Journal of the Egyptian Mathematical Society 25 (2017) 71–78

Contents lists available at ScienceDirect

Journal of the Egyptian Mathematical Society

journal homepage: www.elsevier.com/locate/joems

Original article

Estimations from the exponentiated rayleigh distribution based on generalized Type-II hybrid censored data

M.A.W. Mahmoud[a], M.G.M. Ghazal[b,*]

[a] Mathematics Department, Faculty of Science, Al-Azhar University, Nasr City, Cairo 11884, Egypt
[b] Mathematics Department, Faculty of Science, Minia University, El-Minia, Egypt

ARTICLE INFO

Article history:
Received 31 March 2016
Revised 18 June 2016
Accepted 23 June 2016
Available online 7 July 2016

MSC:
62F10
62F15
62N01
62N02

Keywords:
Exponentiated Rayleigh distribution
Hybrid censored data
Maximum likelihood estimators
Confidence intervals
Bootstrap method
Symmetric and asymmetric loss functions

ABSTRACT

In this article, unknown parameters of exponentiated Rayleigh distribution based on generalized Type II Hybrid censored data, survival function, failure rate function and coefficient of variation are derived by applying the maximum likelihood, Bayes and percentile bootstrap methods. Approximate confidence intervals for the unknown parameters, survival function, failure rate function and coefficient of variation are obtained. We study Bayes estimates under gamma priors distributions depending on symmetric and asymmetric loss functions via the Gibbs within Metropolis-Hasting samplers procedure. Finally, the proposed methods can be understood through illustrating the results of the real data analysis.

Copyright 2016, Egyptian Mathematical Society. Production and hosting by Elsevier B.V.
This is an open access article under the CC BY-NC-ND license.
(http://creativecommons.org/licenses/by-nc-nd/4.0/)

1. Introduction

Epstein [1] has introduced the hybrid censoring scheme (HCS) as a mixture of Type-I and Type-II censoring schemes. In Type-I HCS, the test is terminated at a time $T_1^* = \min(X_{r:n}, T)$ where $X_{r:n}$ is the failure time of the rth item while T is the pre-fixed maximum allowable time to the life-test. In Type-II HCS, the life-test is terminated at a time $T_2^* = \max(X_{r:n}, T)$. According to Chandrasekar et al. [2] Type-I and Type-II HCS's have some essential drawbacks. In Type-I HCS, very few or even no failures are found, while the experiment can last for a too long time in Type-II HCS. As a result, they propose two generalized Type-I and Type-II HCS's. They describe the generalized Type-II HCS as follows: Fix $r \in \{1, \ldots, n\}$ and time T_1 and $T_2 \in \{1, \infty\}$ where $T_2 > T_1$. When the rth failure occurs before time T_1, then the experiment terminate at T_1. When the rth failure occurs between T_1 and T_2, then the experiment terminate at $x_{r:n}$. Finally, the rth failure occurs before time T_2, then the experiment terminates at T_2. Many authors have studied generalized Type-II HCS, see Balakrishnan and Kundu [3] and Shafay [4].

The exponentiated Rayleigh distribution has many characteristics which are quite common to gamma, Weibull and exponentiated exponential distributions. The exponentiated Rayleigh distribution for the distribution function and the density function are found to have closed forms. Consequently, it can be applied very compatibly even on censored data. The exponentiated Rayleigh distribution with parameters β and α denoted by ERD(β, α). The probability density function (PDF), cumulative distribution function (CDF), survival function $R(t)$, and failure rate function $H(t)$ of the two-parameter ERD(β, α) are given, respectively, by

$$f(x; \beta,\ \alpha) = 2\alpha\beta\ x\ e^{-\beta x^2}(1 - e^{-\beta x^2})^{\alpha-1}\ \ x > 0,\ \ \beta > 0,\ \ \alpha > 0, \tag{1.1}$$

$$F(x; \beta,\ \alpha) = (1 - e^{-\beta x^2})^{\alpha}\quad x > 0,\ \ \beta > 0,\ \ \alpha > 0, \tag{1.2}$$

$$R(t) = 1 - (1 - e^{-\beta t^2})^{\alpha}\quad t > 0, \tag{1.3}$$

$$H(t) = \frac{2\alpha\beta\ t\ e^{-\beta t^2}(1 - e^{-\beta t^2})^{\alpha-1}}{1 - (1 - e^{-\beta t^2})^{\alpha}}\quad t > 0. \tag{1.4}$$

* Corresponding author.
E-mail addresses: mawmahmoud11@hotmail.com (M.A.W. Mahmoud), eghazal39@yahoo.com (M.G.M. Ghazal).

http://dx.doi.org/10.1016/j.joems.2016.06.008
1110-256X/Copyright 2016, Egyptian Mathematical Society. Production and hosting by Elsevier B.V. This is an open access article under the CC BY-NC-ND license. (http://creativecommons.org/licenses/by-nc-nd/4.0/)

The coefficient of variation (CV) under the ERD(β,α) is given by

$$CV = \frac{\sqrt{E\left(X^2\right) - (E(X))^2}}{E(X)}, \tag{1.5}$$

where $E(X)$ and $E(X^2)$ are the first and the second moments of the ERD(β, α), can be obtained from

$$\begin{aligned} E(X^r) &= 2\alpha\beta \int_0^\infty x^{r+1} e^{-\beta x^2}\left(1-e^{-\beta x^2}\right)^{\alpha-1} dx \\ &= 2\alpha\beta \sum_{k=0}^{M(\alpha)} (-1)^k \binom{\alpha-1}{k} \int_0^\infty x^{r+1} e^{-(k+1)\beta x^2} dx \\ &= \frac{\alpha}{\beta^{\frac{r}{2}}} \sum_{k=0}^{M(\alpha)} \frac{(-1)^k \binom{\alpha-1}{k}}{(k+1)^{\frac{r}{2}+1}} \Gamma\left(\frac{r}{2}+1\right), \end{aligned} \tag{1.6}$$

where $M(\alpha) = \begin{cases} \alpha-1, & \alpha \in \mathbb{Z}^+, \quad \alpha \ge 1, \\ \infty, & \text{otherwise.} \end{cases}$

The article is organized as follows. In the next section, the maximum likelihood estimators (MLEs) of the unknown parameters, survival function, failure rate function and CV are discussed. In Section 3, asymptotic confidence intervals (CIs) based on the ML estimates are obtained. In Section 4, we demonstrate the percentile bootstrap (Boot-p) method to construct the CIs for the unknown parameters and any function on them. Markov Chain Monte Carlo (MCMC) for estimating the posterior distribution of the unknown parameters, survival function, failure rate function and CV and its interval estimation are obtained under symmetric and asymmetric loss functions in Section 5. Real data set has been analyzed for illustrative purposes in Section 6, while conclusions in Section 7.

2. Maximum likelihood estimation

In this section we derive the MLEs of the unknown parameters of ERD(β, α) under generalized Type II HCS, survival function, failure rate function and CV. According to the generalized Type-II HCS outlined above, we get three cases form of observations as follows:

Case I. $\{X_{1:n} < \ldots < X_{r:n}\}$ if $X_{r:n} < \ldots < X_{D_1:n} < T_1$;
Case II. $\{X_{1:n} < \ldots < X_{D_1:n} < \ldots < X_{r:n}\}$ if $T_1 < X_{r:n} < T_2$;
Case III. $\{X_{1:n} < \ldots < X_{D_1:n} < T_2\}$ if $X_{r:n} \ge T_2$,

Since D_i indicate the number of failures that occur before time T_i, $i = 1, 2$.

The likelihood function of the generalized Type-II hybrid censored sample $\underline{x} = X_{1:n} < \ldots < X_{R:n}$ is

$$L(\phi|\underline{x}) = \frac{n!}{(n-R)!} \prod_{i=1}^{R} f(x_i)\, [1-F(c)]^{n-R}, \tag{2.1}$$

where

$$R = \begin{cases} D_1 & \text{for Case I} \\ r & \text{for Case II} \\ D_2 & \text{for Case III} \end{cases}, \quad c = \begin{cases} T_1 & \text{for Case I} \\ x_r & \text{for Case II} \\ T_2 & \text{for Case III} \end{cases} \tag{2.2}$$

Using (1.1) and (1.2) in (2.1), we get the likelihood function of (β, α) under the generalized Type-II HCS as

$$L(\beta,\ \alpha|\underline{x}) = A\, \alpha^R \beta^R e^{-\beta \sum_{i=1}^{R} x_i^2} \prod_{i=1}^{R}\left(1-e^{-\beta x_i^2}\right)^{\alpha-1} \left[1-\left(1-e^{-\beta C^2}\right)^{\alpha}\right]^{n-R}, \tag{2.3}$$

where $A = \frac{n!}{(n-R)!} \prod_{i=1}^{R} x_i$.

The log-likelihood function for the parameters β and α is

$$\ell = \ln A + R\ln\alpha\beta - \beta\sum_{i=1}^{R} x_i^2 + (\alpha-1)\sum_{i=1}^{R}\ln\left(1-e^{-\beta x_i^2}\right) + (n-R)\ln\left[1-\left(1-e^{-\beta C^2}\right)^{\alpha}\right] \tag{2.4}$$

The MLEs of the parameters β and α are obtained by solving the following likelihood equations simultaneously:

$$\frac{R}{\hat\beta} - \sum_{i=1}^{R} x_i^2 + (\hat\alpha-1)\sum_{i=1}^{R}\frac{x_i^2 e^{-\hat\beta x_i^2}}{1-e^{-\hat\beta x_i^2}} - \frac{\hat\alpha\ (n-R)\ C^2 e^{-\hat\beta x_i^2}\left(1-e^{-\hat\beta C^2}\right)^{\hat\alpha-1}}{1-\left(1-e^{-\hat\beta C^2}\right)^{\hat\alpha}} = 0, \tag{2.5}$$

and

$$\frac{R}{\hat\alpha} + \sum_{i=1}^{R}\ln\left(1-e^{-\hat\beta x_i^2}\right) - \frac{(n-R)\ln\left(1-e^{-\hat\beta C^2}\right)\left(1-e^{-\hat\beta C^2}\right)^{\hat\alpha}}{1-\left(1-e^{-\hat\beta C^2}\right)^{\hat\alpha}} = 0. \tag{2.6}$$

The Eqs. (2.5) and (2.6) cannot be solved analytically for β and α, numerical methods are used.

The ML estimators of $R(t)$, $H(t)$ and CV can be obtained after replacing β and α by their ML estimators $\hat\beta$ and $\hat\alpha$ as

$$\begin{cases} \hat R(t) = 1-\left(1-e^{-\hat\beta t^2}\right)^{\hat\alpha} \\ \hat H(t) = \dfrac{2\hat\alpha\hat\beta\ t\ e^{-\hat\beta\ t^2}\left(1-e^{-\hat\beta t^2}\right)^{\hat\alpha-1}}{1-\left(1-e^{-\hat\beta t^2}\right)^{\hat\alpha}} \\ \widehat{CV} = \dfrac{\sqrt{\frac{\hat\alpha}{\hat\beta}\sum_{k=0}^{M(\alpha)}\frac{(-1)^k\binom{\hat\alpha-1}{k}}{(k+1)^2}\Gamma(2) - \left(\frac{\hat\alpha}{\hat\beta^{\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{(-1)^k\binom{\hat\alpha-1}{k}}{(k+1)^{\frac{3}{2}}}\Gamma\left(\frac{3}{2}\right)\right)^2}}{\frac{\hat\alpha}{\hat\beta^{\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{(-1)^k\binom{\hat\alpha-1}{k}}{(k+1)^{\frac{3}{2}}}\Gamma\left(\frac{3}{2}\right)} \end{cases} \tag{2.7}$$

3. Confidence interval

The asymptotic variances and covariances of the MLE for parameters $\hat\beta$ and $\hat\alpha$ are given by elements of the inverse of the Fisher information matrix defined as

$$I_{ij} = -E\left[\frac{\partial^2\ell}{\partial\varphi_i\partial\varphi_j}\right];\ \varphi_1 = \beta,\ \varphi_2 = \alpha \text{ for } i, j = 1, 2.$$

The Fisher information matrix is replaced by its estimate, the observed information

$$[\hat w] = \begin{bmatrix} \frac{\partial^2\ell}{\partial\beta^2} & \frac{\partial^2\ell}{\partial\beta\partial\alpha} \\ \frac{\partial^2\ell}{\partial\alpha\partial\beta} & \frac{\partial^2\ell}{\partial\alpha^2} \end{bmatrix}^{-1}_{(\hat\beta,\ \hat\alpha)} = \begin{bmatrix} \widehat{\operatorname{var}}\left(\hat\beta\right) & \operatorname{cov}\left(\hat\beta,\ \hat\alpha\right) \\ \operatorname{cov}\left(\hat\beta,\ \hat\alpha\right) & \widehat{\operatorname{var}}\left(\hat\alpha\right) \end{bmatrix}, \tag{3.1}$$

where $\frac{\partial^2\ell}{\partial\varphi_i\partial\varphi_j}$ is the second derivation obtained in (2.4).

Then, the $100(1-\tau)\%$ two sided CIs of β and α, are given by

$$\left(\hat{\beta} \pm z_{\tau/2}\sqrt{\widehat{\text{var}}\left(\hat{\beta}\right)}\right) \text{ and } \left(\hat{\alpha} \pm z_{\tau/2}\sqrt{\widehat{\text{var}}\left(\hat{\alpha}\right)}\right), \tag{3.2}$$

where $z_{\tau/2}$ is the upper $(\tau/2)$ quantile of the standard normal distribution.

Moreover, to construct the asymptotic CIs of the $R(t)$, $H(t)$ and CV, we need to get the variances of them. We use the delta method to find the variance of $\hat{R}(t)$, $\hat{H}(t)$ and $\widehat{CV}$, see Greene [5]. The variance of $\hat{R}(t)$, $\hat{H}(t)$ and $\widehat{CV}$ can be approximated, respectively by $\hat{\sigma}^2_{\hat{R}(t)} = [\nabla\hat{R}(t)]^T[\hat{w}][\nabla\hat{R}(t)]$, $\hat{\sigma}^2_{\hat{H}(t)} = [\nabla\hat{H}(t)]^T[\hat{w}][\nabla\hat{H}(t)]$ and

$$\hat{\sigma}^2_{\widehat{CV}} = \left[\nabla\widehat{CV}\right]^T\left[\hat{w}\right]\left[\nabla\widehat{CV}\right],$$

where $\nabla\hat{R}(t), \nabla\hat{H}(t)$ and $\nabla\widehat{CV}$ are the gradient of $\hat{R}(t)$, $\hat{H}(t)$ and $\widehat{CV}$ with respect to β and α. Then, the $100(1-\tau)\%$ two sided CIs of $R(t)$, $H(t)$ and CV, can be written as

$$\left(\hat{R}(t) \pm z_{\tau/2}\sqrt{\hat{\sigma}^2_{\hat{R}(t)}}\right),\ \left(\hat{H}(t) \pm z_{\tau/2}\sqrt{\hat{\sigma}^2_{\hat{H}(t)}}\right) \text{ and } \left(\widehat{CV} \pm z_{\tau/2}\sqrt{\hat{\sigma}^2_{\widehat{CV}}}\right). \tag{3.3}$$

4. Bootstrap confidence intervals

In this section, we use the parametric Boot-p method proposed by Efron and Tibshirani [6] to construct the Boot-p CIs of β, α, $R(t)$, $H(t)$ and CV. The algorithm of estimating approximate CIs of β, α, $R(t)$, $H(t)$ and CV utilization Boot-p method is illustrated below

(1) Estimate the MLEs β, α, say $\hat{\beta}$ and $\hat{\alpha}$ from the original data $\underline{x} = X_{1:n} < \ldots < X_{n:n}$, using (2.5) and (2.6).
(2) Get a bootstrap sample $\underline{x}^* = X^*_{1:n} < \ldots < X^*_{n:n}$ by resampling with replacement.
(3) Compute the MLEs under the bootstrap sample and indicate bootstrap estimate by $\hat{\psi}^*$ (ψ can be β, α, $R(t)$, $H(t)$ or CV).
(4) Repeat Steps 2 and 3 NBoot times, and get $\hat{\psi}^*_1$, $\hat{\psi}^*_2, \ldots, \hat{\psi}^*_{\text{NBoot}}$, where $\hat{\psi}^*_j = (\hat{\beta}^*_j,\ \hat{\alpha}^*_j,\ \hat{R}^*_j,\ \hat{H}^*_j,\ \widehat{\text{CV}}^*_j)$, $j = 1,\ 2, \ldots, \text{NBoot}$.
(5) Arrange $\hat{\psi}^*_j$, $j = 1,\ 2, \ldots, \text{NBoot}$ in ascending orders and get $(\hat{\psi}^*_{(1)},\ \hat{\psi}^*_{(2)}, \ldots, \hat{\psi}^*_{(\text{NBoot})})$.
(6) Let $\hat{\text{G}}_1(x) = \text{P}(\hat{\psi}^* \le x)$ is the CDF of $\hat{\psi}^*$. Define $\hat{\psi}_{\text{Boot-P}}(x) = \hat{\psi}_1^{-1}(x)$ for a given x. The approximate $100(1-\tau)\%$ confidence interval of ψ is $(\hat{\psi}_{\text{Boot-P}}(\frac{\tau}{2}),\ \hat{\psi}_{\text{Boot-P}}(1-\frac{\tau}{2}))$.

5. Bayes estimation based on MCMC

In this section, we get Bayesian estimates of β and α, in addition to some lifetime parameters $R(t)$, $H(t)$ and CV against the squared error, LINEX and entropy loss functions. Assuming that β and α follows the gamma prior distributions.

$$\pi_1(\beta) \propto\ \beta^{a_1-1}\ e^{-b_1\ \beta},\ \beta>0,\ a_1,\ b_1>0, \tag{5.1}$$

$$\pi_2(\alpha) \propto\ \alpha^{a_2-1}\ e^{-b_2\alpha},\ \alpha>0,\ a_2,\ b_2>0. \tag{5.2}$$

The joint posterior density function of β and α given the data can be written as

$$\pi^*(\beta,\ \alpha|\underline{x}) = \frac{\pi_1(\beta)\ \pi_2(\alpha)\ L(\beta,\ \alpha|\underline{x})}{\int_0^\infty\int_0^\infty \pi_1(\beta)\ \pi_2(\alpha)\ L(\beta,\ \alpha|\underline{x})\ d\beta\ d\alpha}. \tag{5.3}$$

Thus, the Bayes estimate of $g(\beta,\alpha)$ based on squared error loss (SEL) function is

$$\hat{g}_{BS}(\beta,\ \alpha|\underline{x}) = E_{\beta,\ \alpha|\underline{x}}(g(\beta,\alpha)) = \frac{\int_0^\infty\int_0^\infty g(\beta,\alpha)\ \pi_1(\beta)\ \pi_2(\alpha)\ L(\beta,\ \alpha|\underline{x})\ d\beta\ d\alpha}{\int_0^\infty\int_0^\infty \pi_1(\beta)\ \pi_2(\alpha)\ L(\beta,\ \alpha|\underline{x})\ d\beta\ d\alpha} \tag{5.4}$$

Varian [7] introduced the LINEX loss function $L(\Delta)$ for a parameter $\phi = \phi(\beta,\alpha)$ can be written as

$$L(\Delta) \propto e^{a\Delta} - a\Delta - 1,\ a \ne 0,\ \Delta = \hat{\phi} - \phi, \tag{5.5}$$

The sign and magnitude of the shape parameter a represents the direction and degree of symmetry. Several authors examine asymmetric loss functions in reliability and life testing, such as Basu and Ebrahimi [8] and Essam [9].

The Bayes estimate of a function $g(\beta,\alpha)$ according to LINEX loss function in (5.5) is

$$\hat{g}_{BL}(\beta,\ \alpha|\underline{x}) = \frac{-1}{a}\ln\left[\text{E}\left(e^{-ag(\beta,\alpha)}|\underline{x}\right)\right],\ a \ne 0, \tag{5.6}$$

where

$$\text{E}\left(e^{-ag(\beta,\alpha)}|\underline{x}\right) = \frac{\int_0^\infty\int_0^\infty e^{-ag(\beta,\alpha)}\ \pi_1(\beta)\ \pi_2(\alpha)\ L(\beta,\ \alpha|\underline{x})\ d\beta\ d\alpha}{\int_0^\infty\int_0^\infty \pi_1(\beta)\ \pi_2(\alpha)\ L(\beta,\ \alpha|\underline{x})\ d\beta\ d\alpha}. \tag{5.7}$$

Also, Calabria [10] proposed the modified LINEX loss function, called General entropy loss function (GEL), is defined as:

$$L\left(\hat{\phi} - \phi\right) \propto \left(\frac{\hat{\phi}}{\phi}\right)^a - a\ln\left(\frac{\hat{\phi}}{\phi}\right) - 1. \tag{5.8}$$

It may be noted that when $a > 0$, a positive error causes more serious consequences than a negative error. Further, whilst $a < 0$, a negative error causes more serious consequences than a positive error. Under GEL (5.8), the Bayes estimator of $g(\beta,\alpha)$ is given as

$$\hat{g}_{BGE}(\beta,\ \alpha|\underline{x}) = \left[E(g(\beta,\alpha)^{-a}|\underline{x})\right]^{\frac{-1}{a}}, \tag{5.9}$$

provided that $E(g(\beta,\alpha)^{-a}|\underline{x})$ exists and is finite, where

$$E(g(\beta,\alpha)^{-a}|\underline{x}) = \frac{\int_0^\infty\int_0^\infty g(\beta,\alpha)^{-a}\ \pi_1(\beta)\ \pi_2(\alpha)\ L(\beta,\ \alpha|\underline{x})\ d\beta\ d\alpha}{\int_0^\infty\int_0^\infty \pi_1(\beta)\ \pi_2(\alpha)\ L(\beta,\ \alpha|\underline{x})\ d\beta\ d\alpha}. \tag{5.10}$$

It should be noted that, the ratio of two integrals in (5.4), (5.7) and (5.10) cannot be obtained in a closed form. So, we use the MCMC approximation method to generate samples from (5.11) and then calculation the Bayes estimate of β and α and any function of them such as $R(t)$, $H(t)$ and CV and also to construct associated CIs. Gibbs and Metropolis sampler are used to derive the complete set of conditional posterior distribution. From (5.3), the joint posterior up to proportionality can be written as

$$\pi^*(\beta,\alpha|\underline{x}) \propto \beta^{R+a_1-1}\alpha^{R+a_2-1}e^{-\beta(b_1+\sum_{i=1}^R x_i^2)}e^{-\alpha\left[b_2-\sum_{i=1}^R\ln\left(1-e^{-\beta x_i^2}\right)\right]} \times e^{-\sum_{i=1}^R\ln\left(1-e^{-\beta x_i^2}\right)]}\left[1-\left(1-e^{-\beta C^2}\right)^{\alpha-1}\right]^{n-R}. \tag{5.11}$$

From (5.11), the posterior density function of α given β is

$$\pi_1^*(\alpha|\beta,\underline{x}) = \alpha^{R+a_2-1}e^{-\alpha\left[b_2-\sum_{i=1}^R\ln\left(1-e^{-\beta x_i^2}\right)\right]}. \tag{5.12}$$

Thus $\pi_1^*(\alpha|\beta,\underline{x})$ is gamma with parameters as $R+a_2$ and $b_2 - \sum_{i=1}^R\ln(1-e^{-\beta x_i^2})$.

The posterior density function of β given α is

$$\pi_2^*(\beta|\alpha,\underline{x}) = \beta^{R+a_1-1}e^{-\beta(b_1+\sum_{i=1}^R x_i^2)}e^{(\alpha-1)\sum_{i=1}^R\ln\left(1-e^{-\beta x_i^2}\right)]} \times \left[1-\left(1-e^{-\beta C^2}\right)^{\alpha-1}\right]^{n-R}. \tag{5.13}$$

From (5.13) we observe that it is impossible to sample directly by standard methods therefore, we use the Metropolis–Hastings method with normal proposal distribution to generate random numbers from (5.13). We suggest the next MCMC algorithm to draw samples from the posterior density (5.11) and in turn compute the Bayes estimate of β and α and any function of them such as $R(t)$, $H(t)$ and CV and moreover, construct the corresponding CIs.

Algorithm of MCMC method:

(1) Start with initial $\beta^{(0)} = \hat{\beta}$, M =burn-in.
(2) Set $i = 1$.
(3) Generate $\alpha^{(i)}$ using Gamma $(R + a_2,\ b_2 - \sum_{i=1}^{R} \ln(1 - e^{-\beta x_i^2}))$.
(4) According to Metropolis-Hastings (see, Metropolis et al. [11]), generate $\beta^{(i)}$ from $\pi_2^*(\beta|\alpha, \underline{x})$ with the $N(\beta^{(i-1)}, \sigma)$ proposal distribution.
(5) Compute the reliability, hazard function and the coefficient of variation as

$$\begin{cases} R^{(i)}(t) = 1 - \left(1 - e^{-\beta^{(i)} t^2}\right)^{\alpha^{(i)}} \\ H^{(i)}(t) = \dfrac{2\alpha^{(i)}\beta^{(i)}\ t\ e^{-\beta^{(i)} t^2}\left(1 - e^{-\beta^{(i)} t^2}\right)^{\alpha^{(i)}-1}}{1 - \left(1 - e^{-\beta^{(i)} t^2}\right)^{\alpha^{(i)}}} \\ CV^{(i)} = \dfrac{\sqrt{\dfrac{\alpha^{(i)}}{\beta^{(i)}} \displaystyle\sum_{k=0}^{M(\alpha)} \dfrac{(-1)^k \binom{\alpha^{(i)}-1}{k}}{(k+1)^2}\Gamma(2) - \left(\dfrac{\alpha^{(i)}}{\beta^{(i)\frac{1}{2}}} \displaystyle\sum_{k=0}^{\infty} \dfrac{(-1)^k \binom{\alpha^{(i)}-1}{k}}{(k+1)^{\frac{3}{2}}}\Gamma\left(\frac{3}{2}\right)\right)^2}}{\dfrac{\alpha^{(i)}}{\beta^{(i)\frac{1}{2}}} \displaystyle\sum_{k=0}^{\infty} \dfrac{(-1)^k \binom{\alpha^{(i)}-1}{k}}{(k+1)^{\frac{3}{2}}}\Gamma\left(\frac{3}{2}\right)} \end{cases} \tag{5.14}$$

(6) Put $i = i + 1$
(7) Repeat steps 3−6N times and get $\beta^{(i)},\ \alpha^{(i)},\ R^{(i)}(t),\ H^{(i)}(t)$ and $CV^{(i)}$, $i = M+1, \ldots, N$, as $\beta^{(1)} < \ldots < \beta^{(N-M)}$, $\alpha^{(1)} < \ldots < \alpha^{(N-M)}$, $R^{(1)} < \ldots < R^{(N-M)}$, $H^{(1)} < \ldots < H^{(N-M)}$ and $CV^{(1)} < \ldots < CV^{(N-M)}$. Then, the $100(1-\tau)\%$ credible intervals of $\varphi = \beta,\ \alpha,\ S(t),\ H(t)$ or CV is

$$\left(\varphi^{\left((N-M)\frac{\tau}{2}\right)},\ \varphi^{\left((N-M)\left(1-\frac{\tau}{2}\right)\right)}\right). \tag{5.15}$$

The approximate Bayes estimates of $\varphi = \beta,\ \alpha,\ S(t),\ H(t)$ or CV with respect to SEL function, LINEX loss function and GEL, respectively, is given by

$$\hat{\varphi}_{BS} = \frac{1}{N-M} \sum_{j=M+1}^{N} \varphi^{(j)}, \tag{5.16}$$

$$\hat{\varphi}_{BL} = \frac{-1}{a} \ln\left[\frac{1}{N-M} \sum_{j=M+1}^{N} e^{-a\varphi^{(j)}}\right], \tag{5.17}$$

$$\hat{\varphi}_{BGE} = \left[\frac{1}{N-M} \sum_{j=M+1}^{N} \left(\varphi^{(j)}\right)^{-a}\right]^{\frac{-1}{a}}. \tag{5.18}$$

6. Application to real life data

A real data set is taken from Nichols and Padgett [12], these data give 100 observations on breaking stress of carbon fibres (in Gba). From Figs. 1 and 2, it can be show that the fitted ERD(β, α) provides reasonable fits these data.

According to generalized Type-II HCSs, we use these data to obtain three different Schemes:

Scheme 1: if $r = 81,\ T_1 = 3.5$ and $T_2 = 4$. Since $x_{81:100} < T_1$, the testing would have terminated in this case at $T_1 = 3.5$ and we would have obtained the following data: $0.39,\ 0.81, \ldots, 3.31$ and 3.33.

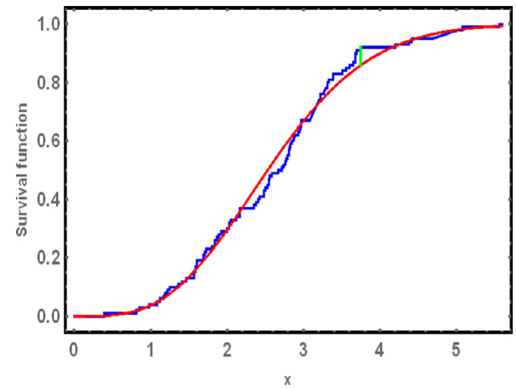

Fig. 1. The empirical and fitted survival functions.

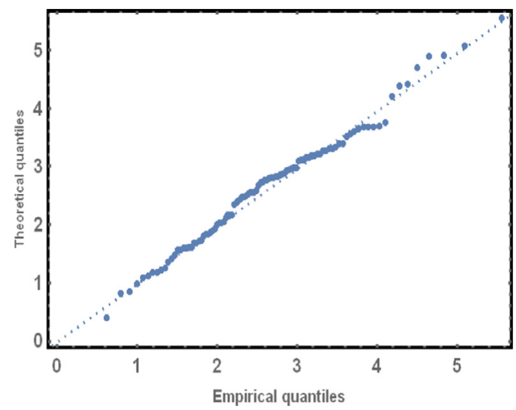

Fig. 2. Quantile-quantile plot.

Table 1
Results of MLEs and Boot-p of $\beta,\ \alpha,\ S(t),\ H(t)$ and CV.

Scheme	Parameters	$(.)_{\text{MLEs}}$	$(.)_{\text{Boot-p}}$
Scheme 1	β	0.17352	0.17465
	α	1.76743	1.80811
	$S(t=4)$	0.10741	0.10756
	$H(t=4)$	1.26955	1.28918
	CV	0.39901	0.39847
Scheme 2	β	0.19252	0.19384
	α	1.93586	1.99613
	$S(t=4)$	0.08703	0.08821
	$H(t=4)$	1.43705	1.47221
	CV	0.38306	0.38138
Scheme 3	β	0.18777	0.18874
	α	1.89225	1.95075
	$S(t=4)$	0.09172	0.09349
	$H(t=4)$	1.39529	1.41472
	CV	0.38696	0.38546

Scheme 2: if $r = 91,\ T_1 = 3.5$ and $T_2 = 4$. Since $T_1 < x_{91:100} < T_2$, the testing would have terminated in this case at $x_{91:100} = 3.70$ and we would have got the data: $0.39,\ 0.81, \ldots, 3.68$ and 3.70.

Scheme 3: if $r = 93,\ T_1 = 3.5$ and $T_2 = 4$. Since $T_2 < x_{93:100}$, the testing would have terminated in this case at $T_2 = 4$ and we would have got the next data: $0.39,\ 0.81, \ldots, 3.70$ and 3.75.

Depending on the above three schemes, we use the MLEs, Bayes and Boot-p methods to estimate $\beta,\ \alpha,\ R(t),\ H(t)$ and CV. Also, we compute the CIs of these parameters. The point estimates of $\beta,\ \alpha,\ R(t),\ H(t)$ and CV using the MLEs and Boot-p methods are presented in Table 1. The 95% approximate CIs using MLEs, Boot-p

Table 2
Two-sided 95% confidence intervals of β, α, $S(t)$, $H(t)$ and CV.

Scheme	Parameters	MLE	Boot-p	MCMC
Scheme 1	β	(0.12890, 0.21814) 0.08924	(0.14682, 0.20525) 0.05843	(0.09546, 0.16041) 0.06495
	α	(1.22619, 2.30868) 1.08249	(1.35302, 2.48211) 1.12909	(0.84574, 1.46063) 0.61488
	$S(t=4)$	(0.05265, 0.16217) 0.10952	(0.08245, 0.13729) 0.05485	(0.09935, 0.20726) 0.10791
	$H(t=4)$	(0.87613, 1.66297) 0.78683	(0.79808, 1.95808) 1.1600	(0.7567, 1.2626) 0.5059
	CV	(0.34377, 0.45426) 0.11049	(0.34358, 0.45127) 0.10769	(0.43556, 0.56866) 0.13311
Scheme 2	β	(0.14841, 0.23663) 0.08822	(0.16184, 0.23313) 0.07129	(0.13546, 0.20694) 0.07148
	α	(1.36419, 2.50752) 1.14333	(1.47197, 2.74697) 1.275	(1.2015, 2.02195) 0.82045
	$S(t=4)$	(0.04225, 0.13181) 0.08956	(0.05803, 0.12239) 0.06436	(0.06581, 0.15046) 0.08465
	$H(t=4)$	(1.0500, 1.82409) 0.77409	(0.91026, 2.48172) 1.57146	(1.0463, 1.6253) 0.5789
	CV	(0.33276, 0.43335) 0.1006	(0.32956, 0.43356) 0.104	(0.37573, 0.47777) 0.10205
Scheme 3	β	(0.14498, 0.23056) 0.08557	(0.14498, 0.23056) 0.07218	(0.13687, 0.2053) 0.06843
	α	(1.33790, 2.44661) 1.1087	(1.33790, 2.44661) 1.32913	(1.2405, 2.04466) 0.80417
	$S(t=4)$	(0.04605, 0.13740) 0.09135	(0.04605, 0.13740) 0.06834	(0.06755, 0.15012) 0.08257
	$H(t=4)$	(1.01937, 1.77122) 0.75185	(0.82795, 1.96915) 1.1412	(1.0791, 1.6349) 0.5558
	CV	(0.33634, 0.43759) 0.10125	(0.33634, 0.43759) 0.11027	(0.37389, 0.47052) 0.09662

Table 3
Bayes MCMC estimates of β, α, $R(t)$, $H(t)$ and CV with $t = 4$.

Scheme	Parameters	SEL	LINEX			GEL		
			$a=-4$	$a=0.5$	$a=4$	$a=-4$	$a=0.5$	$a=4$
Scheme 1	β	0.12664	0.12743	0.12654	0.12586	0.13120	0.12427	0.11834
	α	1.12691	1.20542	1.11829	1.06367	1.17338	1.10400	1.05125
	$R(t)$	0.14935	0.15158	0.14907	0.14720	0.16016	0.14383	0.12020
	$H(t)$	0.99732	1.04622	0.99148	0.95253	1.03201	0.97951	0.93697
	CV	0.49838	0.50176	0.49796	0.49514	0.50341	0.49593	0.49037
Scheme 2	β	0.17019	0.17115	0.17007	0.16924	0.17434	0.16805	0.16289
	α	1.58125	1.72639	1.56583	1.46980	1.64055	1.55182	1.48354
	$R(t)$	0.10386	0.10523	0.10369	0.10255	0.11341	0.09910	0.08812
	$H(t)$	1.32922	1.39178	1.32162	1.27017	1.363	1.31178	1.26961
	CV	0.42380	0.42579	0.42355	0.42187	0.4273	0.42209	0.41822
Scheme 3	β	0.17064	0.17152	0.17053	0.16976	0.17443	0.16868	0.16387
	α	1.61728	1.75113	1.60233	1.50855	1.67313	1.58935	1.52457
	$R(t)$	0.10524	0.10656	0.10507	0.10396	0.11437	0.10071	0.09032
	$H(t)$	1.35124	1.41126	1.34412	1.29652	1.3827	1.33528	1.29752
	CV	0.41912	0.42091	0.41890	0.41739	0.42230	0.41757	0.41404

Table 4
Posterior characteristics under MCMC sample.

Scheme	Parameters	Mean	Median	Mode	S.D	S.E	Sk.
Scheme 1	β	0.12664	0.12603	0.12480	0.01981	0.12818	0.21785
	α	1.12691	1.11262	1.14500	0.18689	1.14230	0.43428
	$R(t=4)$	0.14935	0.14710	0.14920	0.033107	0.15297	0.44946
	$H(t=4)$	0.9973	0.9911	0.9416	0.1533	1.0090	0.2205
	CV	0.49838	0.49598	0.48990	0.04067	0.50000	0.40960
Scheme 2	β	0.17019	0.16976	0.17270	0.02188	0.17159	0.13991
	α	1.58125	1.56575	1.54900	0.25037	1.60095	0.38928
	$R(t=4)$	0.10386	0.10177	0.09008	0.02588	0.10704	0.57206
	$H(t=4)$	1.3292	1.3258	1.3450	0.1746	1.3406	0.1288
	CV	0.42380	0.42173	0.41080	0.03129	0.42495	0.40524
Scheme 3	β	0.17064	0.17007	0.16540	0.02096	0.17192	0.07824
	α	1.61728	1.59764	1.54800	0.24628	1.63592	0.33313
	$R(t=4)$	0.10524	0.10294	0.10160	0.02546	0.10827	0.56376
	$H(t=4)$	1.3512	1.3478	1.3480	0.1692	1.3618	0.1984
	CV	0.41912	0.41783	0.40960	0.02967	0.42017	0.38275

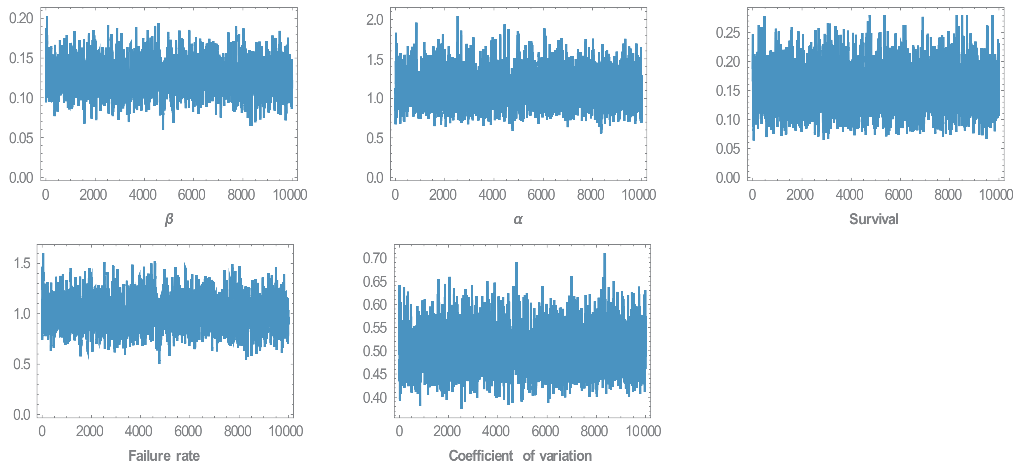

Fig. 3. Trace plot of $\beta,\ \alpha,\ R(t),\ H(t)$ and CV obtained from the Gibbs sampling for Scheme 1.

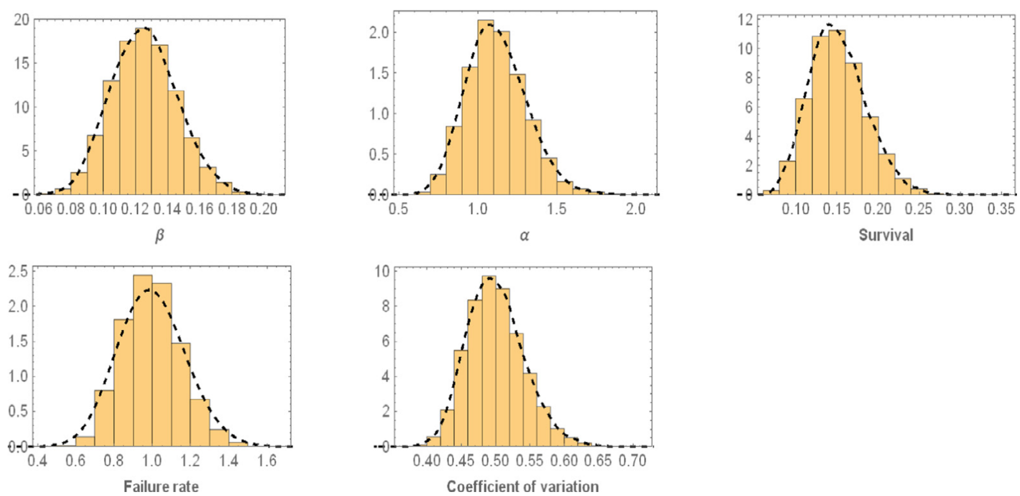

Fig. 4. Histograms of $\beta,\ \alpha,\ R(t),\ H(t)$ and CV obtained from the Gibbs sampling for Scheme 1.

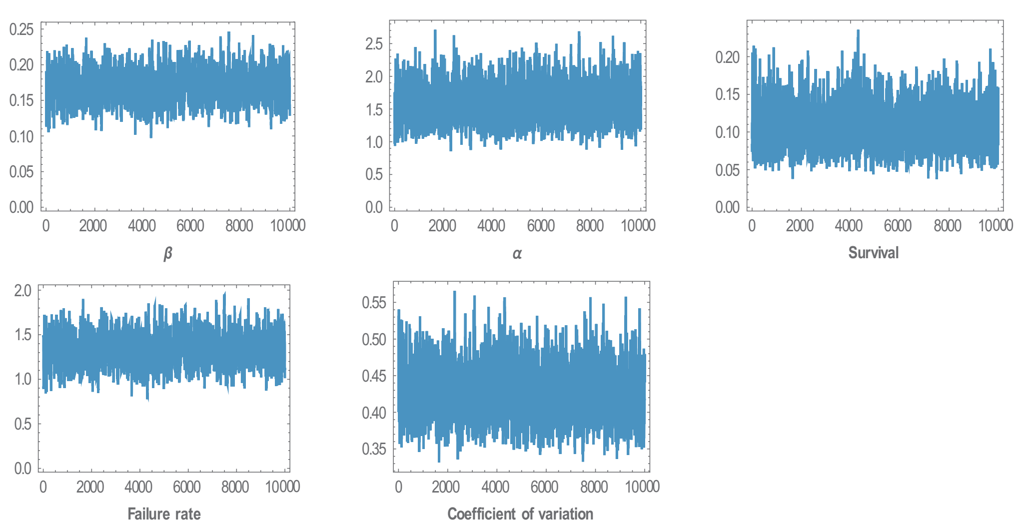

Fig. 5. Trace plot of $\beta,\ \alpha,\ R(t),\ H(t)$ and CV obtained from the Gibbs sampling for Scheme 2.

and MCMC methods of $\beta,\ \alpha,\ R(t),\ H(t)$ and CV are presented in Table 2.

To calculate the Bayes estimates of $\beta,\ \alpha,\ R(t),\ H(t)$ and CV against SE, LINEX and GE loss functions. When the hyperparameters are $a_1 = a_2 = b_1 = b_2 = 0$. We run the Gibbs sampler with in Metropolis–Hasting algorithm to generate a Markov chain with 10,000 observations.

Discarding the first 1000 values as 'burn-in' and taking every tenth variate as iid observations. The descriptive statistics, such as mean, median, mode, standard deviation (SD) and skewness under the MCMC generated sample of $\beta,\ \alpha,\ R(t),\ H(t)$ and CV are

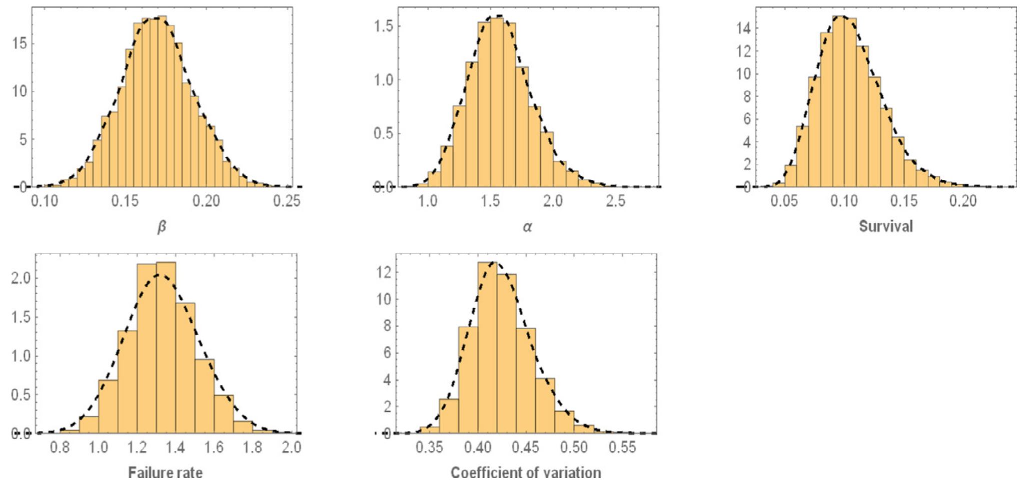

Fig. 6. Histograms of β, α, $R(t)$, $H(t)$ and CV obtained from the Gibbs sampling for Scheme 2.

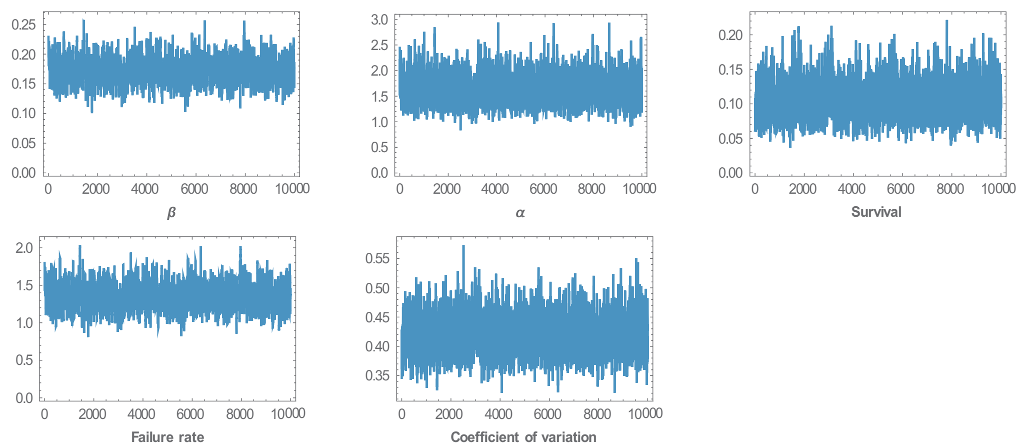

Fig. 7. Trace plot of β, α, $R(t)$, $H(t)$ and CV obtained from the Gibbs sampling for Scheme 3.

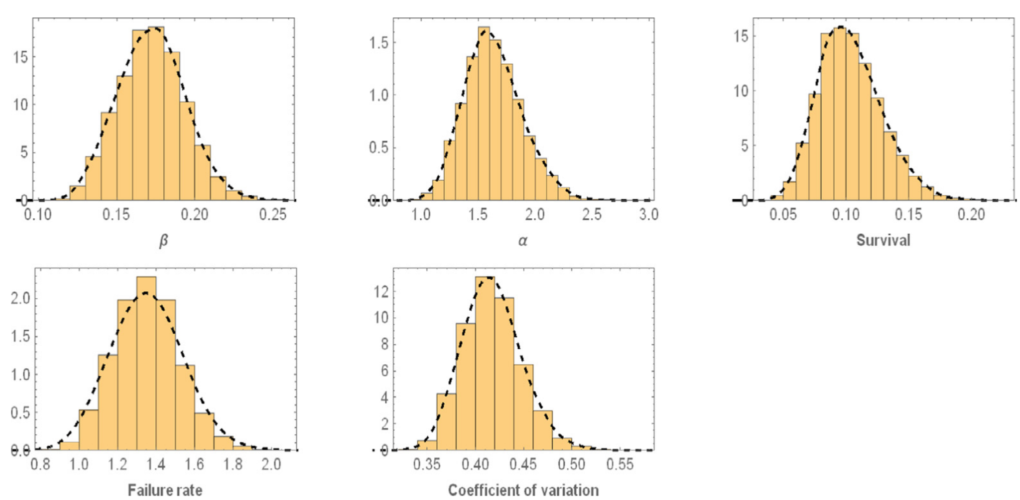

Fig. 8. Histograms of β, α, $R(t)$, $H(t)$ and CV obtained from the Gibbs sampling for Scheme 3.

presented in Table 4. Also, the result of the Bayes estimates of β, α, $R(t)$, $H(t)$ and CV under SEL function, LINEX loss function and GEL are reported in Table 3. Figs. 3, 5 and 7 show the trace plots of the first 1000 MCMC outputs for posterior distribution of β, α, $R(t)$, $H(t)$ and CV based on the above-mentioned schema, as well as, we can see from the histograms of the posterior distributions of β, α, $R(t)$, $H(t)$ and CV in Figs. 4, 6 and 8.

7. Conclusion

In this article, we have discussed different methods to estimate and construct CIs for the parameters besides survival function, failure rate function and coefficient of variation of the exponentiated Rayleigh distribution based on generalized type II Hybrid censored data. The MLEs of the unknown parameters are obtained and suggest different CIs using asymptotic distributions and Boot-p method. We used the MCMC technique to calculate the approximate Bayes estimates and the corresponding credible intervals. Real data set are used to illustrate the proposed methods.

Acknowledgements

The authors would like to express thanks to the anonymous referees for their valuable comments and suggestions, which significantly improved the paper.

References

[1] B. Epstein, Truncated life-test in exponential case, Ann. Math. Stat. 25 (1954) 555–564.
[2] B. Chandrasekar, A. Childs, N. Balakrishnan, Exact likelihood inference for the exponential distribution under generalized type-I and type-II hybrid censoring, Nav. Res. Logist. 51 (2004) 994–1004.
[3] N. Balakrishnan, D. Kundu, Hybrid censoring: models, inferential results and applications, Comput. Stat. Data Anal. 57 (2013) 166–209.
[4] A.R. Shafay, Bayesian estimation and prediction based on generalized type-II hybrid censored sample, J. Stat. Comput. Simul. 32 (2015) 641–661.
[5] W.H. Greene, Econometric Analysis, Prentice Hall, New York, 2000.
[6] B. Efron, R.J. Tibshirani, An Introduction to the Bootstrap, Chapman & Hall, New York, 1993.
[7] H.R. Varian, in: A Bayesian Approach to Real Estate Assessment, North Holland, Amsterdam, 1975, pp. 195–208.
[8] A.P. Basu, N. Ebrahimi, Bayesian approach to life testing and reliability estimation using asymmetric loss function, J. Stat. Plann. Infer. 29 (1991) 21–31.
[9] A.A. Essam, Estimation of some lifetime parameters of generalized Gompertz distribution under progressively type-II censored data, Appl. Math. Modell. 39 (2015) 5567–5578.
[10] R. Calabria, G. Pulcini, Point estimation under asymmetric loss function for left truncated exponential samples, Commun. Stat. Theor. Methods 25 (1996) 585–600.
[11] N. Metropolis, A.W. Rosenbluth, M.N. Rosenbluth, A.H. Teller, E. Teller, Equations of state calculations by fast computing machines, J. Chem. Phys. 21 (1993) 1087–1091.
[12] M.D. Nichols, W.J. Padgett, A bootstrap control chart for Weibull percentiles, Qual. Reliab. Eng. Int. 22 (2006) 141–151.