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On some λ difference sequence spaces of fractional order

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ABSTRACT

In the present paper, the difference sequence spaces $cs_0^{\lambda}(\Delta)$, $cs^{\lambda}(\Delta)$ and $bs^{\lambda}(\Delta)$ of nonabsolute type are generalized by introducing a generalized Λ difference operator $\Lambda(\Delta^{(\tilde{\alpha})})$. Also, their Schauder basis are calculated and α-, β- and γ-duals of these spaces are investigated. Finally, some matrix transformations between these spaces and the basic sequence spaces ℓ_p, c and c_0 are characterized, where $1 \leq p \leq \infty$.

1. Introduction

The gamma function Γ is defined for all real numbers p except the negative integers and zero. It can be expressed as an improper integral as follows:

$$\Gamma(p) = \int_0^{\infty} e^{-t} t^{p-1} dt \tag{1.1}$$

where $p \in \mathbb{R}$.

From the equality (1.1) we deduce following properties:

i. If $n \in \mathbb{N}$, then we have $\Gamma(n+1) = n!$.
ii. If $n \in \mathbb{R} - \{0, -1, -2, -3, \dots\}$ then we have $\Gamma(n+1) = n\Gamma(n)$.
iii. For particular cases, we have $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(3) = 2!, \Gamma(4) = 3!, \dots$

For a positive proper fraction $\tilde{\alpha}$, Baliarsingh and Dutta ([1,2]) (also, see [3–8]) have defined the generalized fractional difference operator $\Delta^{(\tilde{\alpha})}$ as

$$\Delta^{(\tilde{\alpha})}(x_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\,\Gamma(\tilde{\alpha}-i+1)} x_{k-i}. \tag{1.2}$$

Throughout the text, it is assumed that the series defined in (1.2) is convergent for $x \in \omega$. More specially, it is convenient to express the difference operator $\Delta^{(\tilde{\alpha})}$ as an triangle i.e.,

$$(\Delta^{(\tilde{\alpha})})_{nk} = \begin{cases} (-1)^{n-k} \frac{\Gamma(\tilde{\alpha}+1)}{(n-k)!\,\Gamma(\tilde{\alpha}-n+k+1)}, & (0 \leqslant k \leqslant n), \\ 0, & (k > n). \end{cases}$$

The notion of difference sequence space firstly was introduced by Kızmaz [9]. After, it was generalized as Δ^m by Et and Çolak [10]. Thereafter, Malkowsky et al. [11], have introduced the spaces $\Delta_u^{(m)}$. The operator $\Delta^{(\tilde{\alpha})}$ generalizes the operator $\Delta^{(m)}$ introduced by Malkowsky and Parashar [12], Polat and Başar [13], Malkowsky et al. [12], if $\alpha = m$, where m is an integer. Different classes of difference sequences have been studied by Tripathy and Dutta [14], Tripathy et. al [15] and many others.

The main purpose of this paper is to generalize the difference sequence spaces $cs_0^{\lambda}(\Delta)$, $cs^{\lambda}(\Delta)$ and $bs^{\lambda}(\Delta)$ of nonabsolute type by introducing a generalized Λ difference operator $\Lambda(\Delta^{(\tilde{\alpha})})$. Furthermore, their Schauder basis are constructed and α-, β- and γ-duals are computed for these spaces. Finally, the matrix mappings from these spaces to some other sequence spaces are characterized.

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The well known the infinite matrix $\Lambda = (\lambda_{nk})_{n,k=0}^{\infty}$ is defined by the matrix

$$\lambda_{nk} = \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n}, & (0 \leqslant k \leqslant n), \\ 0, & (k > n), \end{cases}$$

where $\lambda = (\lambda_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive reals tending to infinity, that is

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < \dots, \quad \lim_{k\to\infty} \lambda_k = \infty.$$

Combining the Λ mean matrix and the difference matrix of order $\tilde{\alpha}$, we define the product matrix $\Lambda(\Delta^{(\tilde{\alpha})})$ as

$$(\Lambda(\Delta^{(\tilde{\alpha})}))_{nk} = \begin{cases} \sum_{i=k}^{n} (-1)^{i-k} \frac{\Gamma(\tilde{\alpha}+1)}{(i-k)!\Gamma(\tilde{\alpha}-i+k+1)} \frac{\lambda_i - \lambda_{i-1}}{\lambda_n}, & (0 \leqslant k \leqslant n), \\ 0, & (k > n). \end{cases}$$

Furthermore, $\Lambda(\Delta^{(\tilde{\alpha})})$ can be written as follows:

$$\Lambda(\Delta^{(\tilde{\alpha})}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{\lambda_0}{\lambda_1} - \tilde{\alpha}\frac{\lambda_1-\lambda_0}{\lambda_1} & \frac{\lambda_1-\lambda_0}{\lambda_1} & 0 & 0 & \dots \\ \frac{\lambda_0}{\lambda_2} - \tilde{\alpha}\frac{\lambda_1-\lambda_0}{\lambda_2} + \frac{\tilde{\alpha}(\tilde{\alpha}-1)}{2!}\frac{\lambda_2-\lambda_1}{\lambda_2} & \frac{\lambda_1-\lambda_0}{\lambda_2} - \tilde{\alpha}\frac{\lambda_2-\lambda_1}{\lambda_2} & \frac{\lambda_2-\lambda_1}{\lambda_2} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Now, we give the following results involving the inverse of the matrices $\Delta^{(\tilde{\alpha})}$ and $\Lambda(\Delta^{(\tilde{\alpha})})$.

Lemma 1.1. *([4, 5, 6, 7])* The inverse of the difference matrix $\Delta^{(\tilde{\alpha})}$ is given by the triangle

$$(\Delta^{(-\tilde{\alpha})})_{nk} = \begin{cases} (-1)^{n-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(n-k)!\Gamma(-\tilde{\alpha}-n+k+1)}, & (0 \leqslant k \leqslant n), \\ 0, & (k > n). \end{cases}$$

Lemma 1.2. *The inverse of the Λ mean difference matrix $\Lambda(\Delta^{(\tilde{\alpha})})$ is given by a triangle (b_{nk}), where*

$$b_{nk} = \begin{cases} \sum_{j=k}^{k+1} (-1)^{n-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(n-j)!\Gamma(-\tilde{\alpha}-n+j+1)} \frac{\lambda_k}{\lambda_j - \lambda_{j-1}}, & (0 \leqslant k \leqslant n), \\ 0, & (k > n). \end{cases}$$

Proof. Proof follows from Lemma 1.1. □

2. New λ difference sequence spaces

In this section, certain sequence spaces of non-absolute type $cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$, $cs^{\lambda}(\Delta^{(\tilde{\alpha})})$ and $bs^{\lambda}(\Delta^{(\tilde{\alpha})})$ are introduced by combining the mean operator Λ and the fractional difference operator $\Delta^{(\tilde{\alpha})}$ and also the some topological properties of these sequence spaces are examined.

Let $\tilde{\alpha}$ be a positive real number. We define certain classes of λ difference sequence spaces as follows:

$$cs_0^{\lambda}(\Delta^{(\tilde{\alpha})}) = \left\{ x = (x_k) \in \omega : \lim_{m\to\infty} \sum_{n=0}^{m} (\Lambda(\Delta^{(\tilde{\alpha})})x)_n = 0 \right\},$$

$$cs^{\lambda}(\Delta^{(\tilde{\alpha})}) = \left\{ x = (x_k) \in \omega : \lim_{m\to\infty} \sum_{n=0}^{m} (\Lambda(\Delta^{(\tilde{\alpha})})x)_n \text{ exists} \right\},$$

$$bs^{\lambda}(\Delta^{(\tilde{\alpha})}) = \left\{ x = (x_k) \in \omega : \sup_m \left| \sum_{n=0}^{m} (\Lambda(\Delta^{(\tilde{\alpha})})x)_n \right| < \infty \right\},$$

where

$$(\Lambda(\Delta^{(\tilde{\alpha})})x)_n = \frac{1}{\lambda_n} \sum_{j=0}^{n} \sum_{i=j}^{n} (-1)^{i-j} \frac{\Gamma(\tilde{\alpha}+1)}{(i-j)!\Gamma(\tilde{\alpha}-i+j+1)} \times (\lambda_i - \lambda_{i-1})x_j; \; (n \in \mathbb{N}). \tag{2.1}$$

The spaces $cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$, $cs^{\lambda}(\Delta^{(\tilde{\alpha})})$ and $bs^{\lambda}(\Delta^{(\tilde{\alpha})})$ can be redefine as the matrix domains of the triangle $\Lambda(\Delta^{(\tilde{\alpha})})$ in the spaces cs_0, cs and bs by

$$cs_0^{\lambda}(\Delta^{(\tilde{\alpha})}) = (cs_0)_{\Lambda(\Delta^{(\tilde{\alpha})})}; \; cs^{\lambda}(\Delta^{(\tilde{\alpha})}) = (cs)_{\Lambda(\Delta^{(\tilde{\alpha})})};$$
$$bs^{\lambda}(\Delta^{(\tilde{\alpha})}) = (bs)_{\Lambda(\Delta^{(\tilde{\alpha})})}. \tag{2.2}$$

Keeping the above new sets in mind, we define the sequence $y = (y_k)$, which is used as the $\Lambda(\Delta^{(\tilde{\alpha})})$–transform of a sequence $x = (x_k)$; that is, $y = \Lambda(\Delta^{(\tilde{\alpha})})x$, and so we have

$$y_k = \frac{1}{\lambda_k} \sum_{j=0}^{k} \sum_{i=j}^{k} (-1)^{i-j} \frac{\Gamma(\tilde{\alpha}+1)}{(i-j)!\Gamma(\tilde{\alpha}-i+j+1)} (\lambda_i - \lambda_{i-1})x_j, \quad (k \in \mathbb{N}). \tag{2.3}$$

In particular, the above new spaces include the classes defined by Kaya and Furkan [16] for $\tilde{\alpha} = 0$,

Now, we give some interesting results of these spaces concerning their topological structures. We give the proof of only one from these three spaces. The proofs of other spaces may be obtained by using similar arguments.

Theorem 2.1. *For a positive proper fraction $\tilde{\alpha}$, the sequence spaces $cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$, $cs^{\lambda}(\Delta^{(\tilde{\alpha})})$ and $bs^{\lambda}(\Delta^{(\tilde{\alpha})})$ are BK-spaces with the norm $\|x\|_{cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})} = \|x\|_{cs^{\lambda}(\Delta^{(\tilde{\alpha})})} = \|x\|_{bs^{\lambda}(\Delta^{(\tilde{\alpha})})} = \|\Lambda(\Delta^{(\tilde{\alpha})})x\|_{bs}$, that is*

$$\|x\|_{bs^{\lambda}(\Delta^{(\tilde{\alpha})})} = \|\Lambda(\Delta^{(\tilde{\alpha})})x\|_{bs} = \sup_m \left| \sum_{n=0}^{m} (\Lambda(\Delta^{(\tilde{\alpha})})x)_n \right|.$$

Proof. Since (2.2) holds and cs_0, cs and bs are *BK*-spaces with the sup-norm given by $\|x\|_{bs} = \sup_n |\sum_{k=0}^{n} x_k|$ and the matrix $\Lambda(\Delta^{(\tilde{\alpha})})$ is a triangle, Theorem 4.3.12 of Wilansky [17, page 63] gives the fact that $cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$, $cs^{\lambda}(\Delta^{(\tilde{\alpha})})$ and $bs^{\lambda}(\Delta^{(\tilde{\alpha})})$ are *BK*-spaces with the given norms. This completes the proof. □

Now, we may give the following theorem concerning the isomorphism between the spaces $cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$, $cs^{\lambda}(\Delta^{(\tilde{\alpha})})$, $bs^{\lambda}(\Delta^{(\tilde{\alpha})})$ and cs_0, cs, bs, respectively:

Theorem 2.2. *For a positive proper fraction $\tilde{\alpha}$, the sequence spaces $cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$, $cs^{\lambda}(\Delta^{(\tilde{\alpha})})$ and $bs^{\lambda}(\Delta^{(\tilde{\alpha})})$ are linearly isomorphic to the classical spaces cs_0, cs and bs, respectively.*

Proof. We prove the theorem for the space $cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$. We show that there exists a linear bijection between the spaces $cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$ and cs_0. Consider the transformation T defined, with the notation of (2.3), from $cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$ to cs_0 by $x \mapsto y$. Then, $Tx = y = \Lambda(\Delta^{(\tilde{\alpha})})x \in cs_0$ for every $x \in cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$ and the linearity of T is clear. If $Tx = \theta = (0, 0, 0, \dots)$, then $x = \theta$ and hence T is injective. Let $y = (y_k) \in cs_0$ and using Lemma 1.2, define a sequence $x = (x_k)$ via y_k as

$$x_k = \sum_{i=0}^{k} \sum_{j=i}^{i+1} (-1)^{k-i} \frac{\Gamma(-\tilde{\alpha}+1)}{(k-j)!\Gamma(-\tilde{\alpha}-k+j+1)} \frac{\lambda_i}{\lambda_j - \lambda_{j-1}} y_i, \quad (k \in \mathbf{N}). \tag{2.4}$$

Then, we have

$$\sum_{n=0}^{m} \frac{1}{\lambda_n} \sum_{j=0}^{n} \sum_{i=j}^{n} (-1)^{i-j} \frac{\Gamma(\tilde{\alpha}+1)}{(i-j)!\Gamma(\tilde{\alpha}-i+j+1)} (\lambda_i - \lambda_{i-1})x_j = \sum_{n=0}^{m} y_n.$$

This shows that $\Lambda(\Delta^{(\tilde{\alpha})})x = y$ and since $y \in cs_0$ we conclude that $\Lambda(\Delta^{(\tilde{\alpha})})x \in cs_0$. Thus, we deduce that $x \in cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$ and

$Tx = y$. Hence, T is surjective. Furthermore, one can easily show that T is norm preserving. This completes the proof. □

Lemma 2.3. [5] *T be a triangle and S be its inverse. If (b_n) is a basis of the normed sequence space X, then $(S(b_n))$ is a basis of X_T*

Remark 1. [18] The matrix domain X_T of a normed sequence space has a basis if and only if X has a basis.

The Schauder basis of the sequence spaces $cs_0^\lambda(\Delta^{(\tilde{\alpha})})$, $cs^\lambda(\Delta^{(\tilde{\alpha})})$ and $bs^\lambda(\Delta^{(\tilde{\alpha})})$ can be derived by using Lemma 2.3 and Remark 1, as follows:

Corollary 2.4. *Let $\tilde{\alpha}_k = (\Lambda(\Delta^{(\tilde{\alpha})})x)_k$ for all $k \in \mathbf{N}$. Now for fixed $k \in \mathbf{N}$ define the sequence $b^{(k)} = \{b_n^{(k)}\}_{n\in\mathbb{N}}$ by*

$$b_n^{(k)} = \begin{cases} \sum_{j=k}^{k+1}(-1)^{n-k}\frac{\Gamma(-\tilde{\alpha}+1)}{(n-j)!\Gamma(-\tilde{\alpha}-n+j+1)}\frac{\lambda_k}{\lambda_j-\lambda_{j-1}}, & 0 \leqslant k \leqslant n, \\ 0, & k > n, \end{cases}$$

for all $n, k \in \mathbf{N}$. Then, the following statements hold:

i. *The sequence $\{b_n^{(k)}\}_{n\in\mathbb{N}}$ is a basis for the spaces $cs_0^\lambda(\Delta^{(\tilde{\alpha})})$ and $cs^\lambda(\Delta^{(\tilde{\alpha})})$ and every $x \in cs_0^\lambda(\Delta^{(\tilde{\alpha})})$ or $cs^\lambda(\Delta^{(\tilde{\alpha})})$ has a unique representation in the form*

$$x = \sum_k \tilde{\alpha}_k b^{(k)}.$$

ii. *$bs^\lambda(\Delta^{(\tilde{\alpha})})$ has no Schauder basis.*

3. The α-, β- and γ-duals of the spaces $cs_0^\lambda(\Delta^{(\tilde{\alpha})})$, $cs^\lambda(\Delta^{(\tilde{\alpha})})$ and $bs^\lambda(\Delta^{(\tilde{\alpha})})$

In this section, theorems determining the α-, β- and γ-duals of the λ difference sequence spaces are formulated and proved. The generalized difference has been studied by Chandra and Tripathy [19]. The collection of all finite subsets of $\mathbb{N}$ is denoted by $\mathcal{F}$ throughout. For this investigation, some lemmas are needed in proving the next theorems, due to Stieglitz and Tietz [20].

Lemma 3.1. *$A = (a_{nk}) \in (cs_0 : \ell_1)$ if and only if*

$$\sup_{N,K\in\mathcal{F}}\left|\sum_{n\in N}\sum_{k\in K}(a_{nk} - a_{n,k+1})\right| < \infty. \tag{3.1}$$

Lemma 3.2. *$A = (a_{nk}) \in (cs : \ell_1)$ if and only if*

$$\sup_{N,K\in\mathcal{F}}\left|\sum_{n\in N}\sum_{k\in K}(a_{nk} - a_{n,k-1})\right| < \infty. \tag{3.2}$$

Lemma 3.3. *$A = (a_{nk}) \in (bs : \ell_1)$ if and only if (3.1) holds and*

$$\lim_k a_{nk} = 0, \ \forall n \in \mathbf{N}. \tag{3.3}$$

Lemma 3.4. *$A = (a_{nk}) \in (cs_0 : c)$ if and only if*

$$\sup_n \sum_k |a_{nk} - a_{n,k+1}| < \infty, \tag{3.4}$$

and

$$\lim_n (a_{nk} - a_{n,k+1}) \text{ exists for all } k \in \mathbf{N}. \tag{3.5}$$

Lemma 3.5. *$A = (a_{nk}) \in (cs : c)$ if and only if (3.4) holds and*

$$\lim_n a_{nk} \text{ exists for all } k \in \mathbf{N}. \tag{3.6}$$

Lemma 3.6. *$A = (a_{nk}) \in (bs : c)$ if and only if (3.3) and (3.6) hold and*

$$\sum_k |a_{nk} - a_{n,k-1}| \text{ converges}. \tag{3.7}$$

Lemma 3.7. *$A = (a_{nk}) \in (cs_0 : \ell_\infty)$ if and only if (3.4) holds.*

Lemma 3.8. *$A = (a_{nk}) \in (cs : \ell_\infty)$ if and only if*

$$\sup_n \sum_k |a_{nk} - a_{n,k-1}| < \infty. \tag{3.8}$$

Lemma 3.9. *$A = (a_{nk}) \in (bs : \ell_\infty)$ if and only if (3.3) and (3.4) hold.*

Now, we may begin the following result which determines the α-dual of the spaces $cs_0^\lambda(\Delta^{(\tilde{\alpha})})$, $cs^\lambda(\Delta^{(\tilde{\alpha})})$ and $bs^\lambda(\Delta^{(\tilde{\alpha})})$.

Theorem 3.10. *Define the sets $h_1^{\tilde{\alpha}}$ and $h_2^{\tilde{\alpha}}$ as follows:*

$$h_1^{\tilde{\alpha}} = \left\{a = (a_n) \in \omega : \sup_{N,K\in\mathcal{F}}\left|\sum_{n\in N}\sum_{k\in K}(b_{nk}^{\tilde{\alpha}} - b_{n,k+1}^{\tilde{\alpha}})\right| < \infty\right\},$$

$$h_2^{\tilde{\alpha}} = \left\{a = (a_n) \in \omega : \sup_{N,K\in\mathcal{F}}\left|\sum_{n\in N}\sum_{k\in K}(b_{nk}^{\tilde{\alpha}} - b_{n,k-1}^{\tilde{\alpha}})\right| < \infty\right\};$$

where the matrix $B^{\tilde{\alpha}} = (b_{nk}^{\tilde{\alpha}})$ is defined via the sequence $a = (a_n) \in \omega$ by

$$b_{nk}^{\tilde{\alpha}} = \begin{cases} \sum_{j=k}^{k+1}(-1)^{n-k}\frac{\Gamma(-\tilde{\alpha}+1)}{(n-j)!\Gamma(-\tilde{\alpha}-n+j+1)}\frac{\lambda_k}{\lambda_j-\lambda_{j-1}}a_n, & 0 \leqslant k \leqslant n, \\ 0, & k > n, \end{cases}$$

for all $n, k \in \mathbf{N}$. Then $\{cs_0^\lambda(\Delta^{(\tilde{\alpha})})\}^\alpha = \{bs^\lambda(\Delta^{(\tilde{\alpha})})\}^\alpha = h_1^{\tilde{\alpha}}$ and $\{cs^\lambda(\Delta^{(\tilde{\alpha})})\}^\alpha = h_2^{\tilde{\alpha}}$.

Proof. Let $a = (a_n) \in \omega$. Then, by bearing in mind the relation (2.4), we easily obtain that

$$a_n x_n = \sum_{k=0}^{n}\left[\sum_{j=k}^{k+1}(-1)^{n-k}\frac{\Gamma(-\tilde{\alpha}+1)}{(n-j)!\Gamma(-\tilde{\alpha}-n+j+1)}\frac{\lambda_k}{\lambda_j-\lambda_{j-1}}a_n\right]$$
$$y_k = (B^{\tilde{\alpha}}y)_n \tag{3.9}$$

holds for all $n \in \mathbf{N}$. We, therefore, observe by (3.9) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in cs_0^\lambda(\Delta^{(\tilde{\alpha})})$ or $bs^\lambda(\Delta^{(\tilde{\alpha})})$ if and only if $B^{\tilde{\alpha}}y \in \ell_1$ whenever $y = (y_k) \in cs_0$ or bs. This means that the sequence $a = (a_n) \in \{cs_0^\lambda(\Delta^{(\tilde{\alpha})})\}^\alpha$ or $a = (a_n) \in \{bs^\lambda(\Delta^{(\tilde{\alpha})})\}^\alpha$ if and only if $B^{\tilde{\alpha}} \in (cs_0 : \ell_1)$ or $B^{\tilde{\alpha}} \in (bs : \ell_1)$. Then, it is clear that the columns of the matrix $B^{\tilde{\alpha}}$ are in the space c_0, since

$$\lim_{k\to\infty} b_{nk}^{\tilde{\alpha}} = 0$$

for all $n \in \mathbf{N}$. Hence, we obtain by Lemmas 3.1 or 3.3 with $B^{\tilde{\alpha}}$ instead of A that $a = (a_n) \in \{cs_0^\lambda(\Delta^{(\tilde{\alpha})})\}^\alpha = \{bs^\lambda(\Delta^{(\tilde{\alpha})})\}^\alpha$ if and only if

$$\sup_{N,K\in\mathcal{F}}\left|\sum_{n\in N}\sum_{k\in K}(b_{nk}^{\tilde{\alpha}} - b_{n,k+1}^{\tilde{\alpha}})\right| < \infty, \tag{3.10}$$

which leads us to the consequence that $\{cs_0^\lambda(\Delta^{(\tilde{\alpha})})\}^\alpha = \{bs^\lambda(\Delta^{(\tilde{\alpha})})\}^\alpha = h_1^{\tilde{\alpha}}$.

Similarly, we deduce from Lemma 3.2 with (3.9) that $a = (a_n) \in \{cs^\lambda(\Delta^{(\tilde{\alpha})})\}^\alpha$ if and only if $B^{\tilde{\alpha}} \in (cs : \ell_1)$. Therefore, we derive from (3.2) that

$$\sup_{N,K\in\mathcal{F}}\left|\sum_{n\in N}\sum_{k\in K}(b_{nk}^{\tilde{\alpha}} - b_{n,k-1}^{\tilde{\alpha}})\right| < \infty. \tag{3.11}$$

This yields that $\{cs^\lambda(\Delta^{(\tilde{\alpha})})\}^\alpha = h_2^{\tilde{\alpha}}$. This completes the proof. □

Theorem 3.11. *Define the sets* $h_3^{\tilde{\alpha}}$, $h_4^{\tilde{\alpha}}$, $h_5^{\tilde{\alpha}}$, $h_6^{\tilde{\alpha}}$, $h_7^{\tilde{\alpha}}$ *and* $h_8^{\tilde{\alpha}}$ *by*

$$h_3^{\tilde{\alpha}} = \left\{a = (a_n) \in \omega : \sup_n \sum_{k=0}^{n-1} |\tilde{a}_k(n) - \tilde{a}_{k+1}(n)| < \infty\right\},$$

$$h_4^{\tilde{\alpha}} = \left\{a = (a_n) \in \omega : \sup_n \left|\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n\right| < \infty\right\},$$

$$h_5^{\tilde{\alpha}} = \left\{a=(a_n) \in \omega : \lim_{n\to\infty} (\tilde{a}_k(n) - \tilde{a}_{k+1}(n)) \text{ exists for each } k\in\mathbf{N}\right\},$$

$$h_6^{\tilde{\alpha}} = \left\{a = (a_n) \in \omega : \lim_{n\to\infty} \tilde{a}_k(n) \text{ exists for each } k \in \mathbf{N}\right\},$$

$$h_7^{\tilde{\alpha}} = \left\{a = (a_n) \in \omega : \sum_{k=0}^{\infty} |\tilde{a}_k(n) - \tilde{a}_{k-1}(n)| \text{ converges}\right\},$$

$$h_8^{\tilde{\alpha}} = \left\{a = (a_n) \in \omega : \lim_{n\to\infty} \left|\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n\right| \text{ exists}\right\},$$

where

$$\tilde{a}_k(n) = \sum_{i=k}^{n} \left[\sum_{j=k}^{k+1} (-1)^{i-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(i-j)!\Gamma(-\tilde{\alpha}-i+j+1)} \frac{\lambda_k}{\lambda_j - \lambda_{j-1}}\right] a_i; \quad (0 \leqslant k \leqslant n).$$

Then $\{cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})\}^{\beta} = h_3^{\tilde{\alpha}} \cap h_4^{\tilde{\alpha}} \cap h_5^{\tilde{\alpha}}$, $\{cs^{\lambda}(\Delta^{(\tilde{\alpha})})\}^{\beta} = h_3^{\tilde{\alpha}} \cap h_4^{\tilde{\alpha}} \cap h_6^{\tilde{\alpha}}$ *and* $\{bs^{\lambda}(\Delta^{(\tilde{\alpha})})\}^{\beta} = h_6^{\tilde{\alpha}} \cap h_7^{\tilde{\alpha}} \cap h_8^{\tilde{\alpha}}$.

Proof. Consider the equality

$$\begin{aligned}\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left[\sum_{i=0}^{k} \sum_{j=i}^{i+1} (-1)^{k-i} \frac{\Gamma(-\tilde{\alpha}+1)}{(k-j)!\Gamma(-\tilde{\alpha}-k+j+1)} \frac{\lambda_i}{\lambda_j - \lambda_{j-1}} y_i\right] a_k \\ &= \sum_{k=0}^{n} \left[\sum_{i=k}^{n} \sum_{j=k}^{k+1} (-1)^{i-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(i-j)!\Gamma(-\tilde{\alpha}-i+j+1)} \frac{\lambda_k}{\lambda_j - \lambda_{j-1}} a_i\right] y_k \\ &= (T^{\tilde{\alpha}} y)_n; \ (n \in \mathbf{N}),\end{aligned} \tag{3.12}$$

where the matrix $T^{\tilde{\alpha}} = (t_{nk}^{\tilde{\alpha}})$ is defined by

$$t_{nk}^{\tilde{\alpha}} = \begin{cases} \sum_{i=k}^{n} \left[\sum_{j=k}^{k+1} (-1)^{i-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(i-j)!\Gamma(-\tilde{\alpha}-i+j+1)} \frac{\lambda_k}{\lambda_j - \lambda_{j-1}}\right] a_i, & 0 \leqslant k \leqslant n, \\ 0, & k > n, \end{cases}$$

for all k, $n \in \mathbf{N}$. Thus, we deduce from (3.12) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$ if and only if $T^{\tilde{\alpha}} \in (cs_0 : c)$. Therefore, by using Lemma 3.4, we derive from (3.4) and (3.5) that

$$\sup_n \sum_{k=0}^{n-1} |\tilde{a}_k(n) - \tilde{a}_{k+1}(n)| < \infty, \tag{3.13}$$

$$\sup_n \left|\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n\right| < \infty, \tag{3.14}$$

$$\lim_{n\to\infty} (\tilde{a}_k(n) - \tilde{a}_{k+1}(n)) \text{ exists for each } k \in \mathbf{N}, \tag{3.15}$$

which show that $\{cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})\}^{\beta} = h_3^{\tilde{\alpha}} \cap h_4^{\tilde{\alpha}} \cap h_5^{\tilde{\alpha}}$.

Similarly, we deduce from Lemma 3.5 with (3.4) and (3.6) that $a = (a_k) \in \{cs^{\lambda}(\Delta^{(\tilde{\alpha})})\}^{\beta}$ if and only if $T^{\tilde{\alpha}} \in (cs : c)$. Therefore, we derive from (3.4) that (3.13) and (3.14) hold. Further, by using Lemma 3.5, we obtain from (3.6) that

$$\lim_{n\to\infty} \tilde{a}_k(n) \text{ exists for each } k \in \mathbf{N}. \tag{3.16}$$

Hence, we deduce that $\{cs^{\lambda}(\Delta^{(\tilde{\alpha})})\}^{\beta} = h_3^{\tilde{\alpha}} \cap h_4^{\tilde{\alpha}} \cap h_6^{\tilde{\alpha}}$.

Finally, we conclude from Lemma 3.6 with (3.3), (3.6) and (3.7) that $a = (a_k) \in \{bs^{\lambda}(\Delta^{(\tilde{\alpha})})\}^{\beta}$ if and only if $T^{\tilde{\alpha}} \in (bs : c)$. Therefore, it is clear that the columns of the matrix $T^{\tilde{\alpha}}$ are in the space c_0, since

$$\lim_{k\to\infty} t_{nk}^{\tilde{\alpha}} = 0$$

for all $n \in \mathbf{N}$. Also, we derive from (3.6) that (3.16) holds. Further, we get from (3.7) that

$$\sum_{k=0}^{\infty} |\tilde{a}_k(n) - \tilde{a}_{k-1}(n)| \text{ converges} \tag{3.17}$$

and

$$\lim_{k\to\infty} \left|\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n\right| \text{ exists.} \tag{3.18}$$

Therefore, we conclude that $\{bs^{\lambda}(\Delta^{(\tilde{\alpha})})\}^{\beta} = h_6^{\tilde{\alpha}} \cap h_7^{\tilde{\alpha}} \cap h_8^{\tilde{\alpha}}$. □

Theorem 3.12. *Define the set* $h_9^{\tilde{\alpha}}$ *as follow:*

$$h_9^{\tilde{\alpha}} = \left\{a = (a_n) \in \omega : \sup_n \sum_{k=0}^{n} |\tilde{a}_k(n) - \tilde{a}_{k-1}(n)| < \infty\right\}.$$

The γ*-dual of the spaces* $cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$ *and* $bs^{\lambda}(\Delta^{(\tilde{\alpha})})$ *is the set* $h_3^{\tilde{\alpha}} \cap h_4^{\tilde{\alpha}}$ *and the* γ*-dual of the space* $cs^{\lambda}(\Delta^{(\tilde{\alpha})})$ *is the set* $h_4^{\tilde{\alpha}} \cap h_9^{\tilde{\alpha}}$.

Proof. The present theorem may be proved by the technique used in the proof of Theorem 3.11. □

4. Some matrix transformations related to the sequence spaces $cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$, $cs^{\lambda}(\Delta^{(\tilde{\alpha})})$ and $bs^{\lambda}(\Delta^{(\tilde{\alpha})})$

In this final section, some matrix mappings on the spaces $cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$, $cs^{\lambda}(\Delta^{(\tilde{\alpha})})$ and $bs^{\lambda}(\Delta^{(\tilde{\alpha})})$ are characterized. Actually, the necessary and sufficient conditions for matrix transformations from these spaces into the spaces ℓ_p, c and c_0 are given, where $1 \le p \le \infty$.

We shall write throughout for brevity that

$$\tilde{a}_{nk}(m) = \sum_{i=k}^{m} \left[\sum_{j=k}^{k+1} (-1)^{i-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(i-j)!\Gamma(-\tilde{\alpha}-i+j+1)} \frac{\lambda_k}{\lambda_j - \lambda_{j-1}}\right] a_{ni}$$

and

$$\tilde{a}_{nk} = \sum_{i=k}^{\infty} \left[\sum_{j=k}^{k+1} (-1)^{i-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(i-j)!\Gamma(-\tilde{\alpha}-i+j+1)} \frac{\lambda_k}{\lambda_j - \lambda_{j-1}}\right] a_{ni}$$

for all n, k, $m \in \mathbf{N}$ provided the convergence of the series.

Let us give some Lemmas [20], which are essential for deriving the characterization of matrix mappings.

Lemma 4.1. $A = (a_{nk}) \in (cs_0 : c_0)$ *if and only if* (3.4) *holds and*

$$\lim_n (a_{nk} - a_{n,k+1}) = 0 \quad (\forall k \in \mathbf{N}). \tag{4.1}$$

Lemma 4.2. $A = (a_{nk}) \in (cs : c_0)$ *if and only if* (3.4) *holds and*

$$\lim_n a_{nk} = 0 \quad (\forall k \in \mathbf{N}). \tag{4.2}$$

Lemma 4.3. $A = (a_{nk}) \in (bs : c_0)$ *if and only if* (3.3) *holds and*

$$\lim_n \sum_k |a_{nk} - a_{n,k+1}| = 0. \tag{4.3}$$

Lemma 4.4. $A = (a_{nk}) \in (cs_0 : \ell_p)$ *if and only if*

$$\sup_K \sum_n \left|\sum_{k\in K} (a_{nk} - a_{n,k+1})\right|^p < \infty \quad (1 < p < \infty). \tag{4.4}$$

Lemma 4.5. $A = (a_{nk}) \in (cs : \ell_p)$ *if and only if*

$$\sup_K \sum_n \left|\sum_{k\in K} (a_{nk} - a_{n,k-1})\right|^p < \infty \quad (1 < p < \infty). \tag{4.5}$$

Lemma 4.6. $A = (a_{nk}) \in (bs : \ell_p)$ *if and only if* (3.3) *and* (4.4) *hold.*

Now, we state following theorems via matrix transformations of the spaces $cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$, $cs^{\lambda}(\Delta^{(\tilde{\alpha})})$ and $bs^{\lambda}(\Delta^{(\tilde{\alpha})})$ for the spaces ℓ_p, c and c_0, where $1 \le p \le \infty$.

Theorem 4.7. i. *$A = (a_{nk}) \in (cs_0^{\lambda}(\Delta^{(\tilde{\alpha})}) : \ell_\infty)$ if and only if*

$$\sup_m \sum_{k=0}^{m-1} |\tilde{a}_{nk}(m) - \tilde{a}_{n,k+1}(m)| < \infty \ (n \in \mathbb{N}), \tag{4.6}$$

$$\sup_m \left| \frac{\lambda_m}{\lambda_m - \lambda_{m-1}} a_{nm} \right| < \infty, \tag{4.7}$$

$$\lim_{m\to\infty} (\tilde{a}_{nk}(m) - \tilde{a}_{n,k+1}(m)) \text{ exists for each } n,\ k \in \mathbf{N}, \tag{4.8}$$

$$\sup_n \sum_k |\tilde{a}_{nk} - \tilde{a}_{n,k+1}| < \infty. \tag{4.9}$$

ii. *$A = (a_{nk}) \in (cs^{\lambda}(\Delta^{(\tilde{\alpha})}) : \ell_\infty)$ if and only if (4.6), (4.7) hold and*

$$\lim_{m\to\infty} \tilde{a}_{nk}(m) \text{ exists for each } n,\ k \in \mathbf{N}, \tag{4.10}$$

$$\sup_n \sum_k |\tilde{a}_{nk} - \tilde{a}_{n,k-1}| < \infty. \tag{4.11}$$

iii. *$A = (a_{nk}) \in (bs^{\lambda}(\Delta^{(\tilde{\alpha})}) : \ell_\infty)$ if and only if (4.9) and (4.10) hold and*

$$\sum_{k=0}^{\infty} |\tilde{a}_{nk}(m) - \tilde{a}_{n,k+1}(m)| \text{ convergent}, \tag{4.12}$$

$$\lim_{m\to\infty} \left| \frac{\lambda_m}{\lambda_m - \lambda_{m-1}} a_{nm} \right| \text{ exists } (n \in \mathbf{N}), \tag{4.13}$$

$$\lim_{k\to\infty} \tilde{a}_{nk} = 0 \ (\forall\, n \in \mathbf{N}). \tag{4.14}$$

Proof. ***i.*** Suppose that the conditions (4.6), (4.7), (4.8) and (4.9) hold and take any $x = (x_k) \in cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})$. Then, we have by Theorem 3.11 that $(a_{nk})_{k=0}^{\infty} \in \{cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})\}^{\beta}$ for all $n \in \mathbf{N}$ and this implies the existence of the A-transform of x, that is, Ax exists. Further, it is clear that the associated sequence $y = (y_k)$ is in cs_0.

Let us now consider the following equality derived by using the relation (2.4) from the m^{th} partial sum of the series $\Sigma_k a_{nk} x_k$:

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} \tilde{a}_{nk}(m) y_k;\ \ (n,\ m \in \mathbf{N}). \tag{4.15}$$

Therefore, by using (4.6), (4.7) and (4.8) as $m \to \infty$ we obtain that

$$\sum_k a_{nk} x_k = \sum_k \tilde{a}_{nk} y_k \text{ for all } n \in \mathbf{N}. \tag{4.16}$$

Further, since the matrix $\tilde{A} = (\tilde{a}_{nk})$ is in the class $(cs_0\colon\ \ell_\infty)$ by Lemma 3.7 and (4.9); we have $\tilde{A}y \in \ell_\infty$. Therefore, we deduce from (4.16) that $Ax \in \ell_\infty$ and hence $A = (a_{nk}) \in (cs_0^{\lambda}(\Delta^{(\tilde{\alpha})}) : \ell_\infty)$.

Conversely, suppose that $A = (a_{nk}) \in (cs_0^{\lambda}(\Delta^{(\tilde{\alpha})}) : \ell_\infty)$. Then $(a_{nk})_{k=0}^{\infty} \in \{cs_0^{\lambda}(\Delta^{(\tilde{\alpha})})\}^{\beta}$ for all $n \in \mathbf{N}$ and this, with Theorem 3.11, implies (4.6), (4.7) and (4.8). Further, since $Ax \in \ell_\infty$ by the hypothesis; we obtain by (4.16) that $\tilde{A}y \in \ell_\infty$ which shows that $\tilde{A} \in (cs_0 : \ell_\infty)$, where $\tilde{A} = (\tilde{a}_{nk})$. Hence, the necessity of (4.9) is immediate by (3.4). This concludes the proof of part *i*.

Since part *ii* and *iii* can be proved similarly, we omit its proof. □

Corollary 4.8. i. *$A = (a_{nk}) \in (cs_0^{\lambda}(\Delta^{(\tilde{\alpha})}) : c)$ if and only if (4.6), (4.7), (4.8) and (4.9) hold and*

$$\lim_{n\to\infty} (\tilde{a}_{nk} - \tilde{a}_{n,k+1}) \text{ exists for all } k \in \mathbf{N}. \tag{4.17}$$

ii. *$A = (a_{nk}) \in (cs^{\lambda}(\Delta^{(\tilde{\alpha})}) : c)$ if and only if (4.6), (4.7), (4.9) and (4.10) hold and*

$$\lim_{n\to\infty} \tilde{a}_{nk} \text{ exists for all } k \in \mathbf{N}. \tag{4.18}$$

iii. *$A = (a_{nk}) \in (bs^{\lambda}(\Delta^{(\tilde{\alpha})}) : c)$ if and only if (4.10), (4.12), (4.13), (4.14) and (4.18) hold and*

$$\sum_k |\tilde{a}_{nk} - \tilde{a}_{n,k-1}| \text{ convergent}. \tag{4.19}$$

Corollary 4.9. i. *$A = (a_{nk}) \in (cs_0^{\lambda}(\Delta^{(\tilde{\alpha})}) : c_0)$ if and only if (4.6), (4.7), (4.8) and (4.9) hold and*

$$\lim_{n\to\infty} (\tilde{a}_{nk} - \tilde{a}_{n,k+1}) = 0 \ (k \in \mathbf{N}). \tag{4.20}$$

ii. *$A = (a_{nk}) \in (cs^{\lambda}(\Delta^{(\tilde{\alpha})}) : c_0)$ if and only if (4.6), (4.7), (4.9) and (4.10) hold and*

$$\lim_{n\to\infty} \tilde{a}_{nk} = 0 \ (k \in \mathbf{N}). \tag{4.21}$$

iii. *$A = (a_{nk}) \in (bs^{\lambda}(\Delta^{(\tilde{\alpha})}) : c_0)$ if and only if (4.10), (4.12), (4.13) and (4.14) hold and*

$$\lim_{n\to\infty} \sum_k |\tilde{a}_{nk} - \tilde{a}_{n,k+1}| = 0. \tag{4.22}$$

Corollary 4.10. i. *$A = (a_{nk}) \in (cs_0^{\lambda}(\Delta^{(\tilde{\alpha})}) : \ell_1)$ if and only if (4.6), (4.7) and (4.8) hold and*

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in N} \sum_{k\in K} (\tilde{a}_{nk} - \tilde{a}_{n,k+1}) \right| < \infty. \tag{4.23}$$

ii. *$A = (a_{nk}) \in (cs^{\lambda}(\Delta^{(\tilde{\alpha})}) : \ell_1)$ if and only if (4.6), (4.7) and (4.10) hold and*

$$\sup_{N,K\in\mathcal{F}} \left| \sum_{n\in N} \sum_{k\in K} (\tilde{a}_{nk} - \tilde{a}_{n,k-1}) \right| < \infty. \tag{4.24}$$

iii. *$A = (a_{nk}) \in (bs^{\lambda}(\Delta^{(\tilde{\alpha})}) : \ell_1)$ if and only if (4.10), (4.12), (4.13), (4.14) and (4.23) hold.*

Corollary 4.11. i. *$A = (a_{nk}) \in (cs_0^{\lambda}(\Delta^{(\tilde{\alpha})}) : \ell_p)$ if and only if (4.6), (4.7) and (4.8) hold and*

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} (\tilde{a}_{nk} - \tilde{a}_{n,k+1}) \right|^p < \infty,\ \ (1 < p < \infty). \tag{4.25}$$

ii. *$A = (a_{nk}) \in (cs^{\lambda}(\Delta^{(\tilde{\alpha})}) : \ell_p)$ if and only if (4.6), (4.7) and (4.10) hold and*

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} (\tilde{a}_{nk} - \tilde{a}_{n,k-1}) \right|^p < \infty,\ \ (1 < p < \infty). \tag{4.26}$$

iii. *$A = (a_{nk}) \in (bs^{\lambda}(\Delta^{(\tilde{\alpha})}) : \ell_p)$ if and only if (4.10), (4.12), (4.13), (4.14) and (4.25) hold.*

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