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Original Article

Application of the improved *F*-expansion method with Riccati equation to find the exact solution of the nonlinear evolution equations

Md. Shafiqul Islam [a,*], Kamruzzaman Khan [a], M. Ali Akbar [b]

[a] *Department of Mathematics, Pabna University of Science and Technology, Pabna 6600, Bangladesh*
[b] *Department of Applied Mathematics, University of Rajshahi, Rajshahi 6205, Bangladesh*

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Abstract In this article, we pay attention to the analytical method named, the improved *F*-expansion method combined with Riccati equation for finding the exact traveling wave solutions of the Benney–Luke equation and the Phi-4 equation. By means of this method we have explored three classes of explicit solutions-hyperbolic, trigonometric and rational solutions with some free parameters. When the parameters are taken as special values, the solitary wave solutions are originated from the traveling wave solutions. Our outcomes disclose that this method is very active and forthright way of formulating the exact solutions of nonlinear evolution equations arising in mathematical physics and engineering.

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1. Introduction

Nonlinear evolution equations play significant rules for understanding qualitative as well as measurable features of many phenomena and dealings to construct exact traveling wave solutions. Exact solutions visually reveal and make it possible to appreciate the mechanism of complex nonlinear properties. Nonlinear wave phenomena appears in various scientific and engineering fields such as fluid mechanics, quantum mechanics, electricity, plasma physics, meteorology, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry [1]. With the development of the symbolic computation package like Maple and Mathematica, looking for the exact traveling wave solutions of nonlinear evolution equations has long been one of the central themes of perpetual interest in Mathematical physics and engineering.

* Corresponding author. Tel.: +880 1746764468.
E-mail addresses: shafiquemath31@gmail.com, shafique.pust@gmail.com (Md.S. Islam).
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The exact solutions to NLEEs support us to afford information about the structure of complex physical phenomena. Therefore, investigation of exact traveling wave solutions to NLEEs turns into an essential task in the study of nonlinear physical phenomena. It is notable to observe that there is no unique method to solve all kind of NLEEs. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of NLEEs have been proposed and developed. A variety of powerful methods such as, homogeneous balance method [2], auxiliary equation method [3], the Exp-function method [4], Darboux transformation method [5], the tanh-function method [6], the modified extended tanh-function method [7], the first integral method [8,9], the modified simple equation method [10–12], the (G'/G)-expansion method [13–15], the exp $(-\Phi(\xi))$-expansion method [16], the F-expansion method [17], the generalized Kudryashov method [18,19] and so on.

The objective of this article is to look for new study for relating to the improved F-expansion method to explore exact traveling wave solutions for the Benney–Luke equation and the Phi-4 equation. This application displays the simplicity, efficiency and effectiveness of the improved F-expansion method. To the best of our knowledge that the improved F-expansion method has not been applied to the above mentioned equation in the previous literature. The Benney–Luke equation is an approximation of the full water wave equations and formally suitable for describing two-way water wave propagation in presence of surface tension. The phi-4 equation plays a significant role in nuclear and particle physics over the decades.

The arrangement of this paper is systematized as follows. In Section 2, we give the description of the improved F-expansion method. In Section 3, we apply this method to the Benney–Luke equation and the phi-4 equation. In Section 4, we discuss the graphical representation of some obtained solutions. In the Section 5, we discuss the comparison and advantages of our proposed method throughout the other existing method and in Section 6, we briefly make a summary to the results that have been obtained.

2. The improved *F*-expansion method

In this section we simply describe the improved F-expansion method for seeking the exact traveling wave solutions of NLEEs.

Let us examine these methods for a given general nonlinear evolution equation is in the form,

$$\Phi(u,\ u_x\, u_t, u_{xx}, ,\ u_{xt}, \ldots) = 0, \tag{2.1}$$

where $u = u(x,\ t)$ is an unknown function, Φ is a polynomial of $u(x, t)$ and its partial derivatives in which the highest order partial derivatives and the nonlinear terms are involved and the subscripts stands for the partial derivatives.

We introduce the leading steps of this method are as follows:

Step-1: The first step we familiarize the traveling wave transformation,

$$u(x,\ t) = u(\eta), \quad \eta = x \pm c\,t, \tag{2.2}$$

where $c \in \Re - \{0\}$ is the speed of traveling wave, The traveling wave transformation Eq. (2.2) transforms Eq. (2.1) into an ordinary differential equation (ODE) for $u = u(\eta)$:

$$\Theta\big(u,\ u',\ u'',\ u''',\ldots\big) = 0, \tag{2.3}$$

where Θ is a polynomial of u and its derivatives and the superscripts specify the ordinary derivatives with respect to η.

Step-2: According to likelihood, Eq. (2.3) can be integrated term by term one or more times, yields constants of integration. The integral constant may be zero for straightforwardness.

Step-3: We assume the traveling wave solution of Eq. (2.3) can be expressed by a polynomial in $F(\eta)$ as follows:

$$u(\eta) = \sum_{i=0}^{N} \alpha_i (m + F(\eta))^i + \sum_{i=1}^{N} \beta_i (m + F(\eta))^{-i}, \tag{2.4}$$

where either α_N or β_N may be zero, but both of them could not be zero at time, $\alpha_i (i = 0,\ 1, 2, \ldots,\ N)$ and $\beta_i (i = 0,\ 1,\ 2, \ldots,\ N)$ and m are arbitrary constants to be determined later.

We consider the well-known Riccati equation

$$F'(\xi) = k + F^2(\eta), \tag{2.5}$$

where the prime stands for derivatives with respect to η; k is the real parameter. Also $F = F(\eta)$ satisfies the Riccati equation.

We now represent the three cases of the general solution of the Riccati Eq. (2.5) as follows;

Case-I: When $k < 0$, the general solutions are

$$F_1 = -\sqrt{-k}\ \tanh\left(\sqrt{-k}\,\eta\right),$$

$$F_2 = -\sqrt{-k}\ \coth\left(\sqrt{-k}\,\eta\right),$$

Case-II: When $k > 0$, the general solutions are

$$F_3 = \sqrt{k}\ \tan\left(\sqrt{k}\,\eta\right),$$

$$F_4 = -\sqrt{k}\ \cot\left(\sqrt{k}\,\eta\right),$$

Case-III: When $k = 0$, the general solution is

$$F_5 = -\frac{1}{\eta}.$$

Step-4: The positive integer N is usually attained by taking the homogeneous balance between the highest order nonlinear terms and the derivatives of the highest order appearing in Eq. (2.3). If the degree of $u(\eta)$ is $D[u(\eta)] = N$, then the degree of the other expressions will be as follows:

$$D\left[\frac{d^p u(\eta)}{d\eta^p}\right] = N + p,\ D\left[u^p\left(\frac{d^q u(\eta)}{d\eta^q}\right)^s\right] = Np + s(N + p), \tag{2.6}$$

Therefore, we can find the value of N in Eq. (2.4), using Eq. (2.6).

Step-5: Substituting Eq. (2.4) including with Eq. (2.5) into Eq. (2.3) together with the value of N attained in Step 3, we attain polynomials in $(m+F)^i$ and $(m+F)^{-i}$ $(i = 1, 2, 3, ...N)$, then collect each coefficient of the resulted polynomial to zero, yields an over-determined set of algebraic equations for α_N, β_N, m and c.

Step-6: Suppose the value of the constants α_N, β_N, m and c can be determined by solving the algebraic equations attained in Step 4. Since the general solution of Eq. (2.5) is well known to us, inserting the value of α_N, β_N, m and c into Eq. (2.4), we attain more general type and new exact traveling wave solutions of the nonlinear partial differential Eq. (2.1).

3. Applications

In this section, we will construct the improved F-expansion method look for the exact solutions to the Benney–Luke equation and the Phi-4 equation.

3.1. The Benney–Luke equation

In this section, we will construct the improved F-expansion method to obtain the exact solutions for the Benney–Luke equation. Let us consider the Benney–Luke equation is of the form

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0, \qquad (3.1.1)$$

where a and b are nonzero constants. This equation is an approximation of the full water wave equations and formally suitable for describing two-way water wave propagation in presence of surface tension.

Suppose the traveling wave transformation equation

$$u(x,t) = u(\eta),\ \eta = x - ct, \qquad (3.1.2)$$

where c is the speed of the traveling wave. Eq. (3.1.2) transforms Eq. (3.1.1) into the following ODEs:

$$\left(c^2-1\right)u'' + \left(a - c^2 b\right)u^{iv} - 3cu'u'' = 0, \qquad (3.1.3)$$

Eq. (3.1.3) is integrable. Integrating Eq. (3.1.3) with respect to η once and setting the constant of integration to zero, we attain

$$\left(c^2-1\right)u' + \left(a - c^2 b\right)u''' - \frac{3}{2}c\left(u'\right)^2 = 0, \qquad (3.1.4)$$

Balancing the highest-order derivative u''' and nonlinear term of highest order u'^2 in Eq. (3.1.4), we attain $N = 1$. Hence for $N = 1$, Eq. (2.4) reduces to

$$u(\eta) = \alpha_0 + \alpha_1(m + F(\eta)) + \beta_1(m + F(\eta))^{-1}, \qquad (3.1.5)$$

Now substituting Eq. (3.1.5) into Eq. (3.1.4), we obtain a polynomial in $F(\eta)$. Equating the coefficient of same power of $F(\eta)$, we get the following system of nine algebraic equations. Solving the above system of equations for $\alpha_0, \alpha_1, \beta_1, m$ and c, we obtain the following set of values: Set-1: $c = \pm\sqrt{\frac{4ak+1}{4bk+1}}$, $\alpha_1 = 0$, $\beta_1 = \pm\frac{4(bk-am^2-ak+bm^2)}{\sqrt{(4ak+1)(4bk+1)}}$. Set-2: $c = \pm\sqrt{\frac{16ak+1}{16bk+1}}$, $m = 0$, $\alpha_1 = \pm\frac{4(a-b)}{\sqrt{(16ak+1)(16bk+1)}}$, $\beta_1 = \mp\frac{4k(a-b)}{\sqrt{(16ak+1)(16bk+1)}}$. Set-3: $c = \pm\sqrt{\frac{4ak+1}{4bk+1}}$, $\alpha_1 = \pm\frac{4(a-b)}{\sqrt{(4ak+1)(4bk+1)}}$, $\beta_1 = 0$.

Case-I: When $k < 0$, we get the following hyperbolic trigonometric solutions:

For the values of the above sets we attain the following traveling wave solutions for Benney –Luke equation equations.

$$\text{Family}-1{:}u_{1,2}(\eta) = \alpha_0 \pm \frac{4\left(bk - am^2 - ak + bm^2\right)}{\sqrt{(4ak+1)(4bk+1)}\left(m - \sqrt{-k}\tanh\left(\sqrt{-k}\eta\right)\right)}, \qquad (3.1.6)$$

$$u_{3,4}(\eta) = \alpha_0 \pm \frac{4\left(bk - am^2 - ak + bm^2\right)}{\sqrt{(4ak+1)(4bk+1)}\left(m - \sqrt{-k}\coth\left(\sqrt{-k}\eta\right)\right)}, \qquad (3.1.7)$$

where $\eta = x \mp \sqrt{\frac{4ak+1}{4bk+1}}t$.

$$\text{Family}-2{:}u_{5,6}(\eta) = \alpha_0 \mp \frac{4\sqrt{-k}(a-b)}{\sqrt{(16ak+1)(16bk+1)}}\left(\tanh\left(\sqrt{-k}\eta\right) + \coth\left(\sqrt{-k}\eta\right)\right), \qquad (3.1.8)$$

$$u_{7,8}(\eta) = \alpha_0 \mp \frac{4\sqrt{-k}(a-b)}{\sqrt{(16ak+1)(16bk+1)}}\left(\tanh\left(\sqrt{-k}\eta\right) + \coth\left(\sqrt{-k}\eta\right)\right), \qquad (3.1.9)$$

where $\eta = x \mp \sqrt{\frac{16ak+1}{16bk+1}}t$.

$$\text{Family}-3{:}u_{9,10}(\eta) = \alpha_0 \pm \frac{4(a-b)\left(m - \sqrt{-k}\tanh\left(\sqrt{-k}\eta\right)\right)}{\sqrt{(4ak+1)(4bk+1)}}, \qquad (3.1.10)$$

$$u_{11,12}(\eta) = \alpha_0 \pm \frac{4(a-b)\left(m - \sqrt{-k}\coth\left(\sqrt{-k}\eta\right)\right)}{\sqrt{(4ak+1)(4bk+1)}}, \qquad (3.1.11)$$

where $\eta = x \mp \sqrt{\frac{4ak+1}{4bk+1}}t$.

Case-II: When $k > 0$, we get the three corresponding families of plane periodic trigonometric solutions (which are omitted for convenience).

Case-III: When $k = 0$, we get the three corresponding families of rational solutions

$$\text{Family}-7: u_{25}(\eta) = \alpha_0 \mp \frac{4m^2\eta\,(a-b)}{m\eta - 1}, \qquad (3.1.12)$$

where $\eta = x - t$.

$$\text{Family}-8: u_{26}(\eta) = \alpha_0 \mp \frac{4(a-b)}{\eta}, \qquad (3.1.13)$$

where $\eta = x - t$.

$$\text{Family}-9: u_{27}(\eta) = \alpha_0 \pm \frac{4\eta\,(a-b)}{m\eta - 1}, \qquad (3.1.14)$$

where $\eta = x - t$.

Remark. All of these solutions have been verified with Maple software by substituting them into the original solutions.

3.2. The Phi-4 equation

In this section, we will exploit the improved F-expansion method to solve the Phi-4 equation. Let us consider the Phi-4 equation is of the form

$$u_{tt} - u_{xx} + a^2 u + b u^3 = 0, \tag{3.2.1}$$

where a and b are nonzero constants. The phi-4 equation plays a significant role in nuclear and particle physics over the decades.

Suppose the traveling wave transformation equation

$$u(x,t) = u(\eta),\ \eta = x - ct, \tag{3.2.2}$$

where c is the speed of the traveling wave. Eq. (3.2.2) transforms Eq. (3.2.1) into the following ODEs:

$$\left(c^2 - 1\right) u'' + a^2 u + b u^3 = 0, \tag{3.2.3}$$

Balancing the highest-order derivative u''' and nonlinear term of highest order u^3 in Eq. (3.2.3), we attain $N = 1$. Hence for $N = 1$, Eq. (2.4) reduces to

$$u(\eta) = \alpha_0 + \alpha_1 (m + F(\eta)) + \beta_1 (m + F(\eta))^{-1}, \tag{3.2.4}$$

Now substituting Eq. (3.2.4) into Eq. (3.2.3), we obtain a polynomial in $F(\eta)$. Equating the coefficient of same power of $F(\eta)$, we get the seven systems of algebraic equations. Solving the above system of equations for $\alpha_0,\ \alpha_1,\ \beta_1$, m and c, we obtain the following set of values:

Set-1 $c = \pm I\sqrt{\frac{a^2-2k}{2k}},\ m = \mp\frac{1}{a}\alpha_0\sqrt{bk},\ \alpha_1 = 0,\ \beta_1 = \sqrt{\frac{k}{b}}\frac{b\alpha_0^2+a^2}{a}$.

Set-2 $c = \pm I\sqrt{\frac{a^2-2k}{2k}},\ m = 0,\ \alpha_0 = 0,\ \alpha_1 = 0,\ \beta_1 = \pm a\sqrt{\frac{k}{b}}$.

Set-3 $c = \pm\frac{1}{2}\sqrt{\frac{a^2+4k}{k}},\ m = 0,\ \alpha_0 = 0,\ \alpha_1 = \pm\frac{a}{\sqrt{-2bk}},\ \beta_1 = \mp Ia\sqrt{\frac{k}{2b}}$.

Set-4 $c = \pm I\frac{1}{2}\sqrt{\frac{a^2-8k}{2k}},\ m = 0,\ \alpha_0 = 0,\ \alpha_1 = \pm\frac{1}{2}\frac{a}{\sqrt{bk}},\ \beta_1 = \mp\frac{1}{2}a\sqrt{\frac{k}{b}}$.

Set-5 $c = \pm I\sqrt{\frac{a^2-2k}{2k}},\ \alpha_0 = \mp\frac{ma}{\sqrt{bk}},\ \alpha_1 = \pm\frac{a}{\sqrt{bk}},\ \beta_1 = 0$.

For the values of the above sets we obtain the following traveling wave solutions for Phi-4 equations.

Case-I: When $k < 0$, we get the following hyperbolic trigonometric solutions:

$$\text{Family} - 1 : u_{1,2}(\eta) = \frac{a\left(b\alpha_0\sqrt{-k}\tanh\left(\sqrt{-k}\eta\right) \mp \sqrt{bk}a\right)}{b\left(\pm\alpha_0\sqrt{bk} + a\sqrt{-k}\tanh\left(\sqrt{-k}\eta\right)\right)}, \tag{3.2.5}$$

$$u_{3,4}(\eta) = \frac{a\left(b\alpha_0\sqrt{-k}\coth\left(\sqrt{-k}\eta\right) \mp \sqrt{bk}a\right)}{b\left(\pm\alpha_0\sqrt{bk} + a\sqrt{-k}\coth\left(\sqrt{-k}\eta\right)\right)}, \tag{3.2.6}$$

where $\eta = x \mp I\sqrt{\frac{a^2-2k}{2k}}\,t$.

$$\text{Family} - 2 : u_{5,6}(\eta) = \pm I\frac{a}{\sqrt{b}}\coth\left(\sqrt{-k}\eta\right), \tag{3.2.7}$$

$$u_{7,8}(\eta) = \pm I\frac{a}{\sqrt{b}}\tanh\left(\sqrt{-k}\eta\right), \tag{3.2.8}$$

where $\eta = x \mp I\sqrt{\frac{a^2-2k}{2k}}\,t$.

$$\text{Family} - 3 : u_{9,10}(\eta) = \pm\frac{a}{\sqrt{2b}}\operatorname{sech}\left(\sqrt{-k}\eta\right)\operatorname{csch}\left(\sqrt{-k}\eta\right), \tag{3.2.9}$$

$$u_{11,12}(\eta) = \mp\frac{a}{\sqrt{2b}}\operatorname{sech}\left(\sqrt{-k}\eta\right)\operatorname{csch}\left(\sqrt{-k}\eta\right), \tag{3.2.10}$$

where $\eta = x \mp \frac{1}{2}\sqrt{\frac{a^2+4k}{k}}\,t$.

$$\text{Family} - 4 : u_{13,14}(\eta) = \mp I\frac{a}{2\sqrt{b}}\left(\tanh\left(\sqrt{-k}\eta\right) + \coth\left(\sqrt{-k}\eta\right)\right), \tag{3.2.11}$$

$$u_{15,16}(\eta) = \mp I\frac{a}{2\sqrt{b}}\left(\tanh\left(\sqrt{-k}\eta\right) + \coth\left(\sqrt{-k}\eta\right)\right), \tag{3.2.12}$$

where $\eta = x \mp I\frac{1}{2}\sqrt{\frac{a^2-8k}{2k}}\,t$.

$$\text{Family} - 5 : u_{17,18}(\eta) = \mp I\frac{a}{\sqrt{b}}\tanh\left(\sqrt{-k}\eta\right), \tag{3.2.13}$$

$$u_{19,20}(\eta) = \pm I\frac{a}{\sqrt{b}}\coth\left(\sqrt{-k}\eta\right), \tag{3.2.14}$$

where $\eta = x \mp I\sqrt{\frac{a^2-2k}{2k}}\,t$.

Case-II: When $k > 0$, we get the five corresponding families of plane periodic trigonometric solutions (which are omitted for convenience).

Remark. All of these solutions have been verified with Maple software by substituting them into the original solutions.

4. Graphical representation of some obtained solutions

Graphical representation is a significant instrument for announcement and it exemplifies obviously the solutions of the problems. We plot the solutions of Eqs. (3.1.10) and (3.1.13) for the Benney–Luke equation in Figs. 1 and 2 and the solutions of Eqs. (3.2.7) and (3.1.8) for the Phi-4 equation in Figs. 3 and 4 along with $x = 0$. The graphs eagerly have shown the solitary wave form of the solutions.

5. Comparisons and advantages

In this section, we will discuss the comparison and the advantages between our proposed methods over others existences methods. We realized that our methods are more convenient and straightforward, also we get more general exact traveling wave solutions from others methods.

To compare between the improved F-expansion method and the modified simple equation method we choose the Phi-4 equation.

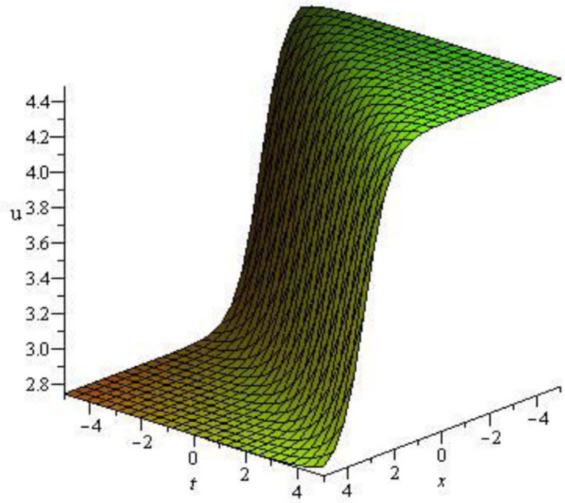

Fig. 1 Kink shaped soliton of Eq. (3.1.10) for $a = 1,\ b = 2,\ k = -1,\ \alpha_0 = 1,\ m = 3$ within the interval $-5 \leq x,\ t \geq 5$.

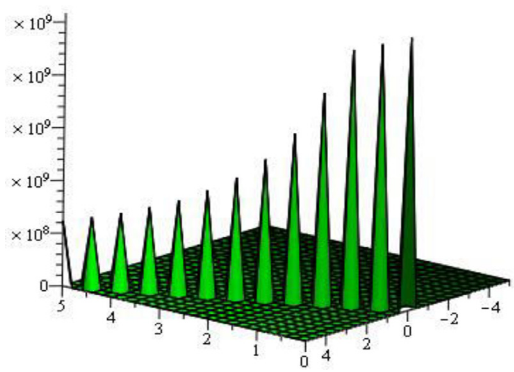

Fig. 2 Singular solution of Eq. (3.1.13) for $a = 3,\ b = 5,\ \alpha_0 = 1$ within the interval $-5 \leq x,\ t \geq 5$.

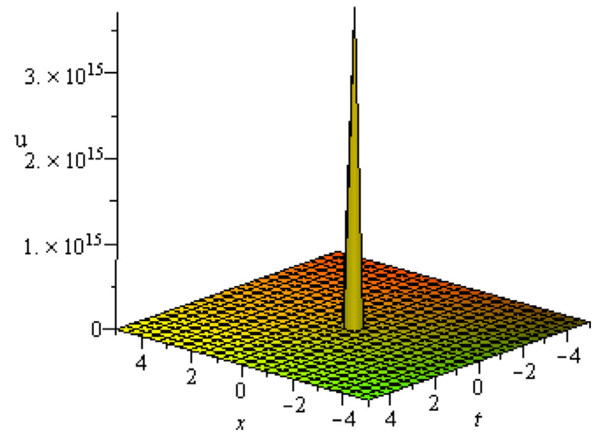

Fig. 3 Single soliton of Eq. (3.2.7) for $a = 1,\ b = -2,\ k = -0.50$ within the interval $-5 \leq x,\ t \geq 5$.

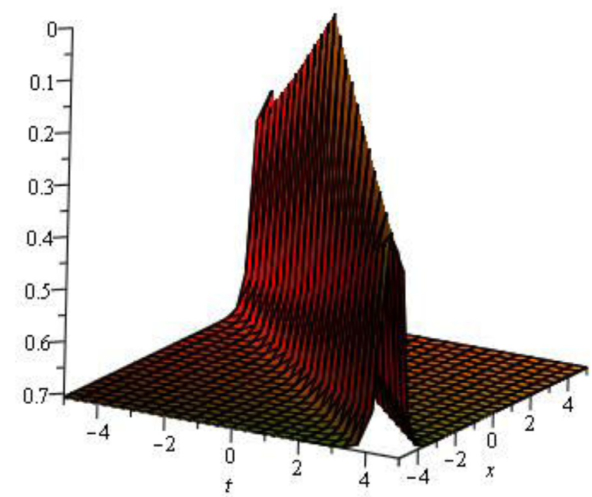

Fig. 4 Soliton solution of Eq. (3.2.8) for $a = 1,\ b = 2,\ k = -5$ within the interval $-5 \leq x,\ t \geq 5$.

In Ref. [20] Akter and Akbar probed exact solutions of the Phi-4 equation throughout the modified simple equation method and attained eight solutions (see Appendix A). On the antagonistic by consuming the improved F-expansion method in this article we attained forty solutions. It is remarkable to point out that some of our solutions are corresponded with already issued results, if we take parameters for particular values which substantiate our solutions. Likewise, if we put $a = 2,\ b = 4$ and $k = -\frac{1}{4}$ in our solutions (3.2.8) and (3.2.7) (Family-2) correspond with the Eq. (A.1) and Eq. (A.2) respectively attained by Akter and Akbar [20] in place of $m = 2,\ \lambda = 4,\ V = 3$.

6. Conclusions

In this present article, we have utilized recently developed the improved F-expansion method combined with Riccati equation for finding the exact traveling wave solutions of some nonlinear evolution equations. We have successfully obtained some exact traveling wave solutions of several nonlinear evolution equations with some free parameters. New explicit solutions are discovered in terms of hyperbolic, trigonometric and rational function solutions to the studied equations through the improved F-expansion method combined with Riccati equation. The presentation of this method confirm that this method is steadfast and effective skill for finding exact traveling wave solutions for a large class of problems in mathematical physics and engineering and can be prolonged to other types of nonlinear evolution equations as well.

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Appendix A

Akter and Akbar [20] inspected the exact solutions of the Phi-4 equation via the modified simple equation method. They gained the following solutions,

$$u(x,t) = \pm\sqrt{-\frac{m^2}{\lambda}}\tanh\left(\frac{1}{2}\sqrt{\frac{2}{V^2-1}}m\,(x - V\,t)\right), \quad \text{(A.1)}$$

$$u(x,t) = \pm\sqrt{-\frac{m^2}{\lambda}}\coth\left(\frac{1}{2}\sqrt{\frac{2}{V^2-1}}m\,(x-V\,t)\right), \tag{A.2}$$

$$u(x,t) = \mp i\sqrt{-\frac{m^2}{\lambda}}\tan\left(\frac{1}{2}\sqrt{\frac{2}{V^2-1}}i\,m\,(x-V\,t)\right), \tag{A.3}$$

$$u(x,t) = \pm i\sqrt{-\frac{m^2}{\lambda}}\cot\left(\frac{1}{2}\sqrt{\frac{2}{V^2-1}}i\,m\,(x-V\,t)\right), \tag{A.4}$$

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