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Analysis and asymptotic stability of uniformly Lipschitzian nonlinear semigroup systems

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ABSTRACT

In this paper, we shall study the asymptotic stability analysis of a special kind of semigroup on $D \times D$, namely, the uniformly Lipschitzian semigroups.

1. Introduction

Stability analysis is one of the extremely important issues in system theory, in stability of equilibrium points of differential equations, in practical computer science, in numerical analysis and in trajectories of dynamical systems. Numerous results on the asymptotic demeanor of nonlinear infinite-dimensional systems are known, for which the non-expansive and the dissipativity properties play an essential role, see e.g. [1,2]. Some years later, Aksikas et al.[3,4] developed a theory concerning the asymptotic stability of infinite-dimensional of nonlinear and semilinear systems in a real reflexive Banach space.

Our contributions in this paper are twofold: (1) we shall introduce a coupled fixed point theorem for uniformly Lipschitzian one parameter semigroup of mappings in Banach space with normal structure; and (2) we establish asymptotically stability for infinite-dimensional nonlinear uniformly Lipschitzian systems in Banach space with normal structure.

2. Preliminaries

Definition 2.1 [3]**.** An operator T_t: $C \to C$, $t \geq 0$, is called nonlinear one parameter semigroups of non-expansive self-mappings on a subset C for a real reflexive Banach space X if it satisfy the following conditions:

(i) $T_{t+s}x = T_t T_s x \ \forall\ s,\ t \geq 0;\ T_0 x = x$, for every $x \in C$;
(ii) $\|T_s x - T_s x_1\| \leq \|x - x_1\| \ \forall\ s \geq 0\ \&\ x, x_1 \in C$.
(iii) $\forall\ t \geq 0$, T_t: $X \to X$ is continuous.

Suppose the following system

$$\begin{cases} \frac{dx}{dt} = Ax(t), & \forall\, t > 0 \\ x(0) = x_0, \end{cases} \tag{1}$$

and $x(t, x_0) = T_t x_0$ is the solution of (1) and A is the generator of T_t on C. The generator A is defined on its domain

$$D(A) = \left\{ x \in C : \lim_{t \to 0+} \frac{T_t x - x}{t}\ exists \right\},$$

by

$$Ax = \lim_{t \to 0+} \frac{T_t x - x}{t}, \text{ for every } x \in D(A).$$

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Definition 2.2 [3]**.** Suppose that X be a real reflexive Banach space and $C \subset X$ be a closed convex subset. Let T_t is a nonlinear Lipschitzian semigroups on C. Then for any $x_0 \in C$, we say that $x \in \omega(x_0)$ if and only if $x \in C$ and there exist a sequence $t_n \to \infty$ such that

$$x = \lim_{n\to\infty} T_{t_n} x_0.$$

Theorem 2.1 [3]**.** *Suppose that T_t be a nonexpansive semigroup of nonlinear self-mappings on C and A satisfy the following properties*

(i) $\|x - x_1\| \le \|(x - x_1) - s(Ax - Ax_1)\|, \ \forall\ x, x_1 \in D(A)$ *and* $\forall\ s > 0$; *(dissipative property)*

(ii) $\overline{convx(D(A))} \subset \bigcap_{\lambda>0} R(I - \lambda A)$;

(iii) $(I - sA)^{-1}$ *is compact for some* $s > 0$.

Assume that $D = \overline{D(A)}$ *and* $\bar{x}$ *is a solution point of system (8). Then for each* $x_0 \in D$, $x(t, x_0) = T_t x_0$ *converges to* $\omega(x_0)$ *as* $n \to \infty$, *where* $\omega(x_0) \subset \{z : \|z - \bar{x}\| = r\}$, *where* $\|x_0 - \bar{x}\| \ge r$. *i.e.*

Moreover if A satisfy the dissipative property with strict inequality, then $\lim_{t\to\infty} x(t, x_0) = \bar{x}$ *i.e.* $\bar{x}$ *is an asymptotically stable of system (8) on C.*

Definition 2.3 [10]**.** Let $\eta = \{T_t : t \in G,\ G = [0, \infty)\}$ be a class of on $C \times C$. Then η is called a lipschitzian semigroup on $C \times C$ if the following properties are satisfied:

(i) $\forall\ u, v \in X,\ T_0(u, v) = u$;

(ii) $T_s T_t(u, v) = T_{s+t}(u, v)\ \ \forall\ s, t \in G$ and $u, v \in X$;

(iii) $\|T_t(u, v) - T_t(u_1, v_1)\| \le \frac{k}{2}[\|u - u_1\| + \|v - v_1\|]$ for every $t \ge 0$ and every $(u, u_1), (v, v_1) \in C \times C$, $k > 0$.

(iv) $\forall\ u, v \in X$, the operator $t \to T_t(u, y)$ and $t \to T_t(v, u)$ from G into X are continuous when G has the relative topology of $[0, \infty)$;

(v) for each $s \in G$, T_s: $C \times C \to C$ is continuous.

The notation normal structure (N.S., for short) of closed subsets of a Banach space is one of the extreme important aspects of fixed point theory. This notation was introduced by Milman and Brodskii[5]. In 1980, Bynum [6] defined the concept of normal structure coefficient $N(X)$ (N. S. C., for short) which was used by Maluta and Casini [7] to get a fixed point for uniformly lipschitzian mappings. The important application of N.S. is in fixed point area and other fields concerning with solutions of differential and integral equations etc.

Assume that C be a bounded nonempty subset of a Banach space X. Then $u \in C$ is called a diametral point of C if $\sup\{||u - x|| : x \in C\} = \sup\{||x - y|| : x, y \in C\} = \delta(C)$ (where $\delta(C)$ denotes the diametral of C) and a nondiametral point of C if $\sup\{||u - x|| : x \in C\} < \sup\{||x - y|| : x, y \in C\}$.

Definition 2.4 [6]**.** Suppose that $E \subset X$ and let F be a family of nonempty subsets of E. The family F is said to be normal structure if $r(C) < \delta(C)$ ($r(C) = \inf_{x\in C} \sup_{y\in C} ||x - y||$) for all nonempty bounded set $C \in F$ with $\delta(C) > 0$. If $0 < c < 1$ such that $r(C) \le c\delta(C)$ for each bounded set $C \in F$ with $\delta(C) > 0$, then F is said to be uniformly normal structure (U. N. S., for short).

Definition 2.5 [6]**.** Let X be a Banach space. Then N. S. C. $N(X)$ defined as follows:

$$N(X) = \inf_{C\subset X} \left\{ \frac{\delta(C)}{r(C)} \right\}, \quad C \text{ a convex bounded closed subset of } X$$

Remark 2.1. $N(X) > 1$ iff X has U. N. S.

We define the modulus of convexity of X by the function δ_X: $[0, 2] \to [0, 1]$ where

$$\delta_X(\varepsilon) = \inf\{1 - ||(u + v)/2|| : ||u|| \le 1, ||v|| \le 1 \text{ and } ||u - v|| \ge \varepsilon\}.$$

Remark 2.2. The following property of modulus of convexity X is quite well-known (see [9])

(e) $[||a - u|| \le r, ||a - v|| \le r$ and $||u - v|| \ge \varepsilon] \Rightarrow ||a - (u + v)/2|| \le r(1 - \delta_X(\varepsilon/r))$.

Let C be a convex closed nonempty subset of a Banach spaces X and let $\{x_t\colon t \in G\}$ be a bounded net of members of X. Then the asymptotic radius and the asymptotic center of $\{x_t\}_{t\in G}$ with respect to C are the values

$$r_C\{x_t\} = \inf_{y\in C} \limsup_t |||x_t - y| \quad \forall \quad i = 1, 2, 3, \dots, n,$$

and respectively, the sets

$$A_C(\{x_t\}) = \{y \in C : \limsup_t |||x_t - y| = r_C\{x_t\} \quad \forall \quad i = 1, 2, 3, \dots, n\}$$

3. Main results

Let us consider the system

$$\begin{cases} \frac{d(u(t), v(t))}{dt} & = A(u(t), v(t)), \quad t > 0 \\ (u(0), v(0)) & = (u_0, v_0), \end{cases} \tag{2}$$

Where A is the generator T_t as follows:

$$D(A) = \{(u, v) \in C : \lim_{t\to 0} t^{-1}[T_t(u, v) - u] \text{ and } \lim_{t\to 0} t^{-1}[T_t(v, u) - v] \text{ exist}\}$$

by

$$A(u, v) = \lim_{t\to 0} t^{-1}[T_t(u, v) - u],$$

$$A(v, u) = \lim_{t\to 0} t^{-1}[T_t(v, u) - v].$$

For any $(u_0, v_0) \in D(\mathcal{A})$, $x(t, (u_0, v_0)) = T_t(u_0, v_0)$ is the solution of system (2).

Remark 3.1. System (2) appear in many fields such as differential equations spatially in periodic boundary value problems [8], system theory, besides numerous others.

Definition 3.1. Suppose the system (2) and let A generates a nonlinear uniformly Lipschitzian semigroup T_t. Consider $(\bar{u}, \bar{v})$ be an equilibrium point of system (2), i.e. $(\bar{u}, \bar{v}) \in D(A)$ and $A(\bar{u}, \bar{v}) = 0$, $A(\bar{v}, \bar{u}) = 0$, i.e. $(\bar{u}, \bar{v})$ is an asymptotically stable point of(2) on C if

$$\lim_{t\longrightarrow\infty} X(t, (u_0, v_0)) = \lim_{t\longrightarrow\infty} T_t(u_0, v_0) = \bar{u},$$

$$\lim_{t\longrightarrow\infty} X(t, (v_0, u_0)) = \lim_{t\longrightarrow\infty} T_t(v_0, u_0) = \bar{v}.$$

For all $(u_0, v_0) \in D \times D$.

The $\omega((u_0, v_0))$ of (u_0, v_0) is the set which for all $(u, v) \in \omega((u_0, v_0))$ if and only if $(u, v) \in C \times C$ and there exists two sequences $t_n \longrightarrow \infty$, $s_n \longrightarrow \infty$ such that

$$u = \lim_{n\longrightarrow\infty} T_{t_n}(u_0, v_0),$$

$$v = \lim_{n\longrightarrow\infty} T_{s_n}(v_0, u_0).$$

Lemma 3.1. *$w((u_0, v_0))$ is closed.*

Proof. Consider a sequence $(u_m, v_m) \in w((u_0, v_0))$ for each $\epsilon > 0$, there exists an $M > 0$ such that $\|u_m - u\| < \frac{\epsilon}{2}$, $\|v_m - v\| < \frac{\epsilon}{2}$ for $n > M$. To each (u_m, v_m) there exist a sequence $\{t_r\}$ such that $\lim_{r\to\infty} t_r = \infty$ and $\|u_m - T(t_r)(u_0, v_0)\| < \frac{\epsilon}{2}$, $\|v_m - T(t_r)(v_0, u_0)\| < \frac{\epsilon}{2}$ for all $\epsilon > 0$. Hence

$$\|T(t_r)(u_0, v_0) - x\| \le \|T(t_r)(u_0, v_0) - u_m\| + \|u_m - u\|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Similarly one can deduce that

$$\|T(t_r)(v_0, u_0) - v\| < \epsilon.$$

This implies that $(u, v) \in w((u_0, v_0))$. $\square$

Proposition 3.1. *For any $(u_0, v_0) \in D \times D$, if $\omega((u_0, v_0))$ is nonempty and if $(\overline{u}, \overline{v})$ is a coupled fixed point of T_t, i.e $T_t(\overline{u}, \overline{v}) = \overline{u}$, $T_t(\overline{v}, \overline{u}) = \overline{v}$ for all $t \geq 0$, then*

$$\omega((u_0, v_0)) \subset \{(z_1, z_2) : \| z_1 - \overline{x} \| = r, \| z_2 - \overline{v} \| = r\},$$
$$\text{with } r \leq \| u_0 - \overline{u} \|, r \leq \| v_0 - \overline{v} \| .$$

Remark 3.2. We note from the Definition 2.3 and Proposition 2.1 that if $(\overline{u}, \overline{v})$ coupled fixed point of T_t then it is equilibrium point of A.

Example 3.1. Let $X = [0, 1]$ and $T_t(u, v) = e^{t[(u-1)+(v-1)]}$. Then (1, 1) is coupled fixed point of T_t and equilibrium point of A_T.

The following coupled fixed point theorem is very important to study asymptotic stability criteria.

Theorem 3.1. *Let X be a real Banach space with U. N. S., C a nonempty convex closed subset of X, $\tau = \{T_s : s \in G\}$ a uniformly Lipschitzian semigroup on $C \times C$ with $k > 1$ and*

$$k < \sqrt{2N(X)\delta_X^{-1}(\eta)}.$$

Where η defined below.

If $\{T_s(u_0, v_0) : s \in G\}$ is bounded for some $(u_0, v_0) \in C \times C$, then there exists $(u, v) \in C \times C$ such that $T_t(u, v) = u, T_t(v, u) = v$, for all $s \in G$.

Proof. Put $\widetilde{N}(X) = N(X)^{-1}$. Since X has a U. N. S, X is reflexive. Due to the boundedness of $\{T_s(u_0, v_0) : s \in G\}$ and by ([11], Lemma 2.1), we get that $A_C(\{(T_t(u_0, v_0), T_t(v_0, u_0))\}_{t\in G})$, is nonempty closed bounded convex subset of $C \times C$. Then we can choose $(u_1, v_1) \in A_C(\{(T_t(u_0, v_0), T_t(v_0, u_0))\}_{t\in G})$, such that

$$\limsup_t ||T_t(u_0, v_0) - u_1|| = \inf_{(w_1,w_2)\in C\times C} \limsup_t ||T_t(u_0, v_0) - w_1||,$$
$$\limsup_t ||T_t(v_0, u_0) - v_1|| = \inf_{(w_1,w_2)\in C\times C} \limsup_t ||T_t(v_0, u_0) - w_2||.$$

Consequently we can choose $(u_2, v_2) \in A_C(\{(T_t(u_1, v_1), T_t(v_1, u_1))\}_{t\in G})$, such that

$$\limsup_t ||T_t(u_1, v_1) - x_2|| = \inf_{(w_1,w_2)\in C\times C} \limsup_t ||T_t(u_1, v_1) - w_1||,$$
$$\limsup_t ||T_t(v_1, u_1) - v_2|| = \inf_{(w_1,w_2)\in C\times C} \limsup_t ||T_t(v_1, u_1) - w_2||.$$

Then, we can construct sequences $\{(u_m, v_m)\}_{m=0}^{\infty}$ in $C \times C$ with the following properties

(i) for each $m \geq 0$, $\{T_t(u_m, v_m)\}_{t\in G}$ and $\{T_t(v_m, u_m)\}_{t\in G}$ are bounded;

(ii) for each $m \geq 0$, $(u_{m+1}, v_{m+1}) \in A_C(\{(T_t(u_m, v_m),, T_t(v_m, u_m))\}_{t\in G})(v_{m+1}, u_{m+1}) \in A_C(\{(T_t(v_m, u_m), T_t(u_m, v_m))\}_{t\in G})$ are points in $C \times C$ such that

$$\lim_t ||T_t(u_m, v_m) - u_{m+1}|| = \inf_{(w_1,w_2)\in C\times C} \lim_t ||T_t(u_m, v_m) - w_1||,$$
$$\lim_t ||T_t(v_m, u_m) - v_{m+1}|| = \inf_{(w_1,w_2)\in C\times C} \lim_t ||T_t(v_m, u_m) - w_2||.$$

Write

$$r_{mu} = r_C(\{T_t(u_m, v_m)\}_{t\in G}),$$
$$r_{mv} = r_C(\{T_t(v_m, u_m)\}_{t\in G}).$$

By using ([11], Lemma 2.2) we get

$$r_{mu} = \limsup_t ||T_t(u_m, v_m) - u_{m+1}|| \leq \widetilde{N}(X)D(\{T_t(u_m, v_m)\}_{t\in G})$$
$$= \widetilde{N}(X) \lim_t (\sup\{||T_i(u_m, v_m) - T_j(u_m, v_m)|| : t \leq i, j \in G\})$$
$$= \widetilde{N}(X) \lim_t (\sup\{||T_i(u_m, v_m) - T_i(T_{j-i}(u_m, v_m))|| : t \leq i, j \in G\})$$
$$= \widetilde{N}(X) \lim_t (\sup\{||T_t(u_m, v_m) - T_i(T_{j-i}(u_m, v_m), T_{j-i}(v_m, u_m)||$$
$$: t \leq i, j \in G\})$$
$$\leq \widetilde{N}(X)\frac{k}{2} \lim_t (\sup\{||u_m - T_{j-i}(u_m, v_m)\| + \|v_m - T_{j-i}(v_m, u_m)||$$
$$: t \leq i, j \in G\}) \leq \widetilde{N}(X).\frac{k}{2}.d(u_m, v_m),$$

that is,

$$r_{mu} \leq \widetilde{N}(X).\frac{k}{2}.d(u_m, v_m) < \widetilde{N}(X).k.d(u_m, v_m). \tag{3}$$

Where

$$d(u_m, v_m) = \sup\{||u_m - T_t(u_m, v_m)\| + \|v_m - T_t(v_m, u_m)|| : t \in G\}).$$

Similarly one can deduce that

$$r_{my} \leq \widetilde{N}(X).\frac{k}{2}.d(u_m, v_m) < \widetilde{N}(X).k.d(u_m, v_m).$$

We may assume that $d(u_m, v_m) > 0$, for all $m \geq 0$ (since otherwise $u_m = T_t(u_m, v_m)$, $v_m = T_t(v_m, u_m)$ i.e., (u_m, v_m) is a coupled fixed point of τ and the proof is finished). Let $m \geq 0$ be fixed and let $\varepsilon > 0$ be small enough. We can choose $j \in G$ such that

$$||T_j(u_{m+1}, v_{m+1}) - u_{m+1}|| > d(u_{m+1}, v_{m+1}) - \varepsilon$$

and then we can choose $s_0 \in G$ so large that

$$||T_s(u_{m+1}, v_{m+1}) - u_{m+1}|| < r_{mu} + \varepsilon, \quad \forall \ s \geq s_0,$$

It turns out, for $s \geq s_0 + j$,

$$||T_s(u_m, v_m) - T_j(u_{m+1}, v_{m+1})|| \leq \|T_j(T_{s-j}(u_m, v_m), T_{s-j}(v_m, u_m))$$
$$-T_j(xu_{m+1}, v_{m+1})\|$$
$$\leq \frac{k}{2}[||T_{s-j}(u_m, v_m) - u_{m+1}|| + ||T_{s-j}(v_m, u_m) - v_{m+1}||$$
$$< \frac{k}{2}(r_{mu} + r_{mv} + \varepsilon) < k(r_{mu} + r_{mv} + \varepsilon).$$

Then it follows from property (e) that

$$||T_s(u_m, v_m) - \frac{1}{2}(u_{m+1} + T_j(u_{m+1}, v_{m+1}))|| \leq k(r_{mu} + r_{mv} + \varepsilon)$$
$$\left(1 - \delta_X\left(\frac{d(u_{m+1}, v_{m+1}) - \varepsilon}{k(r_{mu} + r_{mv} + \varepsilon)}\right)\right)$$

for $s \geq s_0 + j$ and as $\varepsilon \to 0$ we obtain

$$r_{mu} \leq \limsup_s ||T_s(u_m, v_m) - \frac{1}{2}(u_{m+1} + T_j(u_{m+1}, v_{m+1}))|| \leq k$$
$$(r_{mu} + r_{mv})\left(1 - \delta_X\left(\frac{d(u_{m+1}, v_{m+1})}{k(r_{mu} + r_{mv})}\right)\right).$$

This implies that

$$\delta_X\left(\frac{d(u_{m+1}, v_{m+1})}{k(r_{mu} + r_{mv})}\right) \leq 1 - \frac{r_{mu}}{k(r_{mu} + r_{mv})}, \tag{4}$$

$$d(u_{m+1}, v_{m+1}) \leq k(r_{mu} + r_{mv})\delta_X^{-1}(\eta), \tag{5}$$

where $\eta = 1 - \frac{r_{mu}}{k(r_{mu}+r_{mv})} < 1$.

Therefore, utilizing (3) and (5), we obtain

$$d(u_{m+1}, v_{m+1}) \leq k^2\widetilde{N}(X)\delta_X^{-1}(\eta)d(u_m, v_m). \tag{6}$$

Write $A = k^2\widetilde{N}(X)\delta_X^{-1}(\eta)$. Then $A < 1$. Hence, it is follows from (6) that

$$d(u_{m+1}, v_{m+1}) \leq Ad(u_{m-1}, v_{m-1}) \leq \ldots \leq A^m d(u_0, v_0). \tag{7}$$

Since

$$||u_{m+1} - u_m|| + ||v_{m+1} - v_m|| \leq \limsup_t [||u_{m+1} - T_t(u_m, v_m)||$$
$$+||T_t(u_m, v_m) - u_m|| + \limsup_t [||v_{m+1} - T_t(v_m, u_m)||$$
$$+||T_t(v_m, u_m) - v_m|| \leq r_{mu} + r_{mv} + d(u_m, v_m)$$
$$< (\widetilde{N}(X)k + 1)d(u_m, v_m)$$

We get from (7) that $\sum_{m=1}^{\infty}[||u_{m+1}-u_m||+||v_{m+1}-v_m||]<\infty$, and hence $\{u_m\}$, $\{v_m\}$ are Cauchy sequences. Let $u=\lim_{m\to\infty}u_m$, $v=\lim_{m\to\infty}v_m$. Finally, we have for each $s\in G$,

$$\begin{aligned}&||u-T_s(u,v)||+||v-T_s(v,u)||\\ &\leq ||u-u_m||+||T_s(u_m,v_m)-x_m||+||T_s(u_m,v_m)-T_s(u,v)||\\ &+||v-v_m||+||T_s(v_m,u_m)-v_m||+||T_s(v_m,u_m)-T_s(v,u)||\\ &\leq (2k+1)[||u-u_m||+||v-v_m||]+d(u_m,v_m).\to 0 \text{ as } m\to\infty\end{aligned}$$

Hence, $T_t(u,v)=u$, $T_t(v,u)=v$, for all $s\in G$ and the proof is complete. □

Example 3.2. Consider $X=[0,1]$ equipped with the spectral norm. Let η be the set of closed subsets of X. Define the function T_t by

$$T_t(x,y)=\begin{cases}x, & t=0\\ (\frac{x+y}{2})e^{At}, & t>0,\end{cases}$$

for all $x, y\in X$ and

$$A=\begin{pmatrix}-1 & 2\\ 0 & -1\end{pmatrix}$$

Then the following statements are holds

1- A is not strictly dissipative

2- T_t is Lipschitzian one parameter semigroup.

3- (0,0) is a coupled fixed point of T_t.

Lemma 3.2. *$w((x,y))$ is convex for all $(x, y)\in D(T_t)$.*

Lemma 3.3. *Every point in $w((x_0,y_0)$ is coupled fixed point for any $(x_0, y_0)\in D(T_t)$.*

Proof. Since $w((x_0,y_0)$ is convex closed subset of a Banach space $X\times X$ with N. S. and every point of it is convergence point and T_t uniformly Lipschitzian semigroup on $X\times X$. Then by Theorem 3.1 we conclude the desired result. □

Proposition 3.2. *Let T_t be a nonlinear uniformly Lipschitzian semigroup on $D\times D$, generated by $\mathcal{A}_T$. Then $\omega((w,z))=\omega((x,y))$ ∀ $(x,y)\in\omega((w,z))$.*

Proof. Fix $(x,y)\in\omega((w,z))$, say $x=\lim_{n\to\infty}T_{t_n}(w,z)$, $y=\lim_{n\to\infty}T_{t_n}(z,w)$ with $t_n\to\infty$ as $n\to\infty$. Suppose now $(x_1,y_1)\in\omega((w,z))$, say $x_1=\lim_{n\to\infty}T_{\omega_n}(w,z)$, $y_1=\lim_{n\to\infty}T_{\omega_n}(z,w)$ with $\omega_n\to\infty$ as $n\to\infty$. We may assume without loss of generality that $s_n=\omega_n-t_n\geq n$, $n=1,2,\ldots,$ since

$$\begin{aligned}&||T_{s_n}(x,y)-x_1||\leq||T_{s_n}(x,y)-T_{s_n+t_n}(w,z)||\\ &+||T_{s_n+t_n}(w,z)-x_1||\to\ 0\ \ as\ \ n\to\infty.\end{aligned}$$

Similarly one can obtain that

$$||T_{s_n}(y,x)-y_1||\to\ 0\ \ as\ \ n\to\infty.$$

Which implies that $(x_1, y_1)\in\omega((x, y))$, then

$$\omega((w,z))\subset\omega((x,y)). \tag{8}$$

Similarly one can get that

$$\omega((x,y))\subset\omega((w,z)). \tag{9}$$

From (8) and (9) we have

$$\omega((w,z))=\omega((x,y)).$$

□

Proposition 3.3. *Let us consider the system (2) and let T_t be the uniformly Lipschitzian nonlinear semigroup on $C\times C$, generated by $\mathcal{A}_T$. Then*

$$\omega((x,y))\subset D(A_T)\ \text{ for any } (x,y)\in D(A_T).$$

Proof. Let $(x_1, y_1)\in\omega((x, y))$. By Lemma 3.3, (x_1, y_1) is a coupled fixed point of T_t for all $t\geq 0$. Then, $\lim_{t\to 0^+}\frac{1}{t}[T_t(x_1,y_1)-x_1]$ and $\lim_{t\to 0^+}\frac{1}{t}[T_t(y_1,x_1)-y_1]$ are exists. Hence, $(x_1, y_1)\in D(A)$. □

Proposition 3.4. *Suppose the system (2) and let A_T generates a nonlinear uniformly Lipschitzian semigroup T_t. Let $(\bar{x},\bar{y})$ be a solution point of (2), i.e $(\bar{x},\bar{y})\in D(A)$ and $A\bar{x}=(0,0)$. Then*

$$\omega((x,y))=\{(0,0)\}.$$

Proof. Suppose that $(\bar{x},\bar{y})=(0,0)$. Assume that $(x_1, y_1), (x_2, y_2)\in\omega((x, y))$, since

$$||x_1-x_2||\leq||x_1-T_{t_n}(x,y)||+||T_{t_n}(x,y)-x_2||\to 0\quad as\quad n\to\infty \tag{10}$$

Which implies $x_1=x_2$ and similarly one can proof that $y_1=y_2$. Since $(\bar{x},\bar{y})\in\omega((x,y))$, then we obtain $\omega((x,y))=\{(\bar{x},\bar{y})\}$.

Now, we are ready to introduce the following asymptotic stability theorem which is our main goal in this section. □

Theorem 3.2. *Let the system (2) under assumption of Theorem 3.1 and T_t generated by A on $C\times C=\overline{D(\mathcal{A}_T)}$. Let $(\bar{x},\bar{y})$ is an equilibrium point of (2) and $(I-sA)^{-1}$ compact operator for some $s>0$. Then $\lim_{t\to\infty}T_t(x,y)=\bar{x}$, $\lim_{t\to\infty}T_t(y,y)=\bar{y}$.*

Proof. Consider any $(x,y)\in D(\mathcal{A}_T)$. By Proposition 3.3, $\omega((x,y))\subset D(A_T)$. By Proposition 3.4, $\omega((x,y))=\{(\bar{x},\bar{y})\}$. Now, let $(x, y)\in C\times C$, $(x,y)\notin D(A)$. Let $\epsilon>0$ be arbitrarily fixed. Since $D(A_T)$ dense in $C\times C$, there exists $(u,v)\in D(A_T)$ such that $\|x-u\|<\frac{\epsilon}{2k}$, $\|y-v\|<\frac{\epsilon}{2k}$. It follows, by the fact that T_t is a uniformly continuous semigroup, that

$$\|T_t(x,y)-T_t(u,v)\|\leq\frac{k}{2}[\|x-u\|+\|y-v\|]\leq\frac{\epsilon}{2}, \tag{11}$$

for all $t\geq 0$.

Since $(u,v)\in D(A_T)$, by Proposition 3.4 we get $\omega((u,v))=\{(\bar{x},\bar{y})\}$, whence there exists $M>0$ such that,

$$\text{for all } t>M\ ,\ \|T_t(u,v)\|<\frac{\epsilon}{2}. \tag{12}$$

It follows from (11) and (12) that, for each $t>M$,

$$\|T_t(x,y)\|\leq\|T_t(u,v)-T_t(x,y)\|+\|T_t(u,v)\|<\epsilon.$$

Then $\lim_{t\to\infty}T_t(x,y)=\bar{x}$ and similarly $\lim_{t\to\infty}T_t(y,x)=\bar{y}$. □

Open problem. It will be interesting to establish the results of this paper for semilinear systems as in I. Aksikas and J. Fraser Forbes [4].

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References

[1] C.M. Dafermos, M. Slemrod, Asymptotic behavior of nonlinear contraction semigroups, J. Funct. Anal. 13 (1973) 97–106.
[2] Z. Luo, B. Guo, O. Morgül, Stability and Stabilization of Infinite Dimensional Systems with Applications, Springer, Verlag, London, 1999.
[3] I. Aksikas, J. Winkin, D. Dochain, Asymptotic stability of infinite-dimensional semilinear systems: application to a nonisothermal reactor, Syst. Control Lett. 56 (2007) 122–132.
[4] I. Aksikas, J.F. Forbes, On asymptotic stability of semi-linear distributed parameter dissipative systems, Automatica 46 (2010) 1042–1046.
[5] M.S. Brodskii, D.P. Milman, On the center of a convex set, Dokl. Akad. Nauk. SSSR 59 (1948) 837û840. (Russian).
[6] W.L. Bynum, Normal structure coefficients for banach spaces, Pacific J. Math. 86 (1980) 427–436.
[7] E. Casini, E. Maluta, Fixed points of uniformly lipschitzian mappings in spaces with uniformly normal structure, Nonlinear Anal. 9 (1985) 103–108.

[8] D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, Nonlinear Anal. 11 (5) (1987) 623–632.
[9] K. Goeble, S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, Inc., New York and Basel, 1984.
[10] A.H. Soliman, A coupled fixed point theorem for nonexpansive one parameter semigroup, J. Adv. Math. Stud. Vol. 7 (No. 2) (2014) 28–37.
[11] K.K. Tan, H.K. Xu, Fixed point theorems for lipschitzian semigroups in banach spaces, Nonlinear Anal. 20 (1993) 395–404.