Journal of the Egyptian Mathematical Society (2014) **22**, 1–5

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

On posets and independence spaces

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Received 18 July 2012; revised 4 May 2013; accepted 25 May 2013
Available online 4 July 2013

KEYWORDS

Independence space;
Independent set;
Poset;
Boolean lattice

Abstract By constructing a correspondence relationship between independence spaces and posets, under isomorphism, this paper characterizes loopless independence spaces and applies this characterization to reformulate certain results on independence spaces in poset frameworks. These state that the idea provided in this paper is a new approach for the study of independence spaces. We outline our future work finally.

MATHEMATICS SUBJECT CLASSIFICATION: 05B35; 06A06; 06A07

1. Introduction

As one of classes of structures of infinite matroids, independence spaces possess quite fruitful consequences (cf. [1–10]). Additionally, Oxley also points out in [1] that for general cases, there have been three main approaches to the study of infinite matroids, one is primarily the independent-set approach, another is the closure-operator approach, and the third approach is via lattices.

Recalling the results relative to independence spaces in [1–10], we find out that most of them are obtained from primarily independent-set approach and some of them use closure-operator approach. According to the known results, seldom research results on independence spaces for general status are obtained by lattice approach, though some results are produced for some special cases of independence spaces using lattice approach (cf. [2, Chapter 3], [11,12] and [13, Section 1.7 in Chapter 1]).

After we analyze the status quo of independence spaces, we state that if we hope to generalize the applied fields of independence spaces, then we should search out a new approach to study on independence spaces not only independent-set and closure-operator approaches. Perhaps, lattice approach fits to be a key for changing the current situation. However, we believe that many researchers have already tried this approach to study independence spaces for general cases not only for some special ones. The status quo is that very few results are provided for independence spaces by lattice approach. This indicates that lattice approach will perhaps not be a key what we expect. We should find out a new approach to study independence spaces. For this, we observe the following:

(1.1) In view of the definition of an independence space provided in [1,2], we may say that an independence space is uniquely determined by its collection of independent sets.

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2013.05.005

(1.2) For a given independence space M, we can associate a poset whose elements are the independent sets of M ordered by inclusion.
(1.3) The results in [16] illustrates that poset theory is a good way to study Mpi-spaces, though Mpi-spaces are a different class of infinite matroids from independence spaces.

All of these suggest that we may build up the correspondence relationship between independence spaces and posets. This article is motivated by the ideas above.

We narrate the construction of this paper as follows:

Section 2 introduces relevant definitions and properties pertaining to posets and independence spaces. In Section 3, under isomorphism, we establish a correspondence relationship between posets with some pre-conditions and independence spaces without loops. Second, we present the relationship between Boolean lattices and independence spaces in which every member owes a unique maximal independent set. Afterward, a consequence on independence spaces is dealt with poset approach. In Section 4, we outline our future work.

2. Preliminaries

In this section, we may begin by considering the fundamental properties what are needed in the sequel. In addition, some notations used in the sequel are given.

In what follows, we assume that E is an arbitrary – possibly infinite – set; 2^X denotes the family of all the subsets of a set X. For a poset $\{A\}$, $Max\{A\}$ denotes the maximal elements in $\{A\}$. $Y \subset\subset X$ represents Y to be a finite subset of a set X.

We introduce a basic property relative to poset theory. The others are referred to [14,15].

Lemma 2.1 [14]. *Any interval sublattice of a Boolean algebra is a Boolean algebra.*

Analogously to the definition of height function in a poset with finite length in [14, p. 4], this paper will accept the height function of a poset as the following: let P be a poset with the least element 0. Then, the *height* $h(x)$ of an element $x \in P$ is the least upper bound of the lengths of the chains $0 = x_0 < x_1 < \cdots < x_n = x$ between 0 and x. If the least upper bound exists as $n < \infty$, then $h(x)$ is n. Otherwise, $h(x)$ is ∞.

$h(x) = 1$ if and only if x covers 0; such elements are called "*atoms*" of P.

For simplicity, if there is no confusion from the text, then a poset $(P, \leqslant)$ is said to be P. In a poset P, $b \prec a$ stands for "a covers b"; the interval $\{x \in P{:}a \leqslant x \leqslant b\}$ is in notation $[a,b]$; for $H = \{a,b\} \subseteq P$, $\vee H$ sometimes is in notation $a \vee b$. If two posets P_1 and P_2 are isomorphic, then it will be denoted by $P_1 \cong P_2$.

Some notations and terms of independence spaces are reviewed here, the others are referred to [1,2]. The description of finite matroids is seen in [1,2].

Definition 2.1. [2, pp.385-387;& 1]. An *independence space* M is a set E together with a collection $\mathcal{I}$ of subsets of E (called *independent sets*) such that

(i1) $\mathcal{I} \neq \emptyset$.
(i2) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
(i3) If $A, B \in \mathcal{I}$and $|A|, |B| < \infty$ with $|A| = |B| + 1$, then $\exists a \in A \backslash B$ fits $B \cup \{a\} \in \mathcal{I}$.
(i4) If $A \subseteq E$ and every finite subset of A is a member of $\mathcal{I}$, then $A \in \mathcal{I}$.

From Definition 2.1, it is easy to show that $\emptyset \in \mathcal{I}$.

For an independence space $M = (E, \mathcal{I})$, in this article, we sometimes write $\mathcal{I}$ as $\mathcal{I}(M)$.

We define a *loop* of an independence space $M = (E, \mathcal{I})$ to be an element x of E such that $\{x\}$ is not an independent set.

Similarly to finite matroids (cf. [2]), we may present the following definition: for two independence spaces M_i, i.e., $(E_i, \mathcal{I}_i), (i = 1, 2)$, M_1 *is isomorphic to* M_2, in notation, $M_1 \simeq M_2$, if and only if there is a bijection ψ: $E_1 \to E_2$ satisfying $I \in \mathcal{I}_1 \Longleftrightarrow \psi(I) \in \mathcal{I}_2$.

3. Relations

This section will deal with the relationship between an independence space $M = (E, \mathcal{I})$ and a poset.

If M is an independence space on E, we can associate with M a poset $\mathcal{P}(M)$ whose elements are the independent sets of M ordered by inclusion.

Let $\mathcal{A}$ be the atoms of $\mathcal{P}(M)$. By (i2), we know $a \in \mathcal{I}$ for any $a \in \mathcal{A}$ and $\mathcal{A} = \{\{b\} | b \in E, \{b\} \in \mathcal{I}\}$.

In this paper, we do not distinguish an element b and a set $\{b\}$ of single element. Hence, $\mathcal{B} = \{d \in E | \{d\} \in \mathcal{I}\}$ is the same to $\mathcal{A}$.

Therefore, if there is no confusion from the text, then we sometimes denote $\mathcal{A}$ by $\{d \in E | \{d\} \in \mathcal{I}\}$ and $\{d\} \in \mathcal{I}$ by $d \in \mathcal{I}$.

We first present some basic properties of $\mathcal{P}(M)$.

Lemma 3.1. *For an independence space $M = (E, \mathcal{I})$, the poset $(\mathcal{I}, \subseteq)$, i.e., $\mathcal{P}(M)$, has the following properties.*

(m1) *$\emptyset$ is the least element in $(\mathcal{I}, \subseteq)$.*
(m2) *For any $I \in \mathcal{I}$, the interval $[\emptyset, I]$ in $(\mathcal{I}, \subseteq)$ is isomorphic to the poset $(2^I, \subseteq)$. Furthermore, every I in $(\mathcal{I}, \subseteq)$ and $I \in \mathcal{I} \setminus \{\emptyset\}$ is a join of atoms, that is, $I = \bigcup_{a \in \mathcal{A}_I} a = \cup \mathcal{A}_I$, and further $I = \mathcal{A}_I$, where $\mathcal{A}_I$ is the family of atoms in $(\mathcal{I}, \subseteq)$ contained in I.*
(m3) *For any $X, Y \in (\mathcal{I}, \subseteq)$, if $h(X)$, $h(Y) < \infty$ and $h(X) = h(Y) + 1$, then there is $a \in X \backslash Y$ such that $Y \cup a$ covers Y in $(\mathcal{I}, \subseteq)$, where h is the height function of $(\mathcal{I}, \subseteq)$.*
(m4) *Let $X \subseteq \mathcal{A}$. If there is $Y \subset\subset X$ satisfying $Y \notin (\mathcal{I}, \subseteq)$, then $X \notin (\mathcal{I}, \subseteq)$.*

Proof. (m1) is straightforward from $\emptyset \in \mathcal{I}$. (m2) is easily followed from (i2). (i4) implies (m4).

To prove (m3), we first prove that if $|Z| < \infty$ for any $Z \in \mathcal{I}$, then the heigh function $h(Z)$ of Z is $|Z|$.

If $|Z| < \infty$, then the maximum in $\{h'(x) | x \in (2^Z, \subseteq)\}$ is $|Z|$ according to the property of Boolean lattice of $(2^Z, \subseteq)$, where h' is the heigh function of $(2^Z, \subseteq)$. Hence, $h(Z) = |Z|$ holds.

Recalling back (i3), $h(X)$, $h(Y) < \infty$ and the above result, we assure that (m3) holds.

Equivalently to (m3), we can say that for any $X, Y \in (\mathcal{I}, \subseteq)$, if $h(X)$,$h(Y) < \infty$ and $h(X) = h(Y) + 1$, then there is $a \in \mathcal{A}_X \setminus \mathcal{A}_Y$ such that $Y \cup a$ covers Y in $(\mathcal{I}, \subseteq)$.

For an independence space $M = (E, \mathcal{I})$, Lemma 3.1 verifies that $\mathcal{P}(M)$ satisfies (m1)–(m4). However, the following example will demonstrate that $\mathcal{P}(M)$ may not be a lattice.

Example 1. Let $E = \{1, \ldots, n, \ldots\}$, $n \not< \infty$, and $\mathcal{I} = \{\emptyset, \{1\}, \ldots, \{n\}, \ldots\}$. Then, it is easy to testify $(E, \mathcal{I})$ to be an independence space. But, obviously, $(\mathcal{I}, \subseteq)$ is not a lattice.

Conversely, for a poset P, we seek sufficient conditions to assure the existence of an independence space. In light of (m1), we will consider only posets with the least element.

Lemma 3.2. *Let P be a poset with the least element 0, $\mathcal{A}$ be the collection of atoms in P, and $\mathcal{A}_x$ be the atoms contained in $x \in P$. If P satisfies the following (q1)–(q4), then there exists an independence space $M(P)$ such that $P \cong (\mathcal{I}, \subseteq)$ and $M(P)$ has no loops, where $\mathcal{I}$ is the set of independent sets of $M(P)$.*

- (q1) *Every element in $P \backslash \{0\}$ is a join of atoms, i.e., $x = \vee \mathcal{A}_x$ is true for $x \in P \backslash \{0\}$.*
- (q2) *If $x \in P$, then $[0, x] \cong (2^{\mathcal{A}_x}, \subseteq)$.*
- (q3) *For any x, $y \in P$, if $h(x)$, $h(y) < \infty$ and $h(x) = h(y) + 1$, then there exists $a_x \in \mathcal{A}_x \setminus \mathcal{A}_y$ satisfying $y \prec y \vee a_x$ in P, where h is the height function of P.*
- (q4) *For $S \subseteq \mathcal{A}$, if there is $X \subset\subset S$ satisfying $\vee X \notin P$, then $\vee S \notin P$.*

Proof. We will carry out the proof step by step.

Step 1. We prove that if x, $y \in P$ satisfy $x \neq y$, then $\mathcal{A}_x \neq \mathcal{A}_y$.Otherwise, by (q1) and $\mathcal{A}_x = \mathcal{A}_y$, it follows that $\vee \mathcal{A}_x = \vee \mathcal{A}_y$, and so, $x = y$, a contradiction.

Step 2. Let $x, y \in P$. We prove that $x \leqslant y \Longleftrightarrow \mathcal{A}_x \subseteq \mathcal{A}_y$.

($\Rightarrow$) Let $a \in \mathcal{A}_x$. Then, in virtue of the definition of $\mathcal{A}_x$ and $x \leqslant y$, we may state that $a \leqslant x \leqslant y$ holds. Therefore, $a \in \mathcal{A}_y$ holds. Thus, $\mathcal{A}_x \subseteq \mathcal{A}_y$.

($\Leftarrow$) x,$y \in P$ and (q1) together causes $x = \vee \mathcal{A}_x$ and $y = \vee \mathcal{A}_y$ respectively. $\mathcal{A}_x \subseteq \mathcal{A}_y$ brings about $(2^{\mathcal{A}_x}, \subseteq)$ to be a subposet of $(2^{\mathcal{A}_y}, \subseteq)$. Hence, $[0, x]$ is a subposet of $[0, y]$ in light of (q2). Therefore, $x \leqslant y$ is correct.

Step 3. Let $x \in P$. By the induction on $h(x)$ and (q2), it is easy to obtain that if $h(x) < \infty$, then $h(x) = |\mathcal{A}_x|$.

Step 4. Let $X \subseteq \mathcal{A}$ satisfy $\vee X \in P$. This step proves that if $x = \vee X$, then $X = \mathcal{A}_x$.Taken $X \subseteq \mathcal{A}, x = \vee X \in P$ and (q1) together guarantees $X \subseteq \mathcal{A}_x$.Assume $X \subset \mathcal{A}_x$. Considered (q2) with $X \in 2^{\mathcal{A}_x}$, we may indicate that there is $a \in [0, x]$ such that a corresponds to X for $[0, x] \cong (2^{\mathcal{A}_x}, \subseteq)$ and $a < x$. For any $y \in X$, by (q2) and $X \subseteq \mathcal{A}_x$, we may say that y is an atom in P satisfying $y \leqslant a$. Furthermore, $\vee X \leqslant a < x$ follows a contradiction to $\vee X = x$.

Step 5. In virtue of Step 4, we may say that every $\mathcal{A}_x$ is uniquely determined by $x \in P$.

Step 6. Let $\mathcal{I} = \{\mathcal{A}_x | x \in P\}$. We prove that $(\mathcal{A}, \mathcal{I})$ is an independence space with $\mathcal{I}$ as its family of independent sets, and additionally, $(\mathcal{A}, \mathcal{I})$ has no loops.We will denote $(\mathcal{A}, \mathcal{I})$ by $M(P)$.This step will be finished by the following Steps 6.1–6.5.

Step 6.1 Since 0 is the least element in P, it follows that 0 is not a join of atoms. Thus, it gets $\mathcal{A}_0 = \emptyset$. Therefore, $\emptyset \in \mathcal{I}$ holds.

Step 6.2 Let $\mathcal{A}_y \in \mathcal{I}$ and $X \subseteq \mathcal{A}_y$. We prove $X \in \mathcal{I}$.$\mathcal{A}_y \in \mathcal{I}$ and Step 5 will cause that there is a unique $y \in P$ satisfying $y = \vee \mathcal{A}_y$. (q2) guarantees $[0, y] \cong (2^{\mathcal{A}_y}, \subseteq)$. $X \subseteq \mathcal{A}_y$ means $X \in (2^{\mathcal{A}_y}, \subseteq)$. Hence, considered the Boolean lattice property for $(2^{\mathcal{A}_y}, \subseteq)$, (q1), Steps 4 and 5, we gain $X \in \mathcal{I}$.

Step 6.3 To prove (i3) is true for $\mathcal{I}$.Let $A, B \in \mathcal{I}$ and $|A|, |B| < \infty$ with $|A| = |B| + 1$. In view of the definition of $\mathcal{I}$, Steps 4 and 5, there exists uniquely x, $y \in P$ satisfying $A = \mathcal{A}_x$ and $B = \mathcal{A}_y$ respectively. We easily know that for any $t \in P$, if $|\mathcal{A}_t| < \infty$, then the height of $\mathcal{A}_t$ in $(2^{\mathcal{A}_t}, \subseteq)$ is $|\mathcal{A}_t|$. Considered (q2) and the finiteness of $|\mathcal{A}_x|$ and $|\mathcal{A}_y|$, it is easy to get (i3) to be correct for $\mathcal{I}$.

Step 6.4 Let $S \subseteq \mathcal{A}$. We will prove: if there is $X \subset\subset S$ satisfying $X \notin \mathcal{I}$, then $S \notin \mathcal{I}$.Otherwise, in view of $S \in \mathcal{I}$ and (q2), there is $s \in P$ satisfying $s = \vee S$ and $S = \mathcal{A}_s$. However, $X \notin \mathcal{I}$, Step 4 and (q4) together implies $\vee X \notin P$, and further, $\vee S \notin P$. This is a contradiction.

Step 6.5 For any $x \in \mathcal{A}$, it has x to be an atom in P. That is to say, $\mathcal{A}_x = x$. Thus, $x \in \mathcal{I}$ holds. Therefore, $(\mathcal{A}, \mathcal{I})$ has no loops.

Step 7. To prove $P \cong (\mathcal{A}, \mathcal{I})$. □

Let $f: P \rightarrow (\mathcal{I}, \subseteq)$ be defined as $x \mapsto \mathcal{A}_x$ for any $x \in P$. It is easy to know that f is a bijection according to Step 1, Step 4, Step 5 and the definition of $\mathcal{I}$ in Step 6. By Step 2, f is an order-preserving two-sided inverse. Therefore, $P \cong (\mathcal{I}, \subseteq)$ is true according to (1) in Definition 2.1.

Now unfortunately, the structure of an independence space M is not completely specified by the poset $\mathcal{P}(M)$ of its independent sets.

Example 2. Let $a \notin E_1$ and $M_1 = (E_1, \mathcal{I})$ be an independence space satisfying $E_1 = \{x | \{x\} \in \mathcal{I}\}$. Evidently, $M_2 = (E_1 \cup a, \mathcal{I})$ is an independence space and $\{a\}$ is a loop of M_2 though $\mathcal{P}(M_1) = \mathcal{P}(M_2)$.

This indeterminacy of M from $\mathcal{P}(M)$ is due to the existence of loops. The importance of independence spaces without loops lies in the following theorem.

Theorem 3.1. *Suppose that M is an independence space without loops. Then, M is uniquely determined by its poset $\mathcal{P}(M)$ if and only if $\mathcal{P}(M)$ has the least element and satisfies (q1)–(q4).*

Proof. Let $M = (E, \mathcal{I}(M))$ be an independence space without loops. We see that the set of atoms in $\mathcal{P}(M)$ is E. Recalling Lemma 3.1, it follows that $\mathcal{P}(M)$ has the least element and satisfies (q1)–(q4). Further, Lemma 3.2 causes $M(\mathcal{P}(M))$ satisfying $\mathcal{P}(M) \cong (\mathcal{I}(M(\mathcal{P}(M))), \subseteq)$, i.e., $(\mathcal{I}(M), \subseteq) = \mathcal{P}(M) \cong (\mathcal{I}(M(\mathcal{P}(M))), \subseteq)$, and meanwhile, $M(\mathcal{P}(M))$ is defined on the set of atoms of $\mathcal{P}(M)$, that is, on the set $\{\{x\}| x \in E\}$. Hence, $M(\mathcal{P}(M))) \simeq M$ holds. □

Corollary 3.1.

(1) *A poset P is isomorphic to the poset $\mathcal{P}(M)$ of an independence space M without loops if and only if P has the least element and satisfies (q1)–(q4).*
(2) *Let M be an independence space without loops defined on E. Then, $\mathcal{P}(M)$ is a Boolean lattice if and only if $|Max\mathcal{P}(M)| = 1$.*
(3) *Let P be a poset with the least element and P satisfy (q1)–(q4). Then, P is a Boolean lattice if and only if P is bounded, i.e., P does also have the maximum element.*

Proof. The proof of (1) is straightforward from Lemma 3.2 and Theorem 3.1. Both (2) and (3) are routine verification from Theorem 3.1.

Thus, Theorem 3.1 indicates clearly that the study of independence spaces without loops is just the study of posets in which every element has the least element and satisfies (q1)–(q4). Many of the interesting properties of independence spaces are preserved if we just pay attention to independence spaces without loops.

It is also useful to reformulate some of the results on independence spaces in poset frameworks. For an independence space $M = (E, \mathcal{I})$, Oxley proves in [1] that $M| T$, i.e. $(T, \mathcal{I}|T)$, is an independence space on $T \subseteq E$ by independent-set approach. In [8], Mao gets the same with closure-operator approach. Here, under the umbrella of poset frameworks, we will obtain the same result. □

Theorem 3.2. *Let $M = (E, \mathcal{I})$ be an independence space and $T \subseteq E$. Let $\mathcal{I}|T$ be $\{X|X \subseteq T, X \in \mathcal{I}\}$. Then, $M|T = (T, \mathcal{I}|T)$ is an independence space.*

Proof. First, we notice that

(α1) $M = (E, \mathcal{I})$ is an independence space with S to be the collection of loops if and only if $(E \setminus S, \mathcal{I})$ is an independence spaces without loops.
(α2) If M has S to be the collection of loops, then by the definition of $M| T$, $M|T = (T, \mathcal{I}|T)$ will have $S \cap T$ to be the collection of loops.
(α3) Combining (α1) with (α2), we may state that $M|T = (T, \mathcal{I}|T)$ is an independence space on T with $S \cap T$ to be the collection of loops if and only if $M|(T \setminus (S \cap T)) = (T \setminus (S \cap T), \mathcal{I}|T)$ is an independence spaces without loops. □

Thus, we pay our attention only to the independence spaces without loops.

Let $M = (E, \mathcal{I}(M))$ be an independence spaces without loops. By Theorem 3.1, it follows that $\mathcal{P}(M)$ is a poset with the least element and satisfies (q1)–(q4), and in addition, the set $\mathcal{A}$ of atoms in $\mathcal{P}(M)$ is E. $\mathcal{I}(M)|T = \{X|X \subseteq T, X \in \mathcal{I}(M)\}$ implies that $(\mathcal{I}(M)|T, \subseteq)$ is a subposet of $\mathcal{P}(M)$. In the following, we denote $(\mathcal{I}(M)|T, \subseteq)$ by P_T. Moreover, the least element *0* exists in P_T since $\emptyset \subseteq T$.

In $\mathcal{P}(M)$, let $\mathcal{A}_x$ be the set of atoms contained in $x \in \mathcal{P}(M)$. In P_T, let $\mathcal{A}^T$ be the collection of atoms and $\mathcal{A}_y^T$ be the collection of atoms contained in $y \in P_T$. Because $E = \mathcal{A}$ implies $T = \mathcal{A}^T$, it follows $\mathcal{A}_y = \mathcal{A}_y^T$ for any $y \in P_T$. $x \in \mathcal{I}(M)|T$ indicates $x \in \mathcal{I}(M)$, and so $x = \vee\mathcal{A}_x$ holds in $\mathcal{P}(M)$. Hence, $x = \vee\mathcal{A}_y^T$ is correct. That is, (q1) holds for P_T.

Let $a \in P_T$. In P_T, let $[0,x]_T$ be the interval for $x \in P_T$. Then, the following expression is true: $[0,a] = \{x \in \mathcal{P}(M)|0 \leqslant x \leqslant a\} = \{x|x \in \mathcal{I}(M), \emptyset \subseteq x \subseteq a\} = \{x|x \in \mathcal{I}(M), \emptyset \subseteq x \subseteq a \subseteq T\} = [0,a]_T$. On the other hand, $[0,a]$ is a Boolean lattice by (q2), and $[0,a] \cong (2^{\mathcal{A}_a}, \subseteq)$ holds. Hence $[0,a]_T \cong (2^{\mathcal{A}_a^T}, \subseteq)$ holds because of $\mathcal{A}_a = \mathcal{A}_a^T$ and $[0,a] = [0,a]_T$.

Let the height function of P_T be h_T. For any $x \in P_T$, there is $x \in \mathcal{P}(M)$. If we consider with the definition of height function in a poset, then it is easy to obtain $h_T(z) = h(z)$ for any $z \in P_T$ and $h(z) < \infty$.

Let $x, y \in P_T$ such that $h_T(x), h_T(y) < \infty$ and $h_T(x) = h_T(y) + 1$. Evidently, $h_T(x) = h(x)$ and $h_T(y) = h(y)$ hold. In $\mathcal{P}(M)$, using (q3) and considering with $\mathcal{A}_x^T = \mathcal{A}_x$ and $\mathcal{A}_y^T = \mathcal{A}_y$, it follows that (q3) is correct in P_T.

The truth of (q4) in P_T is followed because (q1) holds in P_T and (q4) is correct in $\mathcal{P}(M)$.

In one word, P_T is a poset satisfying (q1)–(q4) and has the least element 0. Therefore, by Theorem 3.1, there is an independence space $M(P_T)$ on T without loops such that under isomorphism, P_T is the family of independent sets of $M(P_T)$. This means that up to isomorphism, $M(P_T) = (T, P_T) = (T, \{X|X \in \mathcal{I}(M), X \subseteq T\}) = M|T$ is correct. That is to say, $M| T$ is an independence spaces without loops.

Theorem 3.1 makes clearly that we also can reformulate some of results on posets in independence space frameworks. We will do if we expect in the future.

4. Conclusion

It is well known that a lattice is a poset but not vice versa. Example 1 demonstrates that $(\mathcal{I}, \subseteq)$ may not be a lattice for some independence space $M = (E, \mathcal{I})$. All of these illustrate that the idea provided in this paper is a new approach for the study on independence spaces. We call this idea poset approach.

Theorem 3.1 builds up a relationship between posets and independence spaces. Applying this relationship, Theorem 3.2 states clearly that poset approach is a way to study on independence spaces. We will approach properties for independence spaces from different angles utilizing Theorem 3.1 in our future work. We hope the relationship provided in Theorem 3.1 to be useful for the other classes of structures of infinite matroids.

Acknowledgements

I would like to thank the two anonymous reviewers for good suggestions and comments. This research is Granted by NSF of China (11101115, 61202178, 61073121) and NSF of Hebei

Province (F2012402037, A2013201119) and SF of Baoding (11ZG005).

References

[1] J. Oxley, Infinite matroid, in: N. White (Ed.), Matroid Application, Cambridge Universtiy Press, Cambridge, 1992, pp. 73–90.
[2] D.J.A. Welsh, Matroid Theory, Academic Press Inc., London, 1976.
[3] H. Bruhn, R. Diestel, A New Look at Infinite Matroids. Version 23 July 2011 <http://www.math.uni-hamburg.de/home/diestel/papers/matroidgraphs.pdf> (accessed 03.10.2011).
[4] H. Bruhn, R. Diestel, M. Kriesell, R. Pendavingh, P. Wollan, Axioms for Infinite Matroids <http://www.math.uni-hamburg.de/home/diestel/papers/matroidaxioms.pdf> (accessed 03.10.2011).
[5] A.W. Ingleton, A geometrical characterization of transversal independence structures, Bull. London Math. Soc. 3 (1971) 47–51.
[6] H. Mao, S. Liu, Remarks on external elements in independence spaces, Southeast Asian Bull. Math. 29 (2005) 939–944.
[7] H. Mao, Independence spaces generated by a graph, East–West J. Math. 9 (1) (2007) 63–68.
[8] H. Mao, G. Wang, The sub-independence-spaces, Chinese Quar. J. Math. 25 (2) (2010) 293–299.
[9] J. Oxley, On a packing problem for infinite graphs and independence spaces, J. Combin. Theory, Series B 26 (2) (1979) 123–130.
[10] G. Wang, H. Mao, Geometric lattices and independence spaces, East–West J. Math. 12 (1) (2010) 1–7.
[11] H. Mao, On geometric lattices and matroids of arbitrary cardinality, Ars Comb. 81 (2006) 23–32.
[12] H. Mao, Paving matroids of arbitrary cardinality, Ars Comb. 90 (2009) 245–256.
[13] J. Oxley, Matroid Theory, Oxford University Press, New York, 2006.
[14] G. Birkhoff, Lattice Theory, third ed., Amer. Math. Soci., Providence, 1967.
[15] G. Grätzer, General Lattice Theory, second ed., Birkhäuser Verlag, Basel, 1998.
[16] H. Mao, S. Liu, Mpi-spaces and posets, Acta Math. Sinica, Chinese Series 54 (6) (2011) 933–942 (in Chinese with English summary).