Journal of the Egyptian Mathematical Society (2013) **21**, 3–10

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Almost periodic solutions to dynamic equations on time scales

Hongtao Zhang [a,b], **Yongkun Li** [b,*]

[a] *Department of Applied Mathematics, Kunming University of Science and Technology, Kunming, Yunnan 650093, People's Republic of China*
[b] *Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China*

Received 10 May 2012; revised 27 September 2012; accepted 13 October 2012
Available online 17 November 2012

KEYWORDS

Almost periodic functions;
Almost periodic solutions;
Time scales;
Dynamic equations;
Liapunov functionals

Abstract In this paper, we first present a notion of almost periodic functions on time scales and study their basic properties. Then by means of Liapunov functionals, we study the existence of almost periodic solutions for an almost periodic dynamic equation on time scales.

MATHEMATICS SUBJECT CLASSIFICATION 2010: 34N05, 34C27, 34D20

1. Introduction

Recently, many researches have studied the existence of periodic solutions for dynamic equations on time scales [1–8]. However, few papers have been published on the existence of almost periodic solutions for dynamic equations on time scales. In fact, the existence of almost periodic, asymptotically almost periodic, pseudo-almost periodic solutions is among the most attractive topics in qualitative theory of differential equations and difference equations due to their applications, especially in biology, economics and physics [9–22]. Therefore, it is interesting to study the existence of almost periodic solutions for dynamic equations on time scales.

Motivated by the above, our main purpose of this paper is to present a notion of almost periodic functions on time scales and study the existence of periodic solutions for almost periodic dynamic equations on time scales. The results in this paper contains some results obtained for differential and difference equations in [11–14].

The organization of this paper is as follows: In Section 2, we introduce some notations and definitions. In Section 3, we study some basic properties about almost periodic functions on time scales. In Section 4, by using the properties of almost periodic functions on time scales and Liapunov functionals, we study the existence of almost periodic solutions to a general almost periodic dynamic equation on time scales.

2. Preliminaries

In this section, we shall recall some basic definitions, lemmas which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the

* Corresponding author.
E-mail address: yklie@ynu.edu.cn (Y. Li).
Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

1110-256X © 2012 Egyptian Mathematical Society. Production and hosting by Elsevier B.V. Open access under CC BY-NC-ND license.
http://dx.doi.org/10.1016/j.joems.2012.10.004

graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{and} \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.

If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided that it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be continuous on $\mathbb{T}$. We define $C[J, \mathbb{R}] = \{u(t)$ is continuous on $J\}$, and $C^1[J, \mathbb{R}] = \{u^\Delta(t)$ is continuous on $J\}$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^\Delta(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|[y(\sigma(s)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

For each $t \in \mathbb{T}$, let N be a neighborhood of t. Then, we define the generalized derivative (or Dini derivative), $D^+u^\Delta(t)$, to mean that, given $\varepsilon > 0$, there exists a right neighborhood $N_\varepsilon \subset N$ such that

$$\frac{u(\sigma(t)) - u(s)}{\mu(t,s)} < D^+u^\Delta(t) + \varepsilon$$

for $s \in N_\varepsilon$, $s > t$, where $\mu(t,s) \equiv \sigma(t) - s$.

If t is right-scattered and u is continuous at t, this reduces to

$$D^+u^\Delta(t) = \frac{u(\sigma(t)) - u(t)}{\sigma(t) - t}.$$

For $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}], D^+V^\Delta(t, x(t))$ to mean that, given $\varepsilon > 0$, there exists a right neighborhood $N_\varepsilon \subset N$ such that

$$\frac{1}{\mu(t,s)}[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t)) - \mu(t,s)f(t, x(t)))] < D^+V^\Delta(t, x(t)) + \varepsilon$$

for each $s \in N_\varepsilon, s > t$, where $\mu(t,s) \equiv \sigma(t) - s$. If t is right-scattered and $V(t, x(t))$ is continuous at t, this reduces to

$$D^+V^\Delta(t, x(t)) = \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(t))}{\sigma(t) - t}.$$

If y is continuous, then y is right-dense continuous, and if y is delta differentiable at t, then y is continuous at t.

Let y be right-dense continuous. If $Y^\Delta(t) = y(t)$, then we define the delta integral by

$$\int_a^t y(s)\Delta s = Y(t) - Y(a).$$

A function $r : \mathbb{T} \to \mathbb{R}$ is called regressive if

$$1 + \mu(t)r(t) \neq 0$$

for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R}$. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \quad \text{for all} \quad t \in \mathbb{T}\}.$$

If r is regressive function, then the generalized exponential function e_r is defined by

$$e_r(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\right\}, \quad \text{for } s, \quad t \in \mathbb{T}$$

with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

Definition 2.1 1. We say that a time scale $\mathbb{T}$ is periodic if there exists $p > 0$ such that if $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale.

Remark 2.1. By the definition above, if a time scale $\mathbb{T}$ is periodic, then $\sup \mathbb{T} = \infty$, and $\mu(t)$ must be bounded, and for any $t \in \mathbb{T}, \mu(t) \leqslant p, \mu(t+p) = \mu(t)$.

Example 2.1. Let $q > 1$, consider the time scale $\mathbb{T} = \{q^n : n \in Z\} \cup \{0\}$. Obviously, it is not a periodic time scale and we have

$$\sigma(t) = \inf\{q^n : n \in [m+1, \infty)\} = q^{m+1} = qq^m = qt$$

if $t = q^m \in \mathbb{T}$ and $\sigma(0) = 0$. So we obtain

$$\sigma(t) = qt \quad \text{for all} \quad t \in \mathbb{T}$$

and consequently

$$\mu(t) = \sigma(t) - t = (q-1)t \quad \text{for all} \quad t \in \mathbb{T}.$$

Hence $\mu(t) \to \infty$ as $t \to \infty$.

Throughout this paper, we always use $\mathbb{T}$ to denote a periodic time scale and $\mathbb{E}^n$ to denote $\mathbb{R}^n$ or $\mathbb{C}^n$, and use the notation:

$$\mathbb{T}_p = \begin{cases} \{np : n \in \mathbb{Z}\}, & \text{if } \mathbb{T} \text{ is a periodic time scale with period } p, \\ \mathbb{R}, & \text{if } \mathbb{T} = \mathbb{R}. \end{cases}$$

For convenience, we denote sequence $\{\alpha_n\}$ by α and

$\beta \subset \alpha$ if sequence $\beta = \{\beta_n\}$ is a subsequence of $\alpha = \{\alpha_n\}$,

$T_\alpha f(t) = \lim_{n\to\infty} f(t + \alpha_n)$, if the limit exists. The mode of convergence will be specified at each use of the symbol.

Definition 2.2 11. Let $A \subset B \subset \mathbb{R}$, we say that A is relatively dense in B if there exists a positive number l such that for all $a \in B$ we have

$$[a, a+l]_B \cap A \neq \emptyset,$$

where $\emptyset$ is the empty set, $[a, a+l]_B = [a, a+l] \cap B \cdot l$ is called the inclusion length.

Definition 2.3. We say that the function $f(t) \in C(\mathbb{T}, \mathbb{E}^n)$ is almost periodic if for any given $\varepsilon > 0$, the set

$$T(f, \varepsilon, \mathbb{T}) = \{\tau \in \mathbb{T}_p : |f(t+\tau) - f(t)| < \varepsilon, \ \forall \ t \in \mathbb{T}\}$$

is relatively dense in $\mathbb{T}_p$; that is, for any given $\varepsilon > 0$, there exist an $l = l(\varepsilon) > 0$ satisfying that each interval of length l contains at least one $\tau = \tau(\varepsilon) \in T(f, \varepsilon, \mathbb{T})$ such that

$$|f(t+\tau) - f(t)| < \varepsilon, \quad \forall \ t \in \mathbb{T}.$$

The set $T(f, \varepsilon, \mathbb{T})$ is called ε-translation set of $f(t)$, τ is called ε-translation of $f(t)$.

Remark 2.2. In above definition, if $\mathbb{T} = \mathbb{R}$, then this definition converts to the definition of almost periodic functions in continuous case, see [11,14]. If $\mathbb{T} = \mathbb{Z}$, then this definition turns to the definition of almost periodic sequence, see [12,13,15,16].

Definition 2.4. $f(t) \in C(\mathbb{T}, \mathbb{E}^n)$ is called an asymptotically almost periodic function on $\mathbb{T}$ if

$$f(t) = p(t) + q(t),$$

where $p(t)$ is an almost periodic function on $\mathbb{T}$, and $q(t) \to 0$ as $t \to \infty$.

Definition 2.5. $f(t, x) \in C(\mathbb{T} \times D, \mathbb{E}^n)$, where D is an open set in $\mathbb{E}^n$ or $D = \mathbb{E}^n$. $f(t,x)$ is said to be almost periodic in t uniformly for $x \in D$, or uniformly almost periodic for short, if for any $\varepsilon > 0$ and compact set S in D, there exists $l = l(\varepsilon, S)$ satisfies that each interval of length $l(\varepsilon, S)$ contains a τ such that

$$|f(t+\tau, x) - f(t, x)| < \varepsilon, \quad \forall\ (t, x) \in \mathbb{T} \times S.$$

$\tau$is called the ε-translation number of $f(t,x)$. The ε-translation set of $f(t,x)$ for $x \in S$ is denoted by

$$T(f, \varepsilon, S, \mathbb{T}) = \{\tau \in \mathbb{T}_p : |f(t+\tau, x) - f(t, x)| < \varepsilon, \ \forall\ (t, x) \in \mathbb{T} \times S\}.$$

Definition 2.6. Let $f(t, x) \in C(\mathbb{T} \times D, \mathbb{E}^n)$, where D is an open set in $\mathbb{E}^n$ or $D = \mathbb{E}^n$. The set $H(f) = \{g(t, x)$: there exits $\alpha \in \mathbb{T}_p$ such that $T_\alpha f(t, x) = g(t, x)$ exists uniformly on $\mathbb{T} \times S$, where $S \subset D$ is any compact set$\}$ is called the hull of f.

3. Properties

In this section, we will give some basic properties about almost periodic, uniformly almost periodic and asymptotically almost periodic functions on time scales, respectively.

To introduce the criteria and the properties of uniformly almost periodic functions on time scales, we first establish the following lemmas.

Lemma 3.1. *Let $f(t, x) \in C(\mathbb{T} \times D, \mathbb{E}^n)$ be almost periodic in t uniformly for $x \in D$. Then for any given sequence $\alpha' = \{\alpha_k\} \subset \mathbb{T}_p$, there exist a subsequence $\beta \subset \alpha'$ and a continuous function $g(t,x)$ such that $T_\beta f(t, x) = g(t, x)$ uniformly on $\mathbb{T} \times S$, where S is any compact set in D. Moreover, $g(t, x)$ is also uniformly almost periodic.*

Proof. If $\mathbb{T} = \mathbb{R}$, then Lemma 3.1 is equivalent to Theorem 2.2 in Ref. [14]. Now we assume that $\mathbb{T} \neq \mathbb{R}$, p is period of $\mathbb{T}$.

First, for any given $\varepsilon > 0$, there exists an $l = l(\varepsilon/2, S)$ such that every interval of length l contains an $\varepsilon/2$-translation number. Therefore, for any sequence $\alpha' = \{\alpha'_n\} \subset \mathbb{T}_p$, there exist $\tau_n \in \mathbb{T}_p$ and $\gamma_n \in \mathbb{T}_p$ such that $\alpha'_n = \tau_n + \gamma_n$, where $\tau_n \in T(f, \varepsilon/2, S, \mathbb{T})$ and $\gamma_n \in \{0, p, \cdots, n_l p\}, n_l p \leqslant l$. Since the set $\{0, p, \ldots, n_l p\}$ is finite, there must be infinite number of γ_n equal to some $\gamma \in \{0, p, \ldots, n_l p\}$. Let $\{\alpha_n\}$ be the set of all α'_n such that $\gamma_n = \gamma$. Then for any two integers p, m and any $(t, x) \in \mathbb{T} \times S$, we have

$$\begin{aligned}
|f(t+\alpha_p, x) - f(t+\alpha_m, x)| &\leqslant \sup_{(t,x)\in\mathbb{T}\times S} |f(t+\alpha_p, x) - f(t+\alpha_m, x)| \\
&\leqslant \sup_{(t,x)\in\mathbb{T}\times S} |f(t+\alpha_p-\alpha_m, x) - f(t, x)| \\
&= \sup_{(t,x)\in\mathbb{T}\times S} |f(t+\tau_p-\tau_m, x) - f(t, x)| \\
&\leqslant \sup_{(t,x)\in\mathbb{T}\times S} |f(t+\tau_p-\tau_m, x) - f(t+\tau_p, x)| \\
&+ \sup_{(t,x)\in\mathbb{T}\times S} |f(t+\tau_p, x) - f(t, x)| \\
&\leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

This proves that for any given $\varepsilon > 0$, compact set S and sequence α', there exists subsequence $\alpha \subset \alpha'$ such that the norm of the difference between any two functions from the function sequence $\{f(t + \alpha_n, x)\}$ less than ε, for $(t, x) \in \mathbb{T} \times S$. By Cauchy convergence principle, this shows that the sequence $\{f(t + \alpha_n, x)\}$ is uniformly convergent on $\mathbb{T} \times S$.

Now we prove that $g(t, x)$ is continuous on $\mathbb{T} \times D$. Suppose that $g(t, x)$ is discontinuous at a point $(t_0, x_0) \in \mathbb{T} \times D$, then there exist a number $\varepsilon_0 > 0$, and sequences $\{\delta_m\}, \{t_m\}, \{x_m\}$, where $\delta_m > 0$, and $\delta_m \to 0$ as $m \to +\infty$, $|t_0 - t_m| + |x_0 - x_m| < \delta_m$ such that

$$|g(t_0, x_0) - g(t_m, x_m)| \geqslant \varepsilon_0. \tag{3.1}$$

Let $X = \{x_m\} \cup \{x_0\}$, then X is a compact set in D. Therefore, there exists positive integer $N = N(\varepsilon_0, X)$ such that for $n > N$, the following inequality holds uniformly for m,

$$|f(t_m + \beta_n, x_m) - g(t_m, x_m)| < \frac{\varepsilon_0}{3}. \tag{3.2}$$

Moreover, for $n > N$ we have

$$|f(t_0 + \beta_n, x_0) - g(t_0, x_0)| < \frac{\varepsilon_0}{3}. \tag{3.3}$$

Since $f(t, x) \in C(\mathbb{T} \times D, \mathbb{E}^n)$ and δ_m can be arbitrary small when m is sufficient large, for a large enough m we have

$$|f(t_0 + \beta_n, x_0) - f(t_m + \beta_n, x_m)| < \frac{\varepsilon_0}{3}. \tag{3.4}$$

From 3.2, 3.3 and 3.4, it is easy to obtain a result which contradicts (3.1). Therefore, $g(t, x)$ is continuous on $\mathbb{T} \times D$.

Finally, for any compact $S \subset D$ and any given $\varepsilon > 0$, take $\tau \in T(f, \varepsilon, S, \mathbb{T})$, then for all $(t, x) \in \mathbb{T} \times S$ we have

$$|f(t + \beta_n + \tau, x) - f(t + \beta_n, x)| < \varepsilon.$$

From this, let $n \to +\infty$, we get

$$|g(t+\tau) - g(t, x)| < \varepsilon, \quad \forall (t, x) \in \mathbb{T} \times S.$$

Hence, $g(t, x)$ is also uniformly almost periodic. □

Lemma 3.2. *Let $f(t, x) \in C(\mathbb{T} \times D, \mathbb{E}^n)$. Suppose that for any sequence $\alpha' \subset \mathbb{T}_p$, there exists a subsequence $\alpha \subset \alpha'$ such that $T_\alpha f(t, x)$ exists uniformly on $\mathbb{T} \times S$, where S is any compact set in D. Then $f(t, x)$ is uniformly almost periodic.*

Proof. Suppose that the conclusion does not hold. Then there exist an $\varepsilon_0 > 0$ and a compact set $S_0 \subset D$ such that for any large $l > 0$, we can find an interval of length l which contains no ε_0-translation number of $f(t, x)$ for $x \in S_0$.

Now we pick a number $\alpha'_1 \in \mathbb{T}_p$ and let $[a_1, a_1 + 2|\alpha'_1|]$ be an interval containing no any ε_0-translation number, where $a_1 \in \mathbb{T}_p$. If we set $\alpha'_2 = a_1 + |\alpha'_1|$, then $a_1 \leqslant \alpha'_2 - \alpha'_1 \leqslant a_1 + 2|\alpha'_1|$ and hence $\alpha'_2 - \alpha'_1$ cannot be an ε_0-translation number. Next, let $[a_2, a_2 + 2|\alpha'_1| + 2|\alpha'_2|]$ be an interval which contains no any ε_0-translation number, where $a_2 \in \mathbb{T}_p$. Set $\alpha'_3 = a_2 + |\alpha'_1| + |\alpha'_2|$, then it is easy to see that $\alpha'_3 - \alpha'_1$ and $\alpha'_3 - \alpha'_2$ are both in $[a_2, a_2 + 2|\alpha'_1| + 2|\alpha'_2|]$. Therefore, $\alpha'_3 - \alpha'_1$ and $\alpha'_3 - \alpha'_2$ are not ε_0-translation numbers. In a similar way, we can define $\alpha'_4, \alpha'_5, \ldots$ so that none of the difference $\alpha'_i - \alpha'_j$ is an ε_0-translation number. Thus, for any i and j, $i \neq j$,

$$\sup_{(t,x)\in\mathbb{T}\times S} \left|f\left(t + \alpha'_j, x\right) - f\left(t + \alpha'_i, x\right)\right|$$
$$= \sup_{(t,x)\in\mathbb{T}\times S} \left|f\left(t + \alpha'_j - \alpha'_i, x\right) - f(t, x)\right| \geqslant \varepsilon_0,$$

which means that the sequence $\{f(t + \alpha'_n, x)\}$ cannot contain any uniformly convergent subsequence. This contradicts the assumption of the theorem. Therefore, $f(t,x)$ must be uniformly almost periodic. The proof of Lemma 3.2 is complete. □

As an immediate consequence of Lemmas 3.1 and 3.2, we obtain that

Proposition 3.1. *$f(t,x) \in C(\mathbb{T} \times D, \mathbb{E}^n)$ is a uniformly almost periodic function if and only if for any sequence $\alpha' \subset \mathbb{T}_p$, there exists a subsequence $\alpha \subset \alpha'$ such that $T_\alpha f(t,x)$ exists uniformly on $\mathbb{T} \times S$, where S is any compact set in D. Furthermore, the limit sequence is also a uniformly almost periodic function.*

Since an almost periodic function can be regarded as a special case of a uniformly almost periodic function, from Proposition 3.1, one has

Proposition 3.2. *$f(t) \in C(\mathbb{T}, \mathbb{E}^n)$ is an almost periodic function if and only if for any sequence $\alpha' \subset \mathbb{T}_p$, there exists a subsequence $\alpha \subset \alpha'$ such that $T_\alpha f(t)$ exists uniformly on $\mathbb{T}$. Furthermore, the limit sequence is also a uniformly almost periodic function.*

From Propositions 3.1, 3.2 and 3.3, it is easy to prove the following proposition.

Proposition 3.3.

(*i*) *Let $f(t), g(t) \in C(\mathbb{T}, \mathbb{E}^n)$ be almost periodic functions, then $f(t) \pm g(t)$, $f(t) \cdot g(t)$ are also almost periodic functions. If $\inf_{t\in\mathbb{T}}|g(t)| > 0$, then the quotient $f(t)/g(t)$ is almost periodic too.*

(*ii*) *Let $f(t,x), g(t,x) \in C(\mathbb{T} \times D, \mathbb{E}^n)$ be almost periodic in t uniformly for $x \in D$, then $f(t,x)) \pm g(t,x)$, $f(t,x) \cdot g(t,x)$ are also uniformly almost periodic functions. If $\inf_{(t,x)\in\mathbb{T}\times S}|g(t,x)| > 0$, where $S \subset D$ be any compact set, then the quotient $f(t,x)/g(t,x)$ is uniformly almost periodic too.*

Proposition 3.4.

(*i*) *If $f(t,x) \in C(\mathbb{T} \times D, \mathbb{E}^n)$ is almost periodic in t uniformly for $x \in D$, then there exists a function $F(r,x) \in C(\mathbb{R} \times D, \mathbb{E}^n)$ which is almost periodic in r uniformly for $x \in D$ such that $F(t,x) = f(t,x)$ for $(t,x) \in \mathbb{T} \times D$;*

(*ii*) *If $F(r,x) \in C(\mathbb{R} \times D, \mathbb{E}^n)$ is almost periodic in r uniformly for $x \in D$, then $F(t,x)$ is also continuous on $\mathbb{T} \times D$ and almost periodic in t uniformly for $x \in D$.*

Proof. (*i*) Assume that $f(t,x) \in C(\mathbb{T} \times D)$ is almost periodic in t for $x \in D$. We define a function $F(r,x)$ on $\mathbb{R} \times D$ as:

$$F(r,x) = \begin{cases} f(t,x), & \text{for} \quad r = t \text{ and } x \in D, \\ & \quad \text{if } t \text{ is right} - \text{dense}, \\ f(t,x) + \frac{r-t}{\sigma(t)-t}[f(\sigma(t),x) - f(t,x)], & \text{for} \quad r \in [t, \sigma(t)) \text{ and } x \in D, \\ & \quad \text{if } t \text{ is right} - \text{scattered}. \end{cases}$$

Clearly, $F(r,x)$ is continuous on $\mathbb{R} \times \mathbb{D}$ and $F(t,x) = f(t,x)$ for $(t,x) \in \mathbb{T} \times D$. Next, we show that the function $F(r,x)$ defined above is almost periodic in r uniformly for $x \in D$.

Since $f(t,x)$ is almost periodic in t uniformly for $x \in D$, for any given $\varepsilon > 0$ and any compact set $S \subset D$ there exists an $l(\varepsilon/3, S)$ such that any interval of length $l(\varepsilon/3, S)$ contains a τ and

$$|f(t + \tau, x) - f(t,x)| < \varepsilon/3, \quad \forall\ (t,x) \in (\mathbb{T} \times S).$$

If t is right-dense, $F(r,x) = f(t,x)$, for $r = t$, then

$$|F(r + \tau, x) - F(r,x)| < \varepsilon/3 < \varepsilon. \tag{3.5}$$

If t is right-scattered, $t \leqslant r < \sigma(t)$, then $t + \tau \leqslant r + \tau < \sigma(t) + \tau$. Noting that $0 \leqslant r - t < \sigma(t) - t = \sigma(t + \tau) - (t + \tau) \leqslant p$, we have

$$\begin{aligned} &|F(r + \tau, x) - F(r,x)| \\ &= \left|f(t + \tau, x) + \frac{(r + \tau) - (t + \tau)}{\sigma(t + \tau) - (t + \tau)}[f(\sigma(t + \tau), x) - f(t + \tau, x)] \right. \\ &\quad \left. - f(t,x) - \frac{r - t}{\sigma(t) - t}[f(\sigma(t), x) - f(t,x)]\right| \\ &= \left|f(t + \tau, x) - f(t,x) + \frac{r - t}{\sigma(t) - t}\{[f(\sigma(t) + \tau, x) \right. \\ &\quad - f(\sigma(t), x)] - [f(t + \tau, x) - f(t,x)]\}| \\ &< |f(t + \tau, x) - f(t,x)| + |f(\sigma(t) + \tau, x) - f(\sigma(t), x)| \\ &\quad + |f(t + \tau, x) - f(t,x)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we conclude that

$$|F(r + \tau, x) - F(r,x)| < \varepsilon, \quad \forall (r,x) \in \mathbb{R} \times S.$$

Thus, $F(t,x)$ is almost periodic in r uniformly for $x \in D$.

(*ii*) Let $F(r,x) \in C(\mathbb{R} \times D, \mathbb{E}^n)$ be uniformly almost periodic, then for any sequence $\alpha' \subset \mathbb{T}_p$, there exists a subsequence $\alpha \subset \alpha'$ such that $T_\alpha F(t + \alpha_n, x)$ exists uniformly on $\mathbb{R} \times S$, where S is any compact set in D. Consequently, $T_\alpha f(t + \alpha_n, x) = T_\alpha F(t + \alpha_n, x)$ exists uniformly on $\mathbb{T} \times S$. In view of Lemma 3.2 and Definition 2.9, this shows that $f(t,x)$ is uniformly almost periodic. □

It follows from Proposition 3.4 that

Proposition 3.5.

(*i*) *If $f(t) \in C(\mathbb{T}, \mathbb{E}^n)$ is an almost periodic function, then there exists an almost periodic function $F(r) \in (\mathbb{R}, \mathbb{E}^n)$ such that $F(r) = f(t)$ for $t \in \mathbb{T}$;*

(*ii*) *If* $F(r) \in C(\mathbb{R}, \mathbb{E}^n)$ *is an almost periodic function, then* $F(t)$ *is an almost periodic function on* $\mathbb{T}$.

Remark 3.1. If $F(r) \in C(\mathbb{R}, \mathbb{E}^n)$ is a periodic function, we cannot obtain that $F(t) \in \mathbb{T} \times \mathbb{E}^n$ is also a periodic function. For example, $F(t) = \sin t$ is periodic on $\mathbb{R}$, but it is only almost periodic and not periodic on $\mathbb{Z}$.

Similar to the proof of Proposition 3.4, one can easily show that

Proposition 3.6.

(*i*) *If* $f(t) \in C(\mathbb{T}, \mathbb{E}^n)$ *is an asymptotically almost periodic function, then there exists an asymptotically almost periodic function* $F(r) \in (\mathbb{R}, \mathbb{E}^n)$ *such that* $F(r) = f(t)$ *for* $t \in \mathbb{T}$.
(*ii*) *If* $F(r) \in C(\mathbb{R}, \mathbb{E}^n)$ *is an asymptotically almost periodic function, then* $F(t)$ *is an asymptotically almost periodic function on* $\mathbb{T}$.

As immediate consequences of above propositions, we have the following results which correspond to the results in the continuous case (see [14,23]).

Proposition 3.7. *If* $f(t,x) \in C(\mathbb{T} \times D, \mathbb{E}^n)$ *is almost periodic in* t *uniformly for* $x \in D$, *then* $f(t,x)$ *is bounded and uniformly continuous on* $\mathbb{T} \times S$, *where* $S \subset D$ *is any compact set.*

Proposition 3.8. *If* $f(t,x) \in C(\mathbb{T} \times D, \mathbb{E}^n)$ *is almost periodic in* t *uniformly for* $x \in D$ *and* $p(t)$ *is an almost periodic function such that* $p(t) \subset S$ *for all* $t \in \mathbb{T}$, *where* $S \subset D$ *is any compact set. Then* $f(t,p(t))$ *is almost periodic in* t.

Proposition 3.9. *If* $f(t) \in C(\mathbb{T}, \mathbb{E}^n)$ *is an asymptotically almost periodic function, then its decomposition*

$$f(t) = p(t) + q(t)$$

is unique, where $p(t) \in C(\mathbb{T}, \mathbb{E}^n)$ *is an almost periodic function and* $\lim_{t\to\infty} q(t) = 0$.

Proposition 3.10. $f(t) \in C(\mathbb{T}, \mathbb{E}^n)$ *is an asymptotically almost periodic function if and only if for any sequence* $\alpha' \in \mathbb{T}_p$ *such that* $\alpha'_n > 0$ *and* $\alpha'_n \to +\infty$ *as* $n \to \infty$, *there exists a subsequence* $\alpha \subset \alpha'$ *such that* $T_\alpha f$ *exists uniformly on* $\mathbb{T}^+ := \mathbb{T} \cap [0, \infty)$.

Proposition 3.11. *If* $0 \in \mathbb{T}$. *Then* $f(t) \in C(\mathbb{T}, \mathbb{E}^n)$ *is an almost periodic function on* $\mathbb{T}$ *if and only if for any sequences* $\alpha' \subset \mathbb{T}_p$ *and* $\beta' \subset \mathbb{T}_p$, *there exist subsequences* $\alpha \subset \alpha'$ *and* $\beta \subset \beta'$ *such that for any* $t \in \mathbb{T}$,

$$T_{\alpha+\beta} f(t) = T_\alpha T_\beta f(t). \tag{3.7}$$

Proof. Suppose $f(t) \in C(\mathbb{T}, \mathbb{E}^n)$ is an almost periodic function on $\mathbb{T}$. Then there exists an almost periodic function $F(r,x)$ on $\mathbb{R}$ such that $F(t) = f(t)$ for all $t \in \mathbb{T}$. For any sequence $\alpha' \subset \mathbb{T}_p$, $\beta' \subset \mathbb{T}_p$, there exist subsequence $\alpha \subset \alpha', \beta \subset \beta'$ such that

$$T_{\alpha+\beta} F(r) = T_\alpha T_\beta F(r) \text{ uniformly holds on } \mathbb{R}.$$

Hence, we obtain

$$T_{\alpha+\beta} f(t) = T_\alpha T_\beta f(t) \text{ uniformly holds on } \mathbb{T}.$$

Conversely, by (3.7) we know that for any sequence $\gamma' \subset \mathbb{T}_p$, there exists a subsequence $\gamma \subset \gamma'$ such that $T_\gamma f(t)$ exists on every $t \in \mathbb{T}$. By Proposition 3.1, it suffices to show that $T_\gamma f(t)$ exists uniformly on $\mathbb{T}$. Otherwise, there must exist $\varepsilon_0 > 0$, subsequences $\alpha' \subset \gamma, \beta' \subset \gamma$ and sequence $s' = \{s'_n\} \subset \mathbb{T}_p$ such that

$$|f(s'_n + \alpha'_n) - f(s'_n + \beta'_n)| \geqslant \varepsilon_0 > 0. \tag{3.8}$$

From (3.7), we know that there exist subsequences $\alpha'' \subset \alpha', s'' \subset s'$ such that

$$T_{s''+\alpha''} f(t) = T_{s''} T_{\alpha''} f(t) \quad \text{holds on } t \in \mathbb{T}.$$

We take $\beta'' \subset \beta'$ such that β'', α'' and s'' are common subsequence of β', α' and s', respectively. From (3.7) it follows that there exist subsequences $\beta \subset \beta'$ and $\alpha \subset \alpha'$ such that

$$T_{s+\beta} f(t) = T_s T_\beta f(t) \quad \text{holds on } t \in \mathbb{T}.$$

We take $\alpha \subset \alpha''$ such that α, β and s are common subsequence of α'', β'' and s'', respectively. Thus, we have

$$T_{s+\alpha} f(t) = T_s T_\alpha f(t) \quad \text{holds on } t \in \mathbb{T}.$$

Since $T_\alpha f(t) = T_\beta f(t) = T_\gamma f(t)$, it follows that

$$T_{s+\alpha} f(t) = T_{s+\beta} f(t) \quad \text{holds on } t \in \mathbb{T},$$

that is, for each $t \in \mathbb{T}$,

$$\lim_{n\to\infty} f(t + s_n + \beta_n) = \lim_{n\to\infty} f(t + s_n + \alpha_n),$$

which contradicts (3.8) if we take $t = 0 \in \mathbb{T}$. This completes the proof. □

Proposition 3.12. $f(t,x) \in C(\mathbb{T} \times D, \mathbb{E}^n)$ *is an uniformly almost periodic function on* $\mathbb{T} \times D$ *if and only if for any sequences* $\alpha' \subset \mathbb{T}_p$ *and* $\beta' \subset \mathbb{T}_p$, *there exist subsequences* $\alpha \subset \alpha'$ *and* $\beta \subset \beta'$ *such that*

$$T_{\alpha+\beta} f(t,x) = T_\alpha T_\beta f(t,x) \quad \text{uniformly holds on } \mathbb{T} \times S, \tag{3.9}$$

where S *is any compact set in* D.

4. Almost periodic dynamic equations on time scales

Consider the following almost periodic dynamic equation on time scale $\mathbb{T}$:

$$x^\Delta = f(t,x), \tag{4.1}$$

where $0 \in \mathbb{T}, f(t,x) \in C(\mathbb{T} \times D, \mathbb{R})$ is almost periodic in t uniformly for $x \in D, D$ denotes $\mathbb{R}^n$ or an open subset of $\mathbb{R}^n$.

Lemma 4.1. *Let* $f(t,x)$ *be uniformly almost periodic and* $g \in H(f)$. *Then there exists a sequence* $\alpha = \{\alpha_n\} \subset \mathbb{T}_p$ *such that* $\alpha_n \to \infty$ *as* $n \to \infty$ *and* $T_\alpha f(t,x) = g(t,x)$ *uniformly on* $\mathbb{T} \times S$ *for any compact set* $S \subset D$.

Proof. Since $g \in H(f)$, there exists a sequence $\alpha' \subset \mathbb{T}_p$ such that $T_{\alpha'} f(t,x) = g(t,x)$ uniformly on $\mathbb{T} \times S$. If $\alpha'_n \to \infty$ as $n \to \infty$, then we are done by letting $\alpha = \alpha'$. Otherwise, let $\varepsilon_n = 1/n$ and choose $\sigma'_n \in [-\alpha'_n + n, -\alpha_n + k + l(\varepsilon_n)]$ such that

$$\left|f(t + \sigma'_n, x) - f(t,x)\right| \leqslant 1/n \quad \text{for all } (t,x) \in \mathbb{T} \times S.$$

Then it follows that $f(t+\sigma'_n, x)$ uniformly converges to $f(t,x)$ on $\mathbb{T}\times S$ as $n\to\infty$, that is,

$$T_{\sigma'}f(t,x) = f(t,x) \quad \text{uniformly on } \mathbb{T}\times S.$$

Now by Proposition 3.11, for the sequences α' and σ' there exist subsequences $\alpha\subset\alpha'$ and $\sigma\subset\sigma'$ such that

$$T_{\alpha+\sigma}f(t,x) = T_\sigma T_\alpha f(t,x) = T_{\sigma'}T_\alpha f(t,x) = T_\alpha f(t,x) = T_{\alpha'}f(t,x)$$
$$= g(t,x) \text{ uniformly on } \mathbb{T}\times S.$$

By the choice of $\sigma' = \sigma'_n$, it is trivial that $\alpha'_n + \sigma' \to\infty$ as $n\to\infty$. Therefore, by replacing α' with $\alpha+\sigma$, which is a subsequence of $\alpha'+\sigma'$, we can fulfill the requirement. □

Definition 4.1. If $g\in H(f)$, we say that

$$x^\Delta = g(t,x)$$

is a hull equation of (4.1).

Theorem 4.1. *If $\phi(t)$ is a asymptotically almost periodic solution of (4.1) on $\mathbb{T}^+$, then (4.1) has an almost periodic solution.*

Proof. Since $\phi(t)$ is asymptotically almost periodic, it has the decomposition

$$\phi(t) = p(t) + q(t),$$

where $p(t)$ is almost periodic in t and $q(t)\to 0$ as $t\to\infty$. By Lemma 4.1, there exists a sequence $\alpha' = \{\alpha'_n\}\subset\mathbb{T}_p$ such that $\alpha'_n\to\infty$ as $n\to\infty$, and $T_{\alpha'}f(t,x) = g(t,x)$ uniformly on $\mathbb{T}\times S$. By Proposition 3.1, there exists a sequence $\alpha\subset\alpha'$ such that $T_\alpha p(t) = \psi(t)$ uniformly on $\mathbb{T}$. Since $T_\alpha f(t,x) = T_{\alpha'}f(t,x) = g(t,x)$ uniformly on $\mathbb{T}\times S$, we have that $T_\alpha\phi(t) = T_\alpha p(t) = \psi(t)$ is an almost periodic solution of the corresponding hull equation

$$x^\Delta = g(t,x)$$

on $\mathbb{T}$. Now $T_{-\alpha}g(t,x) = f(t,x)$ uniformly on $\mathbb{T}\times S$ and $T_{-\alpha}\psi(t) = p(t)$ uniformly on $\mathbb{T}$, hence $p(t)$ is an almost periodic solution of (4.1). □

Theorem 4.2. *If for each $g\in H(f)$, the hull equation $x^\Delta(t) = g(t,x(t))$ has a unique solution in S, then these solutions are almost periodic. In particular, system (4.1) has an almost periodic solution in S.*

Proof. Let $\phi(t)$ be the unique solution of $x^\Delta(t) = g(t,x(t))$ in S with $g\in H(f)$. For any given sequence $\alpha'\subset\mathbb{T}_p$, we will show that there is a subsequence $\alpha\subset\alpha'$ such that $T_a f(t+\alpha_n)$ exists uniformly on $t\in\mathbb{T}$, and hence by Proposition 3.2 we conclude that $\phi(t)$ is almost periodic.

Note that $g\in H(f)$ is uniformly almost periodic by Lemma 2.1. Then for a given sequence $\alpha'\subset\mathbb{T}_p$ we can pick $\alpha\subset\alpha'$ such that $T_a(t,x) = h(t,x)$ uniformly on $T\times S$. Trivially, $h\in H(g)\subset H(f)$. Since $\{\phi(t+\alpha_n)\}\in S$, we may choose a subsequence of α, denoted by α again, such that

$$\phi(t+\alpha_n)\to\phi^*(t) \quad \text{on any finite interval of } \mathbb{T}.$$

Obviously, $\phi^*(t)$ is a solution of

$$x^\Delta = h(t,x).$$

Suppose that $\phi(t+\alpha_n)$ is not convergent uniformly on $\mathbb{T}$ as $n\to\infty$. Then for some $\varepsilon_0>0$, there exist sequences $s' = \{s'_n\}\subset\mathbb{T}_p, \{m'_n\}\subset\mathbb{Z}$ and $\{k'_n\}\subset\mathbb{Z}$ such that

$$m'_n\to\infty, \quad k'_n\to\infty \quad \text{as } n\to\infty,$$

while

$$\left|\phi(s'_n+\alpha_{m'_n}) - \phi(s'_n+\alpha_{k'_n})\right| \geqslant \varepsilon_0. \tag{4.2}$$

Then by Proposition 3.11, for sequence $\{s'_n\}, \{\alpha_{m'_n}\}$, and $\{\alpha_{k'_n}\}$, there exist $\{s_n\}\subset\{t'_n\}, \{\alpha_{m_n}\}\subset\{\alpha_{m_n}\}$, and $\{\alpha_{k_n}\}\subset\{\alpha_{k'_n}\}$ such that

$$T_{s+\alpha_m}g(t,x) = T_{\alpha_m}T_s(t,x)$$

and

$$T_{s+\alpha_k}g(t,x) = T_{\alpha_k}T_s(t,x)$$

hold uniformly on $\mathbb{T}\times S$.

On the other hand, there exist some functions $\varphi(t)$ and $\psi(t)$ such that $T_{s+\alpha_m}\phi(t) = \varphi(t)$, $T_{s+\alpha_k}\phi(t) = \psi(t)$ on any interval of $\mathbb{T}$. Since that $T_{\alpha_m}g(t,x) = T_{\alpha_k}g(t,x) = h(t,x)$, we have $T_{s+\alpha_m}g(t,x) = T_{s+\alpha_k}g(t,x) = T_s h(t,x) = l(t,x)$ for some $l\in H(h)$, taking a subsequence if necessary.

Thus, $\varphi(t)$ and $\psi(t)$ are both the solutions in S of

$$x^\Delta(t) = l(t,x(t)).$$

Note that $l\in H(h)\subset H(g)\subset H(f)$, by the assumption we must have $\varphi(t)\equiv\psi(t)$.

However, it follows from (4.2) that

$$|\varphi(0)-\psi(0)|\geqslant\varepsilon_0.$$

This is a contradiction. Therefore, $\phi(t)$ is an almost periodic solution in S of $x^\Delta(t) = g(t,x(t))$.

In particular, since $f\in H(f)$, we conclude that (4.1) has an almost periodic solution in S. □

Lemma 4.2 24. *Let $y,f\in C_{rd}$ and $p\in\mathcal{R}^+$, then*

$$D^+y^\Delta(t)\leqslant p(t)y(t)+f(t) \quad \text{for all} \quad t\in\mathbb{T}$$

implies

$$y(t)\leqslant y(t_0)e_p(t,t_0) + \int_{t_0}^t e_p(t,\sigma(\tau))f(\tau)\Delta\tau \quad \text{for all} \quad t\in\mathbb{T},$$

where $t_0\in\mathbb{T}$.

Now, by using Liapunov functions on time scales, we investigate the existence of an almost periodic solution of (4.1) which is uniformly asymptotically stable in the whole, that is, every solution which remains in D in the future approaches the almost periodic solution as $t\to\infty$. To this end, for system (4.1), we consider its product system:

$$x^\Delta = f(t,x), \quad y^\Delta = f(t,y). \tag{4.3}$$

Theorem 4.3. *Suppose that there exists a Liapunov function $V(t,x,y)\in C(\mathbb{T}^+\times D\times D,\mathbb{R})$ satisfying the following conditions:*

(i) *$a(|x-y|)\leqslant V(t,x,y)\leqslant b(|x-y|)$, where $a,b\in\mathcal{K}$ with $\mathcal{K} = \{a\in C(R^+,R^+) : a(0)=0$ and a is increasing$\}$;*

(ii) $|V(t,x_1,y_1) - v(t,x_2,y_2)| \leqslant L[|x_1 - x_2| + |y_1 - y_2|]$, *where* $L > 0$ *is a constant;*

(iii) $D^+V^{\Delta}_{(4.3)}(t,x,y) \leqslant -cV(t,x,y)$ *where* $-c \in \mathcal{R}^+$ *and* $c > 0$.

Moreover, if there exists a solution $x(t)$ *of* (4.1) *such that* $x(t) \in S$ *for* $t \in \mathbb{T}^+$, *where* $S \subset D$ *is a compact set. Then there exists a unique uniformly asymptotically stable almost periodic solution* $p(t)$ *of* (4.1) *in* S. *Furthermore, if* $f(t,x)$ *is periodic with period* ω *in* t, *then* $p(t)$ *is a periodic solution of* (4.1) *with period* ω.

Proof. Let $\alpha = \{\alpha_n\} \subset \mathbb{T}_p$ be a sequence such that $\alpha'_n \to \infty$ as $n \to \infty$ and $T_\alpha f(t,x) = f(t,x)$ uniformly on $\mathbb{T} \times S$. Assume that $\phi(t) \subset S$ is a solution of (6.1) for $t \in \mathbb{T}^+$. Then $\phi(t + \alpha_n)$ is a solution of the dynamic equation

$$x^{\Delta} = f(t + \alpha_n, x),$$

which is also in S. For a given $\varepsilon > 0$, choose an integer $k_0(\varepsilon)$ so large that if $m \geqslant k > k_0(\varepsilon)$, we have

$$b(2B)e_{-c}(\alpha_k, 0) < a(\varepsilon)/2 \tag{4.4}$$

and

$$|f(t + \alpha_k, x) - f(t + \alpha_m, x)| < \frac{ca(\varepsilon)}{2L}, \tag{4.5}$$

where B is a positive constant such that $S \subset \{x : |x| \leqslant B\}$. It follows from (*ii*) and (*iii*) that

$$\begin{aligned} D^+V^{\Delta}(t, \phi(t), \phi(t - \alpha_m - \alpha_k)) &\leqslant -cV(t, \phi(t), \phi(t - \alpha_m - \alpha_k)) \\ &\quad + L|f(t + \alpha_m - \alpha_k, \phi(t + \alpha_m \\ &\quad - \alpha_k)) - f(t, \phi(t + \alpha_m - \alpha_k)))|. \end{aligned}$$

In virtue of (4.4), we have

$$\begin{aligned} &D^+V^{\Delta}(t, \phi(t), \phi(t - \alpha_m - \alpha_k)) \\ &\quad \leqslant -cV(t, \phi(t), \phi(t - \alpha_m - \alpha_k)) + \frac{ca(\varepsilon)}{2}. \end{aligned} \tag{4.6}$$

If $m \geqslant k \geqslant {}_0(\varepsilon)$, by Lemma 4.2, condition (*i*) and (4.3) and (4.5) imply that

$$\begin{aligned} V(t + \alpha_k, \phi(t + \alpha_k), \phi(t - \alpha_m)) &\leqslant e_{-c}(t + \alpha_m, 0)V(0, \phi(0), \phi(\alpha_m \\ &\quad - \alpha_k)) + \frac{a(\varepsilon)}{2}(1 - e_{-c}(t \\ &\quad + \alpha_k, 0)) \\ &\leqslant e_{-c}(t + \alpha_m, 0)V(0, \phi(0), \phi(\alpha_m \\ &\quad - \alpha_k)) + \frac{a(\varepsilon)}{2} \\ &< a(\varepsilon) \quad \text{for } t \in \mathbb{T}^+. \end{aligned}$$

Therefore, by condition (*i*), we have

$$|\phi(t + \alpha_k) - \phi(t + \alpha_m)| < \varepsilon \quad \text{for all } t \in \mathbb{T}^+ \quad \text{if } m \geqslant k \geqslant k_0.$$

This shows that $\phi(t)$ is asymptotically almost periodic, and thus system (4.1) has an almost periodic solution $p(t) \subset S$ in virtue of Theorem 4.1.

By using the Liapunov function $V(t,x,y)$, with the standard arguments, it is easy to show that $p(t)$ is uniformly asymptotically stable and every solution remaining in D approaches $p(t)$ as $t \to \infty$, which also implies the uniqueness of $p(t)$.

In the case where $f(t,x)$ is periodic in t with period $\omega, p(t + \omega)$ is also a solution of (4.1) which remains in S. By the uniqueness of solution, we know that $p(t + \omega) = p(t)$. This completes the proof. □

Example 4.1. Consider the following dynamic equation on time scale $\mathbb{T}$:

$$x^{\Delta} = a(t)f(x) + p(t), \tag{4.7}$$

where $a(t)$, $p(t)$ is almost periodic on $\mathbb{T}, f(x)$ is monotone increasing and $f(-\infty) = -\infty, f(+\infty) = +\infty$. Let $\mu^* = \sup_{t\in\mathbb{T}}\{\mu(t)\}$, if there exist constants α, β such that $f'(x) \geqslant \alpha > 0, a(t) \leqslant -\beta < 0$ and $2 - \mu^*\alpha\beta > 0$, then there exists a unique uniformly asymptotically stable almost periodic solution of (4.7).

Proof. We first prove that (4.7) has bounded solutions on $\mathbb{T}^+$. Denote the right-hand of (4.7) by $F(t,x)$, it is easy to see that there exists a constant $M_0 > 0$ such that $F(t, M_0) < 0, f(t, -M_0) > 0$ for $t \in \mathbb{T}^+$. Therefore, the solution of (4.7) with initial value $(0,x_0)$ satisfies $|x(t;0,x_0)| < M_0$ for $t \in \mathbb{T}^+$ while $|x_0| < M_0$. For any $M > M_0$, we obtain that the set $\Omega = \{x(t) : x(t)$ is a solution of (4.7) and $|x(t)| < M$ for $t \in \mathbb{T}^+\} \neq \emptyset$.

Consider product system:

$$x^{\Delta} = a(t)f(x) + p(t), \quad y^{\Delta} = a(t)f(y) + p(t).$$

For $x, y \in \Omega$, let Liapunov function $V(t,x,y) = (x - y)^2$. Obviously,

$$\frac{1}{2}|x - y|^2 \leqslant V(t,x,y) \leqslant 2|x - y|^2.$$

Set $a = \frac{1}{2}x^2, b = 2x^2$, condition (*i*) of Theorem 4.3 is satisfied.

Next,

$$\begin{aligned} |V(t,x_1,y_1) - V(t,x_2,y_2)| &= |(x_1 - y_1)^2 - (x_2 - y_2)^2| \\ &= |(x_1 - y_1) + (x_2 - y_2)||(x_1 - y_1) \\ &\quad - (x_2 - y_2)| \leqslant 4M(|x_1 - x_2| + |y_1 - y_2|). \end{aligned}$$

Let $L = 4M$, condition (*ii*) of Theorem 4.3 is satisfied.

At last,

$$\begin{aligned} D^+V^{\Delta}_{(4.3)}(t,x,y) &= [2(x - y) + \mu(t)(x - y)^{\Delta}](x - y)^{\Delta} \\ &= [2(x - y) + \mu(t)(a(t)(f(x) \\ &\quad - f(y)))](a(t)(f(x) - f(y))) \\ &= [2(x - y) + \mu(t)(a(t)f'(\xi)(x - y))](a(t)f'(\xi) \\ &\quad \times (x - y)) \leqslant -\alpha\beta(2 - \mu^*\alpha\beta)(x - y)^2. \end{aligned}$$

Set $c = \alpha\beta(2 - \mu^*\alpha\beta)$, then $c > 0$ and $-c \in \mathcal{R}^+$, so condition (*iii*) of Theorem 4.3 is also satisfied.

Therefore, by using Theorem 4.3, there exists a unique uniformly asymptotically stable almost periodic solution of (4.7). The proof is complete. □

Acknowledgments

This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grant 10971183, and this work is also supported by IRTSTYN.

References

[1] E.R. Kaufmann, Y.N. Raffoul, Periodic solutions for a neutral nonlinear dynamical equation on a time scale, J. Math. Anal. Appl. 319 (2006) 315–325.

[2] C. Wang, Y. Li, Yu Fei, Three positive periodic solutions to nonlinear neutral functional differential equations with impulses and parameters on time scales, Math. Comput. Modell. 52 (2010) 1451–1462.

[3] H.T. Zhang, Y.K. Li, Existence of positive periodic solutions for functional differential equations with impulse effects on time scales, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 19–26.

[4] L. Zhang, H.X. Li, X.B. Zhang, Periodic solutions of competition Lotka-Volterra dynamic system on time scales, Comput. Math. Appl. 57 (2009) 1204–1211.

[5] X.L. Liu, W.T. Li, Periodic solutions for dynamic equations on time scales, Nonlinear Anal. 67 (2007) 1457–1463.

[6] Y.K. Li, H.T. Zhang, Existence of periodic solutions for a periodic mutualism model on time scales, J. Math. Anal. Appl. 343 (2008) 818–825.

[7] M. Fazly, M. Hesaaraki, Periodic solutions for predator-prey systems with Beddington-DeAngelis functional response on time scales, Nonlinear Anal. Real World Appl. 9 (2008) 1224–1235.

[8] F. Geng, Y. Xu, D. Zhu, Periodic boundary value problems for first-order impulsive dynamic equations on time scales, Nonlinear Anal. 69 (2008) 4074–4087.

[9] G. Seifert, On uniformly almost periodic sets of functions for almost periodic differential equations, Tohoku Math. J. 34 (1982) 301–309.

[10] S. Bochner, A new approach to almost periodicity, Proc. Nat. Acad. Sci. USA 48 (1962) 2039–2043.

[11] A.M. Fink, Almost Periodic Differential Equations, Lecture Notes in Mathematics, vol. 377, Springer-Verlag, New York, Berlin, 1974.

[12] A.M. Fink, G. Seifert, Liapunov functions and almost periodic solutions for almost periodic systems, J. Diff. Eqs. 5 (1969) 307–313.

[13] D. Cheban, C. Mammana, Invariant manifolds, global attractors and almost periodic solutions of nonautonomous difference equations, Nonlinear Anal. 56 (4) (2004) 465–484.

[14] C.Y. He, Almost Periodic Differential Equations, Higher Education Press, Beijing, 1992 (in Chinese).

[15] C. Corduneanu, Almost Periodic Functions, second ed., Chelsea, New York, 1989.

[16] G.M. N'Guérékata, Almost Automorphic Functions and Almost Periodic Functions in Abstract Spaces, Kluwer Academic, Plenum Publishers, New York, London, Moscow, 2001.

[17] C.Y. Zhang, Almost Periodic Type Functions and Ergodicity, Science Press, Kluwer Academic Publishers, New York, 2003.

[18] C.Y. Zhang, Pseudo almost periodic solutions of some differential equations, J. Math. Anal. Appl. 151 (1994) 62–76.

[19] A. Fink, J. Gatica, Positive almost periodic solutions of some delay epidemic equations, J. Diff. Eqs. 83 (1990) 166–178.

[20] Y. Hino, N.V. Minh, J.S. Shin, Almost Periodic Solutions of Differential Equations in Banach Spaces, Taylor & Francis, London, New York, 2002.

[21] E.H.A. Dads, P. Cieutat, K. Ezzinbi, The existence of pseudo-almost periodic solutions for some nonlinear differential equations in Banach spaces, Nonlinear Anal. 69 (2008) 1325–1342.

[22] Y.K. Li, T.W. Zhang, Permanence and almost periodic sequence solution for a discrete delay logistic equation with feedback control, Nonlinear Anal. Real World Appl. 12 (2011) 1850–1864.

[23] T. Yoshizawa, Stability theory and the existence of periodic solutions and almost periodic solutions, Appl. Math. Sciences, 14, Springer-Verlag, 1975.

[24] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.