Journal of the Egyptian Mathematical Society 25 (2017) 86–89

Contents lists available at ScienceDirect

Journal of the Egyptian Mathematical Society

journal homepage: www.elsevier.com/locate/joems

N-dimensional Schrödinger equation at finite temperature using the Nikiforov–Uvarov method

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ARTICLE INFO

Article history:
Received 23 December 2015
Revised 18 June 2016
Accepted 22 June 2016
Available online 18 July 2016

MSC:
81Q05
81V25
81V05

Keywords:
Schrödinger equation
Nikiforov–Uvarov method
Finite temperature
Quarkonium spectra

ABSTRACT

The *N*-radial Schrödinger equation is analytically solved. The Cornell potential is extended to finite temperature. The energy eigenvalues and the wave functions are calculated in the *N*-dimensional form using the Nikiforov–Uvarov (NV) method. At zero temperature, the energy eigenvalues and the wave functions are obtained in good agreement with other works. The present results are applied on the charmonium and bottomonium masses at finite temperature. The effect of dimensionality number is investigated on the quarkonium masses. A comparison is discussed with other works, which use the QCD sum rules and lattice QCD. The present approach successfully generalizes the energy eigenvalues and corresponding wave functions at finite temperature in the *N*-dimensional representation. In addition, the present approach can successfully be applied to the quarkonium systems at finite temperature.

1. Introduction

The development of the radial Schrödinger equation (SE) in quantum mechanics and its solutions play a major role in the many fields of modern physics, in particular, in the high energy physics. The solutions of the SE can be found only when the potential of the system is determined [1]. There are several potentials such as the Cornell potential as in Refs. [2,3] or mixed between the Cornell potential and the harmonic oscillator potential as in Refs. [4,5] or Morse potential [6] are suggested for solving the SE. The theoretical studies of the heavy-meson systems such as bottomonium and charmonium are one of the special interest because of its relies on entirely on the QCD theory as in Ref. [7] and references therein.

Heavy quarkonia have been suggested as hard probes of the quark-gluon plasma [8] since the modification of static interactions at finite temperature eventually implies a dissolution of heavy quarkonia bound states into the continuum of scattering states (Mott effect). This effect results is studied in a suppression of heavy quarkonia production in heavy-ion collisions as an observable signal [9].

At finite temperature, there are several works for solving the SE such as in Refs. [10–12] using different techniques in comparison with the present method. In Ref. [10], the authors employed the modified internal potential as a function of temperature to study the quark-gluon plasma using the Mayer's expansion and phenomenology thermodynamic model. In Ref. [11], the finite temperature SE was solved by using Funke–Hecke theorem and the application on electron and proton systems. In Ref. [12], the authors obtained the generalized form of SE based on the first low of thermodynamics. In Ref. [13], the authors numerically solved the SE at finite temperature by employing an effective temperature dependent given by a linear combination of color singlet and internal energies.

Recently, some authors focus to extend the SE to the higher-dimensional space which gives more detail about the systems under study. Moreover, the energy eigenvalues and wave functions are obtained in the lower-dimensional space [1].

The aim of the present work is to study the *N*-dimensional radial SE to obtain the energy eigenvalues and wave functions at finite temperature by using the Nikiforov–Uvarov method. So far no attempt has been made to solve the *N*-radial SE using the Nikiforov–Uvarov method when finite temperature is included. Additionally, the effect of dimensionality number is investigated on the quarkonuim masses at finite temperature.

The paper is organized as follows: In Section 2, the NU method is briefly explained. In Section 3, The energy eigenvalue and wave

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http://dx.doi.org/10.1016/j.joems.2016.06.006

function are calculated in the N-dimensional space. In Section 4, the results are discussed. In Section 5, the summary and conclusion are presented.

2. Theoretical description of the Nikiforov–Uvarov (NU) method

In this section, the NU method [14] is briefly given to solve the second-order differential equation which takes the following form:

$$\Psi''(s)+\frac{\bar{\tau}(s)}{\sigma(s)}\Psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\Psi(s)=0, \quad (1)$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials of maximum second degree and $\bar{\tau}(s)$ is a polynomial of maximum first degree with an appropriate $s=s(r)$ coordinate transformation. To find particular solution of Eq. (1) by separation of variables, if one deals with the transformation

$$\Psi(s)=\Phi(s)\chi(s), \quad (2)$$

it reduces to an equation of hypergeometric type as follows

$$\sigma(s)\chi''(s)+\tau(s)\chi'(s)+\lambda\chi(s)=0, \quad (3)$$

where

$$\sigma(s)=\pi(s)\frac{\Phi(s)}{\Phi'(s)}, \quad (4)$$

$$\tau(s)=\bar{\tau}(s)+2\pi(s); \quad \tau'(s)<0, \quad (5)$$

and

$$\lambda=\lambda_n=-n\tau'(s)-\frac{n(n-1)}{2}\sigma''(s), n=0,1,2,\ldots \quad (6)$$

$\chi(s)=\chi_n(s)$ which is a polynomial of n degree which satisfies the hypergeometric equation, taking the following form

$$\chi_n(s)=\frac{B_n}{\rho_n}\frac{d^n}{ds^n}(\sigma''(s)\rho(s)), \quad (7)$$

where B_n is a normalization constant and $\rho(s)$ is a weight function which satisfies the following equation

$$\frac{d}{ds}\omega(s)=\frac{\tau(s)}{\sigma(s)}\omega(s); \quad \omega(s)=\sigma(s)\rho(s), \quad (8)$$

$$\pi(s)=\frac{\sigma'(s)-\bar{\tau}(s)}{2}\pm\sqrt{\left(\frac{\sigma'(s)-\bar{\tau}(s)}{2}\right)^2-\tilde{\sigma}(s)+K\sigma(s)}, \quad (9)$$

and

$$\lambda=K+\pi'(s), \quad (10)$$

the $\pi(s)$ is a polynomial of first degree. The values of K in the square-root of Eq. (9) is possible to calculate if the expressions under the square root are square of expressions. This is possible if its discriminate is zero. (for detail, see Ref. [14]).

3. The Schrödinger equation with the Cornell potential at finite temperature

The SE for two particles interacting via a spherically symmetric (central) potential $V(r)$ in the N-dimensional space, where r is inter-particle distance, is given by [2]

$$\left[\frac{d^2}{dr^2}+\frac{(N-1)}{r}\frac{d}{dr}-\frac{L(L+N-2)}{r^2}+2\mu(E-V(r))\right]\Psi(r)=0, \quad (11)$$

where L, N, and μ are the angular momentum quantum number, the dimensionality number and the reduced mass for the quarkonium particle (for charmonium $\mu=\frac{m_c}{2}$ and for bottomonium $\mu=\frac{m_b}{2}$), respectively. Setting the wave function $\Psi(r)=\frac{R(r)}{r}$, the following radial SE is obtained

$$\left[\frac{d^2}{dr^2}+2\mu\left[(E-V(r))-\frac{L(L+N-2)}{2\mu r^2}\right]\right]R(r)=0, \quad (12)$$

where $V(r)$ is the Cornell potential which takes following form

$$V(r)=ar-\frac{b}{r}, \quad (13)$$

where a and b are arbitrary constants will be determined later. The potential has distinctive features of strongly interaction: The confinement and the asymptotic freedom which are represented in the first and the second terms, respectively. Eq. (13) is modified to finite temperature [15] as follows

$$V(r)=a(T,r)r-\frac{b(T,r)}{r}, \quad (14)$$

where $a(T,r)=\frac{a}{m_D(T)r}(1-e^{-m_D(T)r})$ and $b(T,r)=be^{-m_D(T)r}$ where $m_D(T)$ is the Debye mass that vanishes at $T\to 0$ (for detail, see Ref. [15]). By substituting Eq. (14) into Eq. (12) and using approximation $e^{-m_D(T)r}=\sum_{j=0}^{\infty}\frac{(-m_D(T)r)^j}{j!}$ up to second-order, which gives a good accuracy when $m_D r\ll 1$. We obtain

$$\left[\frac{d^2}{dr^2}+2\mu\left(E-A+\frac{b}{r}-Cr+Dr^2-\frac{L(L+N-2)}{2\mu r^2}\right)\right]R(r)=0. \quad (15)$$

where, $A=b\,m_D(T), C=a-\frac{1}{2}bm_D^2(T)$, and $D=\frac{1}{2}a\,m_D(T)$.

By taking $r=\frac{1}{x}$, Eq. (15) takes the following form

$$\left[\frac{d^2}{dx^2}+\frac{2x}{x^2}\frac{d}{dx}+\frac{2\mu}{x^4}\left(E-A+bx-\frac{C}{x}+\frac{D}{x^2}-\frac{L(L+N-2)}{2\mu}x\right)\right]$$
$$R(x)=0. \quad (16)$$

The scheme is based on the expansion of $\frac{C}{x}$ and $\frac{D}{x^2}$ in a power series around the characteristic radius r_0 of meson up to the second order. Setting $y=x-\delta$, where $\delta=\frac{1}{r_0}$, thus, we expand the $\frac{C}{x}$ and $\frac{D}{x^2}$ into a series of powers around $y=0$.

$$\frac{C}{x}=\frac{C}{y+\delta}=\frac{1}{\delta}(1+\frac{y}{\delta})^{-1}$$
$$=\frac{C}{\delta}\left(1-\frac{y}{\delta}+\frac{y}{\delta^2}\right),$$
$$=C\left(\frac{3}{\delta}-\frac{3x}{\delta^2}+\frac{x^2}{\delta^3}\right). \quad (17)$$

Similarly,

$$\frac{D}{x^2}=D\left(\frac{6}{\delta^2}-\frac{8x}{\delta^3}+\frac{3x^2}{\delta^4}\right). \quad (18)$$

By substituting Eqs. (17) and (18) into Eq. (16). Eq. (16) takes the following form

$$\left[\frac{d^2}{dx^2}+\frac{2x}{x^2}\frac{d}{dx}+\frac{2\mu}{x^4}(-D_1+D_2x-D_3x^2)\right]R(x)=0, \quad (19)$$

where, $D_1=-\mu(E-A-\frac{3C}{\delta}+\frac{6D}{\delta^2})$, $D_2=\mu(\frac{3C}{\delta^2}-\frac{8D}{\delta^3}+b)$, and $D_3=\mu(\frac{C}{\delta^3}-\frac{3D}{\delta^4}+\frac{L(L+N-2)}{2\mu})$. The $\frac{1}{x}$ expansion gives a good accuracy when δ tends to x. In Tables (1) and (2), δ is determined which gives a good accuracy in comparison with experimental data (for detail, see Refs. [2,3]).

By comparing Eqs. (19) and (1), we find $\bar{\tau}(s)=2x$, $\sigma(s)=x^2$, and $\tilde{\sigma}(s)=2\mu(-D_1+D_2x-D_3x^2)$. Hence, the Eq. (16) satisfies the

Table 1
Mass spectra of charmonium (in GeV) ($m_c = 1.209$ GeV [1], $a = 0.2$ GeV^2, $\delta = 0.231$ GeV, $b = 1.244$ and $T = 0$) .

State	1S	1P	2S	1D	2P	3S	4S
Results	3.096	3.255	3.686	3.504	3.779	4.040	4.269
Exp. [16]	3.096	–	3.686	–	3.773	4.040	4.263

Table 2
Mass spectra of bottomonium (in GeV) ($m_b = 4.823$ GeV [1], $a = 0.2$ GeV^2, $\delta = 0.378$ GeV, $b = 1.569$ and $T = 0$) .

State	1S	1P	2S	1D	2P	3S	4S
Results	9.460	9.619	10.023	9.864	10.114	10.355	10.567
Exp. [16]	9.460	–	10.023	–	–	10.355	10.580

conditions in Eq. (1). By following the NU method that mentioned in Section 2, therefore

$$\pi = \pm\sqrt{(K + 2C_1)x^2 - 2Bx + 2A}. \tag{20}$$

The constant K is chosen such as the function under the square root has a double zero, i.e. its discriminant $\Delta = 4B^2 - 8A(K + 2C_1) = 0$. Hence,

$$\pi = \pm\frac{1}{\sqrt{2A}}(2A - Bx). \tag{21}$$

Thus,

$$\tau = 2x \pm \frac{2}{\sqrt{2A}}(2A - Bx). \tag{22}$$

For bound state solutions, we choose the positive sign in above equation so that the derivative

$$\tau' = 2 - \frac{2B}{\sqrt{2A}}. \tag{23}$$

By using Eq. (10), we obtain

$$\lambda = \frac{B^2}{2A} - 2C_1 - \frac{B}{\sqrt{2A}}, \tag{24}$$

and Eq. (6), we obtain

$$\lambda_n = -n\left(2 - \frac{2B}{2\sqrt{A}}\right) - n(n-1). \tag{25}$$

From Eq. (6); $\lambda = \lambda_n$. The energy eigenvalues of Eq. (15) at finite temperature in the N-dimensional space is given

$$E^N_{nL} = A + \frac{3C}{\delta} - \frac{6D}{\delta^2} - \frac{2\mu(\frac{3C}{\delta^2} + b - \frac{8D}{\delta^3})^2}{[(2n+1) \pm \sqrt{1 + \frac{8\mu C}{\delta^3} + 4((L + \frac{N-2}{2})^2 - \frac{1}{4}) - \frac{24\mu D}{\delta^4}}]^2}. \tag{26}$$

The radial of wave function of Eq. (15) takes the following form

$$R_{nL}(r) = C_{nL}\, r^{-\frac{D_2}{\sqrt{2D_1}}-1} e^{\sqrt{2D_1}r}\left(-r^2\frac{d}{dr}\right)^n \left(r^{-2n+\frac{D_2}{\sqrt{2D_1}}} e^{-2\sqrt{2D_1}r}\right). \tag{27}$$

C_{nL} is the normalization constant that is determined by $\int |R_{nL}(r)|^2 dr = 1$. We note that the radial wave function in Eq. (24) does not explicitly depend on the number of dimensions. Hence, $\int |R_{nL}(r)|^2 dr = 1$ remains unchanged. For detail, see Ref. [3].

4. Discussion of results

In this section, we calculate spectra of the heavy quarkonium system such as the charmonium and bottomonium mesons at finite temperature that have quark and anitquark flavor. The mass of quarkonium is calculated in the 3-dimensional space ($N = 3$). We apply the following relation as in Refs. [1,2]

$$M = 2m + E^{N=3}_{nL}, \tag{28}$$

where m is quarkonium bare mass for the charmonium or bottomonium mesons. By using Eq. (26), we write Eq. (28) as follows:

$$M = 2m + A + \frac{3C}{\delta} - \frac{6D}{\delta^2} - \frac{2\mu(\frac{3C}{\delta^2} + b - \frac{8D}{\delta^3})^2}{[(2n+1) \pm \sqrt{1 + \frac{8\mu C}{\delta^3} + 4L(L+1) - \frac{24\mu D}{\delta^4}}]^2} \tag{29}$$

Eq. (29) represents the quarkonium masses at finite temperature in the 3-dimensional space. One can obtain the quarkonium masses at zero temperature by taking $T = 0$ leads to $A = D = 0$ and $C = a$. Therefore, Eq. (29) takes the following form

$$M = 2m + \frac{3a}{\delta} - \frac{2\mu(\frac{3a}{\delta^2} + b)^2}{[(2n+1) \pm \sqrt{1 + \frac{8\mu a}{\delta^3} + 4L(L+1)}]^2}. \tag{30}$$

Eq. (30) coincides with Ref. [2], in which the authors obtained the quarkonium mass at zero temperature. Free parameters a, b, and δ are fitted with experimental data with using Eq. (30) as in Refs. [2,3]. All parameters are fixed as in Ref. [2] to check accuracy of the present results in comparison with the results in Ref. [2]. The present results are in good agreement with available experimental data for all states of charmonium and bottomonium mesons and close with results in Ref. [2] as in Tables (1) and (2)

At finite temperature, the behavior of the quarkonium states is discussed. The parameters values at zero temperature are used as the initial parameters at finite temperature. To calculate the mass of spectra of charmonium and bottomonium masses at finite temperature, we define the explicit form of $m_D(T)$ as follows as in Ref. [10]:

$$m_D(T) = \gamma\alpha_s(T)T, \tag{31}$$

where γ is fixed parameter and $\alpha_s(T)$ is running coupling constant which takes the following form at finite temperatures

$$\alpha_s(T) = \frac{2\pi}{\left(11 - \frac{2}{3}n_f\right)\ln\left(\frac{T}{0.104T_c}\right)}. \tag{32}$$

In Fig. (1), the mass of spectrum 1S of bottomonium is plotted as a function of ratio temperature $\frac{T}{T_c}$ where the critical temperature $T_c = 170$ MeV [17] at different values of N . In three dimensions space, the curve decreases with increasing temperature. By increasing the dimensional number N, we note that the curves shift to higher values. This indicates that the binding energy increases with increasing dimensionality number and qualitative agreement is noted for all values of N. In comparison with Ref. [18], the authors found that the spectrum of bottomonium mass decreases at higher-values of temperature in the framework of the QCD sum rules. In addition, the behavior of bottomonium mass is in agreement with lattice QCD. Therefore, the behavior of bottomonium is in qualitative agreement with the QCD sum rules and lattice QCD. In Ref. [13], the authors found that the spectrum of bottomonium mass decreases with increasing temperature in the framework of the SE, in which the different method is used in comparison with the present work.

In Fig. 2, the mass spectrum of 1S of charmonium is plotted as a function of $\frac{T}{T_c}$ at different values of N. We note that the

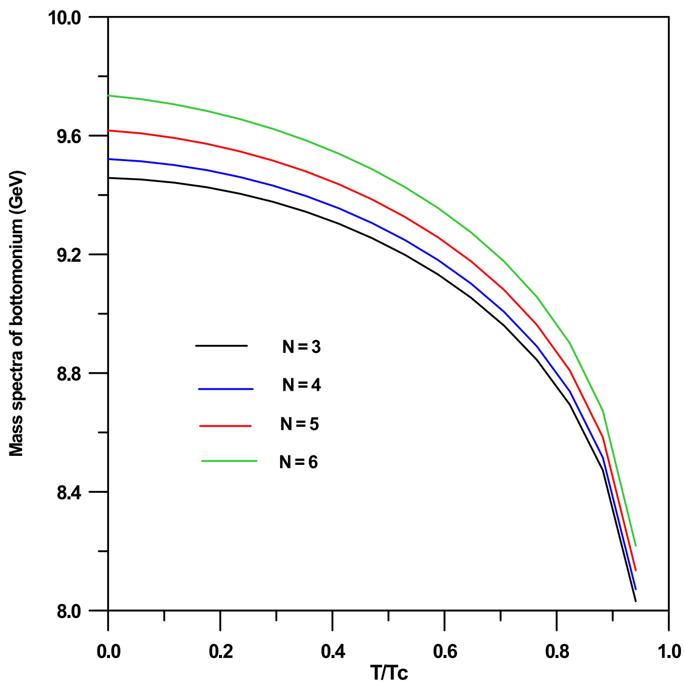

Fig. 1. Mass Specrum 1S of bottomonium is plotted as a function of ratio temperature $\frac{T}{T_c}$ for parameters $m_b = 4.823$ GeV, $a = 0.2$ GeV^2, $b = 1.569$, $\delta = 0.378$ GeV at different values of N.

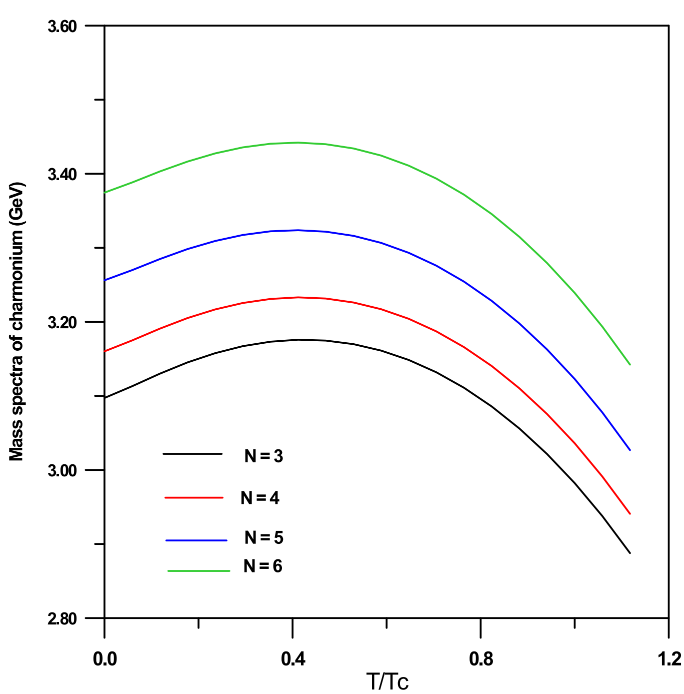

Fig. 2. Mass spectrum of 1S of charmonium mass is plotted as a function of ratio temperature $\frac{T}{T_c}$ for parameters $m_c = 1.209$ GeV, $a = 0.2$ GeV ^{2} $b = 1.244$ $\delta = 0.231$ GeV at different values of N.

curves increase with increasing temperature up to $T = 0.5T_c$, then the curves decrease with increasing temperature. In addition, the curves shift to higher values. This indicates that the binding energy increases with increasing the number of dimensionality and also qualitative agreement is noted for all values of N . This behavior is in qualitative agreement with QCD sum rules and lattice QCD as in Ref. [18] and references therein.

5. Summary and conclusion

In the present work, the N-dimensional Schrödinger equation is analytically solved using the NV method. The Cornell potential is extended to include finite temperature based on Ref. [15]. The energy eigenvalues and corresponding wave functions are obtained in the N-dimensional form at finite temperature. The energy eigenvalues and corresponding wave functions are obtained in lower dimensions and zero temperature which coincide with other works. We apply the present results on the quarkonium masses at finite temperature and find the qualitative agreement with the QCD sum rules, lattice QCD, and other approaches. We add future investigations by studying the effect of dimensionality number on the quarkonium masses at finite temperature. We find that the quarkonium masses increase with increasing dimensionality number (N) at finite temperature.

The novelty in this work that NU method successfully applies to find the solution of the N-radial SE at finite temperature. Additionally, the dimensionality number plays an important role at finite temperature. The present results are in agreement with the QCD sum rules, lattice QCD, and other approaches. We hope to extend this work for further investigations of other characteristics of quarkonium at finite temperature.

Acknowledgement

The author thanks the referees for useful comments that supported this work.

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