Journal of the Egyptian Mathematical Society 25 (2017) 48–52

Contents lists available at ScienceDirect

Journal of the Egyptian Mathematical Society

journal homepage: www.elsevier.com/locate/joems

Introducing Weingarten cyclic surfaces in R^3

Fathi M. Hamdoon, M.A. Abd-Rabo*

Math. Dept., Faculty of Science, Al-Azhar Univ., Assuit branch, Assiut 71524, Egypt

ARTICLE INFO

Article history:
Received 11 April 2016
Revised 3 June 2016
Accepted 4 June 2016
Available online 15 November 2016

MSC:
53A05

Keywords:
Circle of curvature
Weingarten surfaces

ABSTRACT

In this paper, Cyclic surfaces are introduced using the foliation of circles of curvature of a space curve. The conditions on a space curve such that these cyclic surfaces are of type Weingarten surfaces or HK-quadric surfaces are obtained. Finally, some examples are given and plotted.

1. Introduction

A surface M in Euclidean 3-space R^3 is called Weingarten surface if there is a relation between its two principal curvatures k_1, k_2. If there exist a linear function S of two variables such that $S(k_1, k_2) = 0$, or in the form

$$k_1 = mk_2 + n, \tag{1.1}$$

where m, n are constants, the surfaces is called a linear Weingarten surface and it abbreviate by LW-surface [1]. In particular if K and H denote the Gaussian and mean curvatures respectively of a surface M, and are related through the linear relation

$$aH + bK = c, \tag{1.2}$$

where a, b and c are constants and $a^2 + b^2 \neq 0$, in this case then that M is a special Weingarten surface and we abbreviate it by SW-surface [2]. Weingarten introduce this type of surfaces in the context of the problem of finding all surfaces isometric to a given surface of revolution [3,4]. Study of Weingarten surfaces has a long history [5,6], and more recently [7,8]. Application of Weingarten surfaces on computer aided geometric design and shape investigation can seen in [9]. Cyclic surfaces in R^3 are one-parametric family of circles [10]. R. López in [1] proved that a cyclic surfaces is linear Weingarten surfaces with $(m, n) = (m, 0)$ must be Riemann type. In [2] he proved that all special Weingarten cyclic surface with $aH + bK = c$ must be a surface of revolution, a Riemann minimal surface or generalized cone. In [11] we investigated a cyclic surfaces by using circle of curvature of a space curve. We obtained some conditions on this curve to ensure that this cyclic surface is of zero or nonzero constant Gaussian curvature. We presented a procedure to determine a geodesic curves on this surface. In this paper, we show that what conditions should be imposed on the space curve such that the cyclic surface generated by circle of curvature of this curve is a LW-surface or a SW-surface. For more details see [5–8]

2. Basic concepts

Consider a surface M in R^3 with Gaussian curvature K and mean curvature H. Let $\mathbf{X} = \mathbf{X}(u, v)$ be a local parametrization of M. The tangent vectors to the parametric curves of the surface M are

$$\mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u}, \qquad \mathbf{X}_v = \frac{\partial \mathbf{X}}{\partial v} \tag{2.1}$$

The unit normal vector filed on M is given by

$$\mathbf{N} = \frac{\mathbf{X}_u \wedge \mathbf{X}_v}{|\mathbf{X}_u \wedge \mathbf{X}_v|} \tag{2.2}$$

where $\wedge$ means the cross product in R^3. The first fundamental quadratic form on the surface M is

$$\mathbf{I} =< dX, dX >= g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2 \tag{2.3}$$

where $g_{\alpha\beta}$ are the first fundamental coefficients, where

$$g_{11} = \langle \mathbf{X}_u, \mathbf{X}_u \rangle, \qquad g_{12} = \langle \mathbf{X}_u, \mathbf{X}_v \rangle, \qquad g_{22} = \langle \mathbf{X}_v, \mathbf{X}_v \rangle, \tag{2.4}$$

The second fundamental quadratic form is given by

$$II = -\langle d\mathbf{N}, d\mathbf{X} \rangle = h_{11}du^2 + 2h_{12}dudv + h_{22}dv^2, \tag{2.5}$$

* Corresponding author. Tel.: +201014765860.
E-mail addresses: f_hamdoon@yahoo.com (F.M. Hamdoon), mahmoud_abdrabo86@yahoo.com (M.A. Abd-Rabo).

http://dx.doi.org/10.1016/j.joems.2016.06.001
1110-256X/

where $g_{\alpha\beta}$ are the second fundamental coefficients, where

$$h_{11} = \langle \mathbf{N}, \mathbf{X}_{uu} \rangle, \qquad h_{12} = \langle \mathbf{N}, \mathbf{X}_{vu} \rangle, \qquad h_{22} = \langle \mathbf{N}, \mathbf{X}_{vv} \rangle, \tag{2.6}$$

The Gaussian and mean curvatures are given by

$$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2}, \tag{2.7}$$

$$H = \frac{h_{11}g_{22} - 2h_{12}g_{12} + h_{22}g_{11}}{2(g_{11}g_{22} - g_{12}^2)}, \tag{2.8}$$

Let $\Psi : I \to \mathrm{R}^3$ be a unit speed curve with $\Psi' \neq 0$ where $\Psi' = d\Psi/du$. The arc-length parameter u of a curve Ψ is determined such that $\|\Psi'(u)\| = 1$. Let us denote $\mathbf{t}(u) = \Psi'(u)$ where $\mathbf{t}(u)$ is a unit tangent vector of Ψ at u. We define the curvature of Ψ by $k(u) = \|\Psi''(u)\|$. If $k(u) \neq 0$, then the unit principal normal vector $\mathbf{n}(u)$ of the curve Ψ at u is given by $\Psi''(u) = k(u)\mathbf{n}(u)$. The unit binormal vector of Ψ at u is defined as $\mathbf{b}(u) = \mathbf{t}(u) \times \mathbf{n}(u)$. Then we have the Frenet-Serret formulae:

$$\begin{pmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}, \quad ' = \frac{d}{du} \tag{2.9}$$

where $\tau(u)$ is the torsion of the curve Ψ at u

Definition 2.1. The circle of curvature of a curve $\Psi = \Psi(u)$ at a point m is the limiting position of a circle drawn through m and two other points of the curve, such that these two points tends to m [12].

The circle of curvature of the curve Ψ lies on the osculating plane and the center of curvature at any point is given by

$$\mathbf{c}(u) = \Psi(u) + \rho(u)\mathbf{n} \tag{2.10}$$

where $\rho(u) = 1/k(u)$ is the radius of curvature of the curve Ψ [12].

Definition 2.2. [10] A cyclic surface in Euclidean space R^3 is a surface determined by a smooth uniparametric family of pieces of circle.

Remark 2.3. A cyclic surface can be generated by a circles of curvature of a space curve [13].

The representation of cyclic surfaces foliated by circles of curvature of the space curve Ψ is given by

$$M : \mathbf{X}(u, v) = \mathbf{c}(u) + \rho(u)(\mathbf{t}\cos v + \mathbf{n}\sin v) \tag{2.11}$$

where $\mathbf{c}(u)$ are the center of curvatures given by (2.10)

3. Cyclic Weingarten-surfaces

In this section, we obtained the condition such that the cyclic surface parameterized by Eq. (2.11) is LW-surface or SW-surface. Our method depends on, equations reduces (1.1), (1.2) to an expression that can be rewrite as a linear combination of the functions $\cos(iv)$, $\sin(iv)$ whose coefficients A_i, B_i are function of the u-variable. Therefore, they must be vanish in some u-interval. The Gaussian and mean curvatures of a surface M are

$$K = \frac{K_1}{W^2}, \tag{3.1}$$

$$H = \frac{H_1}{2W^{3/2}}, \tag{3.2}$$

where $H_1 = g_{22}[X_u, X_v, X_{uu}] - 2g_{12}[X_u, X_v, X_{uv}] + g_{11}[X_u, X_v, X_{vv}]$, $K_1 = [X_u, X_v, X_{uu}][X_u, X_v, X_{vv}] - [X_u, X_v, X_{uv}]^2$. $W = g_{11}g_{22} - g_{12}^2$,

The principal curvatures k_1, k_2 are given by

$$k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K} \tag{3.3}$$

3.1. Cyclic LW-surfaces

If the cyclic surface M which given by (2.11) is LW-surface , then it satisfies the condition (1.1). Using Eqs. (3.1), (3.2) and (3.3) this condition take the form

$$(1 - m)H_1 - 2W^{3/2}n = -(1 + m)\sqrt{H_1^2 - 4WK_1} \tag{3.4}$$

After some computations, the condition (1.1) can be written as the following

$$(-mH_1^2 + (1 + m)^2WK_1 + n^2W^3)^2 - n^2(1 - m)^2H_1^2W^3 = 0 \tag{3.5}$$

Remark 3.1. The surface M is totally umbilical if $m = 1$, $n = 0$, and the condition (1.1) becomes

$$H_1^2 - 4WK_1 = 0 \tag{3.6}$$

3.1.1. Case $n = 0$

In this case, the Eq. (3.5) take the form

$$mH_1^2 - (1 + m)^2WK_1 = 0 \tag{3.7}$$

By using Eqs. (2.9) and (2.10), we can expressed (3.7) by trigonometric polynomial on $\cos(iv)$, $\sin(iv)$, $0 \leq i \leq 6$, these coefficients A_i, B_i, are functions on the u-variable. Therefore, these coefficients must vanish in some u-interval. The work then is to compute explicitly these coefficients by successive manipulations. We used the symbolic program mathematica to check out our computations.

$$\sum_{i=0}^{6}(A_i\cos(iv) + B_i\sin(iv)) = 0. \tag{3.8}$$

Since this is an expression on the independent trigonometric terms $\cos nv$ and $\sin nv$, all coefficients A_i , B_i vanish identically.

After some computation, the values for A_6, B_6 are

$$A_6 = \frac{1}{32}(-1 + m)^2\rho(u)^6\tau(u)^2(\rho(u)^2\tau(u)^2 + \rho'(u)^2)^2$$

$$B_6 = 0$$

From A_6 we have the two possibilities

Case (1) $\tau(u) = 0$, which leads to zero coefficients identically

Case (2) $m = 1$, then

$$A_6 = \ldots = B_5 = 0$$

$$A_4 = \frac{\rho(u)^8}{8}(\rho(u)\tau(u)^3 - \rho'(u)\tau'(u) + \tau(u)\rho''(u))^2$$

Thus, the solution of the differential equation $A_4 = 0$ is

$$\tau(u) = \pm\frac{\rho'(u)}{\sqrt{c^2 - \rho^2(u)}}$$

$$\Rightarrow \rho(u) = c\sin\left(\int \tau(u)du\right) + constant$$

$$or \quad \rho(u) = c\cos\left(\int \tau(u)du\right) + constant \tag{3.9}$$

where c constant, this leads to all coefficients (3.8) are vanished. Indeed we have proved the following theorem:

Theorem 3.2. *The cyclic surface generated by circles of curvature of the space curve is a linear Weingarten surface with condition $k_1 = mk_2$ if the curve is a plane curve or its torsion and radius of curvature are related by the relation (3.9) and the surface M is part of a sphere (totally umbilical).*

3.1.2. Case $n \neq 0$

Similarly as in Section (3.1.1), we can express (3.5) in the following form

$$\sum_{i=0}^{12}(A_i\cos(iv)+B_i\sin(iv))=0. \tag{3.10}$$

After some computation, the values for A_{12}, B_{12}, A_{11} are given as

$$A_{12}=\frac{\rho(u)^{12}}{2048}(\rho(u)^2\tau(u)^2+\rho'(u)^2)^4 \times((-(-1+m)^2+n^2\rho(u)^2)\tau(u)^2+n^2\rho'(u)^2)^2$$

$$B_{12}=A_{11}=0$$

From A_{12} We have the two possibilities

Case(1) $(\rho(u)^2\tau(u)^2+\rho'(u)^2)=0$ that is $\rho'(u)=0,\ \tau(u)=0$

Then, all the coefficients are vanished identically.

In this case, the center of curvature of the circle is fixed, i.e., $c(u)=c_0$, then

$$|X(u,v)-c_0|=\rho=constant \tag{3.11}$$

Case(2)

$$(-(-1+m)^2+n^2\rho(u)^2)\tau(u)^2+n^2\rho'(u)^2=0.$$

that is $\tau(u)=\pm\frac{n\rho'(u)}{\sqrt{(-1+m)^2-n^2\rho(u)^2}}$

$$\Rightarrow \rho(u)=\frac{m-1}{n}\sin\left(\int\tau du\right)+constant$$

$$or\quad \rho(u)=\frac{m-1}{n}\cos\left(\int\tau du\right)+constant \tag{3.12}$$

Then, $A_{12}=B_{12}=A_{11}=\ldots=A_0=0$ identically.

Thus, we have proved the following:

Theorem 3.3. *The cyclic surface generated by circles of curvature of a space curve is a linear Weingarten surface with condition if the curve is a circle or the radius of curvature and the torsion are related through (3.12).*

Remark 3.4. The cyclic surface generated by a circle of curvature (3.11) of a circle is contained in a sphere.

3.2. Cyclic SW-surfaces

Without loss of generality, we can take $a=1$ in the condition (1.2). By using the Eqs. (3.1) and (3.2), the condition (1.2), can be written in the following form

$$\frac{H_1}{2W^{3/2}}+b\frac{K_1}{W^2}=c, \tag{3.13}$$

$$H_1^2W-4(cW^2-bK_1)^2=0. \tag{3.14}$$

3.2.1. Case $c=0$

In this case (3.14) takes the form

$$H_1^2W-4(bK_1)^2=0 \tag{3.15}$$

As in the previous section, we can expressed (3.15) as follows

$$\sum_{i=0}^{8}(A_i\cos(iv)+B_i\sin(iv))=0. \tag{3.16}$$

After some computation, the non vanishing coefficients of Eq. (3.16) are given as

$$A_8=\frac{1}{32}\rho(u)^8\tau(u)^2(\rho(u)^2\tau(u)^2+\rho'(u)^2)^2 \times((-b^2+\rho(u)^2)\tau(u)^2+\rho'(u)^2)$$

Thus, we have the two possibilities, we have all the coefficients are vanished identically.

Case (1) $\tau(u)=0$,

Case (2) $((-b^2+\rho(u)^2)\tau(u)^2+\rho'(u)^2)=0$

that is, $\tau(u)=\pm\frac{\rho'(u)}{\sqrt{b^2-\rho(u)^2}}$

$$\Rightarrow \rho(u)=b\sin\left(\int\tau du\right)+constant$$

$$or\quad \rho(u)=b\cos\left(\int\tau du\right)+constant \tag{3.17}$$

Thus, we have the proof at the following theorem:

Theorem 3.5. *The cyclic surface generated by circles of curvature of the space curve is a special Weingarten surface with condition $H+bK=0$ if the curve is a plan curve or its torsion and radius of curvature are related by the relation (3.17).*

3.2.2. Case c $\neq 0$

Similarly, we can express (3.14) as the following form

$$\sum_{i=0}^{8}(A_i\cos(iv)+B_i\sin(iv))=0. \tag{3.18}$$

After some computation, the non vanishing coefficients of Eq. (3.18) are

$$A_8=-\frac{1}{32}\rho(u)^8(\rho(u)^2\tau(u)^2+\rho'(u)^2)^2 \times((b^2-(1+2bc)\rho(u)^2+c^2\rho(u)^4)\tau(u)^4 -(1+2bc-2c^2\rho(u)^2)\tau(u)^2\rho'(u)^2+c^2\rho'(u)^4)$$

From A_8, We discuss the following three cases for the vanishing all the coefficients identically

Case (1) $(\rho(u)^2\tau(u)^2+\rho'(u)^2)=0$ then, that is $\rho'(u)=0$, $\tau(u)=0$

Case (2) $((b^2-(1+2bc)\rho(u)^2+c^2\rho(u)^4)\tau(u)^4-(1+2bc-2c^2\rho(u)^2)\tau(u)^2\rho'(u)^2+c^2\rho'(u)^4)=0$ after simplification, we have

$$\tau(u)=\pm\frac{\rho'(u)}{\sqrt{(\frac{d}{\sqrt{2}})^2-\rho(u)^2}}$$

where $d^2=\frac{1+2bc\pm\sqrt{1+4bc}}{c^2}$

By integration, we have

$$\rho(u)=\frac{d}{\sqrt{2}}\sin\left(\int\tau(u)du\right)+constant$$

$$or\quad \rho(u)=\frac{d}{\sqrt{2}}\cos\left(\int\tau(u)du\right)+constant \tag{3.19}$$

Thus, we have the proof at the following theorem:

Theorem 3.6. *The cyclic surface generated by circles of curvature of a space curve is a special Weingarten surface with condition $H+bK=c$ if the curve is a circle or the radius of curvature and the torsion are related by (3.19).*

3.3. Cyclic W-surfaces

A surface M in 3-dimensional Euclidean space R^3 is called a Weingarten surface if there is a relation between its two curvatures K and H, that is, if the jacobian determinant is identically zero [14], i.e.,

$$\varphi(K,H)=\left|\frac{\partial(K,H)}{\partial(u,v)}\right|\equiv 0 \tag{3.20}$$

Or the following form

$$\frac{\partial H}{\partial u}\frac{\partial K}{\partial v}-\frac{\partial H}{\partial v}\frac{\partial K}{\partial u}=0, \tag{3.21}$$

After some computations we have

$$\left(2\frac{\partial H_1}{\partial u}W - 3H_1\frac{\partial W}{\partial u}\right)\left(\frac{\partial K_1}{\partial v}W - 2K_1\frac{\partial W}{\partial v}\right) - \left(2\frac{\partial H_1}{\partial v}W - 3H_1\frac{\partial W}{\partial v}\right)\left(\frac{\partial K_1}{\partial u}W - 2K_1\frac{\partial W}{\partial u}\right) = 0 \tag{3.22}$$

Using the same technique as in the previous section we have

$$\sum_{i=0}^{9}(A_i\cos(iv) + B_i\sin(iv)) = 0. \tag{3.23}$$

After some computation, the non vanishing coefficient of Eq. (3.23) is

$$A_9 = -\frac{1}{128}\rho(u)^{13}\rho(u)'(\rho(u)^2\tau(u)^2 + \rho'(u)^2) \times (\rho(u)\tau(u)^3 - \rho'(u)\tau'(u) + \tau(u)\rho''(u))^3$$

Thus, we have three cases for the vanishing of A_9 identically as

Case (1) $\rho'(u) = 0$ **Case** (2) $\tau(u) = 0$

Case (3)

$$\rho(u)\tau(u)^3 - \rho'(u)\tau'(u) + \tau(u)\rho''(u) = 0,\ \tau(u) \neq 0$$

or in the following form

$$\tau(u) = \pm\frac{\rho'(u)}{\sqrt{d - \rho(u)^2}}$$

By integration we have

$$\rho(u) = d\sin\left(\int \tau(u)du\right) + constant$$

$$or \quad \rho(u) = d\cos\left(\int \tau(u)du\right) + constant \tag{3.24}$$

where d is constant.

Thus, we have the proof at the following theorem:

Theorem 3.7. *The cyclic surface generated by circles of curvature of the space curve is a W-surface with condition (3.21) if the curve is a plan curve or its radius of curvature is constant or the radius of curvature and the torsion is related by (3.24).*

4. Cyclic HK-quadratic surfaces

In this section we study surface satisfying some algebraic equation in the mean curvature and the Gaussian curvatures.

$$aH^2 + 2bHK + cK^2 = constant, \quad a \neq 0. \tag{4.1}$$

This type of surfaces is called HK- quadratic surface [15]. Using (3.1), (3.2) the HK-quadratic surface satisfies the condition

$$a\left(\frac{H_1}{2W^{\frac{3}{2}}}\right)^2 + 2b\left(\frac{H_1}{2W^{\frac{3}{2}}}\right)\left(\frac{K_1}{W^2}\right) + c\left(\frac{K_1}{W^2}\right)^2 = 1$$

or, equivalently,

$$aH_1^2W + 4bW^{\frac{1}{2}}H_1K_1 + 4cK_1^2 = 4W^4$$

equivalently, we have

$$4b\sqrt{W}H_1K_1 - (4W^4 - aH_1^2W - 4cK_1^2) = 0, \quad a \neq 0. \tag{4.2}$$

Using the Eqs. (2.9) and (2.10), we can expressed (4.2) by trigonometric polynomial on $\cos(iv)$, $\sin(iv)$. Exactly, there exist smooth functions on u, namely A_i, B_i, such that (4.2) writes as

$$\sum_{i=0}^{8}(A_i\cos(iv) + B_i\sin(iv)) = 0. \tag{4.3}$$

Since this is an expression on the independent trigonometric terms $\cos nv$ and $\sin nv$, all coefficients A_i , B_i must vanish identically.

After some computation, the non vanishing coefficient of Eq. (4.3) is

$$A_8 = \frac{1}{32}\rho(u)^8(\rho'(u)^2 + \rho(u)^2\tau(u)^2)^2\big(\tau(u)^4(a\rho(u)^2 + c - \rho(u)^4) + \tau(u)^2(a - 2\rho(u)^2)\rho'(u)^2 - \rho'(u)^2\big)$$

Thus, we have the two possibilities for vanishing the coefficients identically as in the following.

Case (1)$(\rho(u)^2\tau(u)^2 + \rho'(u)^2) = 0$ i.e. $\tau(u) = 0,\ \rho'(u) = 0$

Case (2)

$$(\tau(u)^4(a\rho(u)^2 + c - \rho(u)^4) + \tau(u)^2(a - 2\rho(u)^2)\rho'(u)^2 - \rho'(u)^4) = 0,$$

$\tau(u) \neq 0$ *and* $\rho'(u) \neq 0$ *at the same time*

the solution of this equation is

$$\tau_{1,2}(u) = \pm\frac{\sqrt{-\frac{\rho'(u)^2\left(\sqrt{a^2+4c}+a-2\rho(u)^2\right)}{a\rho(u)^2+c-\rho(u)^4}}}{\sqrt{2}},$$

$$\tau_{3,4}(u) = \pm\frac{\sqrt{\frac{\rho'(u)^2\left(\sqrt{a^2+4c}-a+2\rho(u)^2\right)}{a\rho(u)^2+c-\rho(u)^4}}}{\sqrt{2}} \tag{4.4}$$

(a) then, from $\tau_1(u)$ we have

$B_7 = \frac{-b\rho(u)^7\rho'(u)^4}{64(a\rho(u)^2+c-\rho(u)^4)^2}((a\sqrt{a^2+4c} - a^2 - 2c)\rho(u)^4 +$
$2c(\sqrt{a^2+4c} - a)$
$\rho(u)^2 - 2c^2)(-\frac{\rho'(u)^2(\sqrt{a^2+4c}+a-2\rho(u)^2)}{a\rho(u)^2+c-\rho(u)^4})^{3/2}$
$\sqrt{\frac{\rho(u)^2\rho'(u)^2(\sin(\frac{v}{2})+\cos(\frac{v}{2}))^4((a-\sqrt{a^2+4c})\rho(u)^2+2c)}{a\rho(u)^2+c-\rho(u)^4}}$

Thus,

(i) $\rho'(u) \neq 0$, $b = 0$
then, Thus we have all coefficients vanishes.

(ii) $\rho'(u) \neq 0, \quad c = 0, \quad a = 1$
Then, $\tau(u) = -\sqrt{-\frac{\rho'(u)^2}{\rho(u)^2}}$, this is contradiction

(b) From $\tau_{2,\,3,\,4}(u)$ we have the same results.

Theorem 4.1. *The cyclic surface generated by circles of curvature of a circle is a HK-quadratic surface. The cyclic surfaces satisfied equation $aH^2 + cK^2 = 1$ if the torsion of a space curve is given by Eq. (4.4).*

5. Example

Example 5.1 (Surface of type LW-surface and HK-quadratic)**.** Consider a plane curve given by

$$\mathbf{\Psi}(u) = \{\sin(au), \cos(au), 0\}, \tag{5.1}$$

Thus, the center of circle of curvature is

$$\mathbf{c}(u) = \{(1-a)\sin(au), (1-a)\cos(au), 0\}, \tag{5.2}$$

Therefore, the equation of the cyclic surface that is generated by the circle of curvature of a plane curve (5.1) is

$$\mathbf{X}(u, v) = \mathbf{c}(u) + \{a\cos(au + v), -a\sin 9au + v), 0\},$$

According to Theorems 3.2 and 4.1 , this is a cyclic surface satisfying conditions of LW-surface and HK-quadratic surface. This surface plotted as in Fig. 1.

Example 5.2 (W-surface)**.** Consider the space curve (helix) given by

$$\Psi(u) = \{a\sin(u), a\cos(u), au\}, \tag{5.3}$$

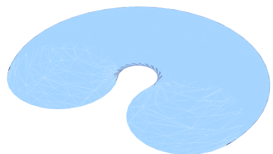

Fig. 1. Cyclic LW-surface and HK-quadratic surface, with $\tau(u) = 0,\ \rho'(u) = 0$.

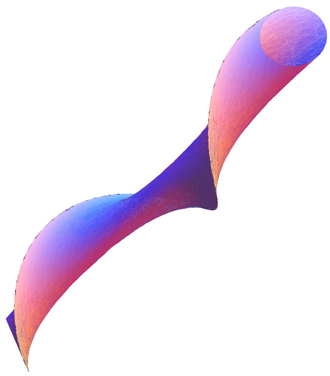

Fig. 2. Cyclic W-surface with $\rho'(u) = 0$.

Thus, the center of circle of curvature is

$$\mathbf{c}(u) = \left\{-\frac{\sin(u)}{\sqrt{2}} + a\sin(u), -\frac{\cos(u)}{\sqrt{2}} + a\cos(u), au\right\},\ \rho = \frac{1}{\sqrt{2}} \tag{5.4}$$

Therefore, the equation of the cyclic surface that is generated by the circle of curvature of a space curve (5.3) is

$$\mathbf{X}(u,v) = \mathbf{c}(u) + \left\{\frac{1}{2}(\cos(u)\cos(v) - \sqrt{2}\sin(u)\sin(v)), -\frac{1}{2}\cos(v)\sin(u) - \frac{\cos(u)\sin(v)}{\sqrt{2}}, \frac{1}{2}\cos(v)\right\},$$

According to Theorem 3.7, this is a cyclic surface of type Weingarten surface, which display in Fig. 2.

Acknowledgments

We wish to express our profound thanks and appreciation to professor, Dr. Nassar H. Abdel All, Department of Mathematics, Assiut University, Egypt, for his strong support, continuous encouragement, revising this paper carefully and for his helpful suggestions. We would like also to thank Dr. H. N. Abd-Ellah, Department of Mathematics, Faculty of Science, Assiut University, Egypt, for his critical reading of this manuscript, helpful suggestions and making several useful remarks. Also, the authors thank the editor and the referees for their comments and suggestions to improve the paper.

References

[1] R. López, On linear Weingarten surfaces, Int. J. Math. 19 (4) (2008a) 439–448.
[2] R. López, Special Weingarten surfaces foliated by circles, Monatsh Math. 154 (2008b) 289–302.
[3] W. J., Ueber eine Klasse auf einander abwickelbarer Fläachen, J. Reine Angew. Math. 59 (1861) 382–393.
[4] J. Weingarten, Ueber die Flächen, derer Normalen eine gegebene Fläche berühren, J. Reine Angew. Math. 62 (1863) 61–63.
[5] K. Voss, Über geschlossene Weingartensce Flächen, Math. AAnnalen 138 (1959) 42–54.
[6] H. Hopf, Über flächen mit einer relation zwischen den hauptkrümmungen, Math. Nachr4 (1951) 232–249.
[7] H. Rosenberg, R. Sa Earp, The geometry of properly embedded special surfaces in R^3, e.g., surfaces satisfying $ah + bk = 1$, where a and b are positive, Duke Math. J. 73 (1994) 291–306.
[8] W. Kühnel, M. Steller, On closed Weingarten surfaces, Monatsh Math. 146 (2005) 113–126.
[9] B. van Brunt, K. Grant, Potential applications of weingarten surfaces in CAGD. I: Weingarten surfaces and surface shape investigation, Comput. Aided Geom. Des. 13 (1996) 569–582.
[10] R. López, Cyclic surfaces of constant gauss curvature, Houston J. Math. 1.27 (4) (2001) 799–805.
[11] N.H. Abdel-All, F.M. Hamdoon, H.N. Abd-Ellah, M.A. Abd-Rabo, Investigation of cyclic surfaces with constant curvatures, Ciencia e Tecnica Vitivinicola J. 30 (2) (2015) 214–224.
[12] M.D. Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1976.
[13] F.M. Hamdoon, Spiral surfaces enveloped by one-parametric sets of spheres, J. Geom. 98 (2010) 51–58.
[14] H.N. Abd-Ellah, Translation l/w-surfaces in euclidean 3-space e^3, J. Egyptian Math. Soc. 23 (2015) 513–517.
[15] Y.H. Kim, D.W. Yoon, On non-developable ruled surfaces in lorentz-minkowski 3-space, Taiwanese J. Math. 11 (1) (2007) 197–214.