqrt{1-m^2\theta^2}}{m\theta} = \cot[nt],$$

in the Eq. (16) we have the explicit parametric representation of such curve as follows:

$$\begin{cases} \psi_1(t) = \frac{n}{4m}\left[\frac{n-1}{2n+1}\sin[(2n+1)t] + \frac{n+1}{2n-1}\sin[(2n-1)t] - 2n\sin[t]\right], \\ \psi_2(t) = \frac{n}{4m}\left[\frac{1-n}{1+2n}\cos[(1+2n)t] - \frac{1+n}{1-2n}\cos[(1-2n)t] + 2n\cos[t]\right], \\ \psi_3(t) = \frac{n}{4m^2}(2nt - \sin[2nt]), \end{cases} \quad (36)$$

where $\theta = \frac{\sqrt{1-m^2s^2}}{m}$ and $t = \frac{1}{n}\arcsin(m\theta)$. One can see a special examples of such curves in the Fig. 2.

Remark 4.3. A family of curves with constant curvature but non-constant torsion is called Salkowski curves and a family of curves with constant torsion but non-constant curvature is called anti-Salkowski curves [19]. Monterde [18] studied some of characterizations of these curves and he proved that the principal normal vector makes a constant angle with fixed straight line. So that: Salkowski and anti-Salkowski curves are important examples of slant helices.

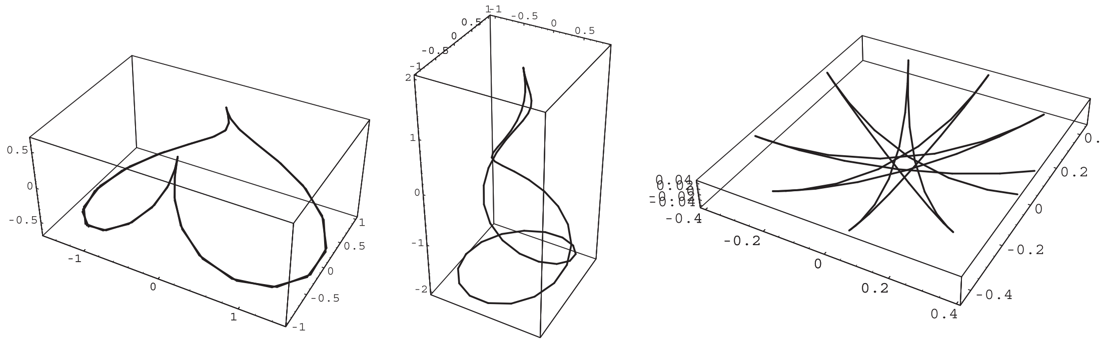

Figure 1 Slant helices with $\kappa = 1$ and $n = \frac{1}{3}, \frac{1}{8}, \frac{10}{11}$.

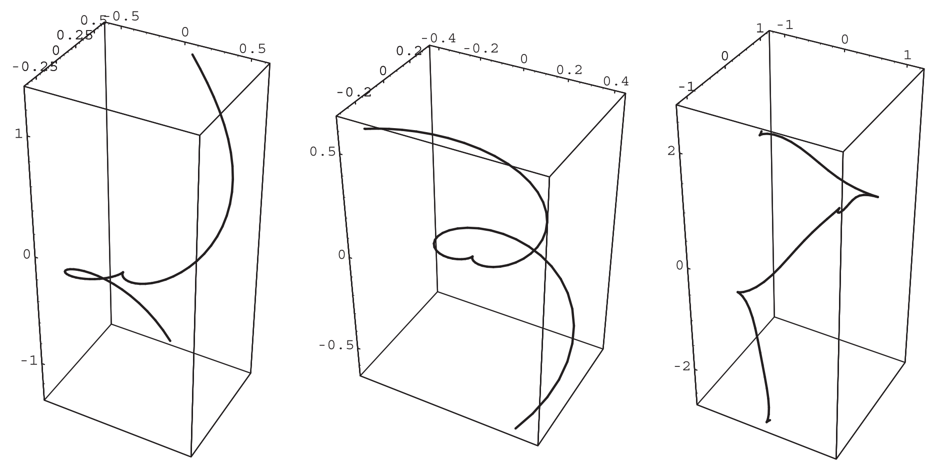

Figure 2 Slant helices with $\tau = 1$ and $n = \frac{1}{5}, \frac{1}{13}, \frac{2}{3}$.

Example 4.4. The case of a slant helix with

$$\kappa = \frac{\mu}{m}\cos[\mu\ s], \quad \tau = \frac{\mu}{m}\sin[\mu\ s]. \tag{37}$$

Substituting $\kappa = \frac{\mu}{m}\cos[m\ s]$ in the Eq. (14) we have the natural representation of such curve as follows:

$$\begin{cases} \psi_1(s) = -\frac{m^2}{n\ \mu}\left[(1+n^2)\cos[\mu\ s]\cos[\frac{\mu\ s}{n}] + 2n\sin[\mu\ s]\sin[\frac{\mu\ s}{n}]\right], \\ \psi_2(s) = -\frac{m^2}{n\ \mu}\left[(1+n^2)\cos[\mu\ s]\sin[\frac{\mu\ s}{n}] - 2n\sin[\mu\ s]\cos[\frac{\mu\ s}{n}]\right], \\ \psi_3(s) = -\frac{n}{m\ \mu}\ \cos[\mu\ s]. \end{cases} \tag{38}$$

The above curve is a geodesic of the tangent developable of a general helix [11]. One can see a special examples of such curves in the Fig. 3.

Remark 4.5. A unit speed curve of constant precession is defined by the property that its (Frenet) Darboux vector

$$W = \tau\ \mathbf{T} + \kappa\ \mathbf{B}$$

revolves about a fixed line in space with angle and constant speed. A curve of constant precession is characterized by having

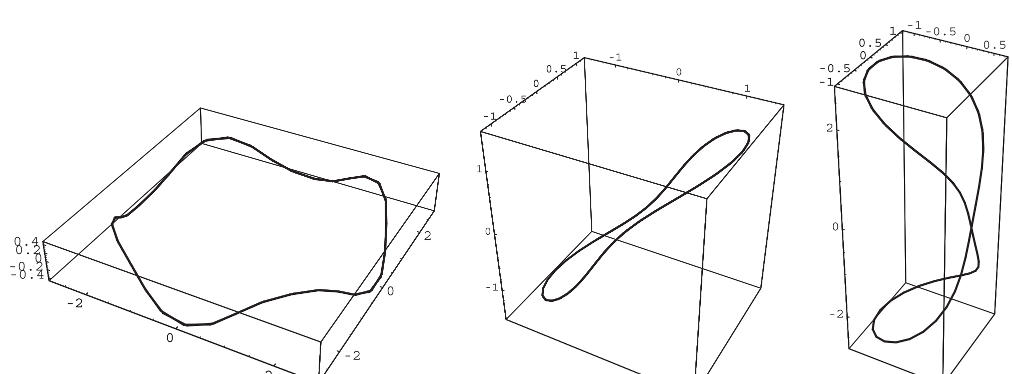

Figure 3 Slant helices with $\kappa = \frac{\mu}{m}\cos[\mu\ s], \mu = m$ and $n = \frac{4}{5}, \frac{1}{2}, \frac{1}{3}$.

$$\kappa = \frac{\mu}{m}\sin[\mu\ s], \quad \tau = \frac{\mu}{m}\cos[\mu\ s]$$

or

$$\kappa = \frac{\mu}{m}\cos[\mu\ s], \quad \tau = \frac{\mu}{m}\sin[\mu\ s]$$

where μ and m are constants. This curve lie on a circular one-sheeted hyperboloid

$$x^2 + y^2 - m^2\ z^2 = 4m^2.$$

The curve closed if and only if $n = \frac{m}{\sqrt{1+m^2}}$ is rational [20]. Kula and Yayli [13] have proved that the geodesic curvature of the spherical image of the principal normal indicatrix of a curve of constant precession is a constant function equal $-m$. So that: the curves of constant precessions are important examples of slant helices.

The curves which considered in examples (4.1), (4.2) and (4.4) are plated in Figs. 1–3, respectively.

References

[1] A.T. Ali, Inclined curves in the Euclidean 5-space $\mathbf{E}^5$, J. Adv. Res. Pure Math. 1 (2009) 15–22.
[2] A.T. Ali, Position vector of general helices in Euclidean 3-space, Bull. Math. Anal. Appl. 3 (2011) 1–8.
[3] M. Barros, General helices and a theorem of Lancret, Proc. Am. Math. Soc. 125 (1997) 1503–1509.
[4] C. Boyer, A History of Mathematics, Wiley, New york, 1968.
[5] N. Chouaieb, A. Goriely, J.H. Maddocks, Helices, PANS 103 (2006) 9398–9403.
[6] L.P. Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces, Ginn and Co., 1909.
[7] S. Falcon, A. Plaza, The k-Fibonacci hyperbolic function, Chaos, Solitons and Fractals 38 (2008) 409–420.
[8] S. Falcon, A. Plaza, On the 3-dimensional k-Fibonacci spirals, Chaos, Solitons and Fractals 38 (2008) 993–1003.
[9] H.H. Hacisalihoglu, Differential Geometry, Ankara University, Faculty of Science Press, 2000.
[10] K. Ilarslan, O. Boyacioglu, Position vectors of a space like W-cuerve in Minkowski space $\mathbf{E}_1^3$, Bull. Korean Math. Soc. 44 (2007) 429–438.
[11] S. Izumiya, N. Takeuchi, New special curves and developable surfaces, Turk. J. Math. 28 (2004) 531–537.
[12] W. Kuhnel, Differential Geometry: Curves – Surfaces – Manifolds, Braunchweig, Wiesdaden, 1999.
[13] L. Kula, Y. Yayli, On slant helix and its spherical indicatrix, Appl. Math. Comp. 169 (2005) 600–607.
[14] L. Kula, N. Ekmekci, Y. Yayli, K. Ilarslan, Characterizations of slant helices in Euclidean 3-space, Turk. J. Math. 33 (2009) 1–13.
[15] M.M. Lipschutz, Schum's Outline of Theory and Problems of Differential Geometry, McGraw-Hill Book Company, New York, 1969.
[16] A.A. Lucas, P. Lambin, Diffraction by DNA, carbon nanotubes and other helical nanostructures, Rep. Prog. Phys. 68 (2005) 1181–1249.
[17] J. Monterde, Curves with constant curvature ratios, Bulletin of Mexican Mathematic Society, 3a serie 13 (2007) 177–186.
[18] J. Monterde, Salkowski curves revisted: A family of curves with constant curvature and non-constant torsion, Comput. Aided Geomet. Design 26 (2009) 271–278.
[19] E. Salkowski, Zur transformation von raumkurven, Mathematische Annalen 66 (1909) 517–557.
[20] P.D. Scofield, Curves of constant precession, Am. Math. Monthly 102 (1995) 531–537.
[21] D.J. Struik, in: Lectures in Classical Differential Geometry, Addison-Wesley, Reading, MA, 1961.
[22] C.D. Toledo-Suarez, On the arithmetic of fractal dimension using hyperhelices, Chaos, Solitons and Fractals 39 (2009) 342–349.
[23] M. Turgut, S. Yilmaz, Contributions to classical differential geometry of the curves in $\mathbf{E}^3$, Scientia Magna 4 (2008) 5–9.
[24] J.D. Watson, F.H.C. Crick, Generic implications of the structure of deoxyribonucleic acid, Nature 171 (1953) 964–967.
[25] X. Yang, High accuracy approximation of helices by quintic curve, Comput. Aided Geomet. Design 20 (2003) 303–317.