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Original Article

Solitary wave solution of the generalized Hirota–Satsuma coupled KdV system

Mostafa M.A. Khater [a,*], Emad H.M. Zahran [b], Maha S.M. Shehata [c]

[a] *Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt*
[b] *Department of mathematical and physical engineering, University of Benha, Faculty of engineering Shubra, Egypt*
[c] *Department of mathematics, Faculty of science, Zagazig University, Zagazig, Egypt*

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Abstract In this research, we find the exact traveling wave solutions involving parameters of the generalized Hirota–Satsuma couple KdV system according to the modified simple equation method with the aid of Maple 16. When these parameters are taken special values, the solitary wave solutions are derived from the exact traveling wave solutions. It is shown that the modified simple equation method provides an effective and a more powerful mathematical tool for solving nonlinear evolution equations in mathematical physics. Comparison between our results and the well-known results will be presented.

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1. Introduction

No one can deny the important role which played by the nonlinear partial differential equations in the description of many and a wide variety of phenomena not only in physical phenomena, but also in plasma, fluid mechanics, optical fibers, solid state physics, chemical kinetics and geochemistry phenomena. So that, during the past five decades the nonlinear partial differential equations of mathematical physics are major subjects in physical science [1], a lot of method was discovered by a diverse group of scientists to solve the nonlinear partial differential equations. For example, tanh–sech method [2–4], extended tanh-method [5–7], sine–cosine method [8–10], homogeneous balance method [11], the exp($\varphi(\xi)$)-expansion Method [12], Jacobi elliptic function method [13–16], F-expansion method [17–19], exp-function method [20] and [21], trigonometric function series method [22], $(\frac{G'}{G})-$ expansion method

* Corresponding author. Tel.: +201149206914.
E-mail addresses: mostafa.khater2024@yahoo.com, Darsh_2024@yahoo.com (M.M.A. Khater).
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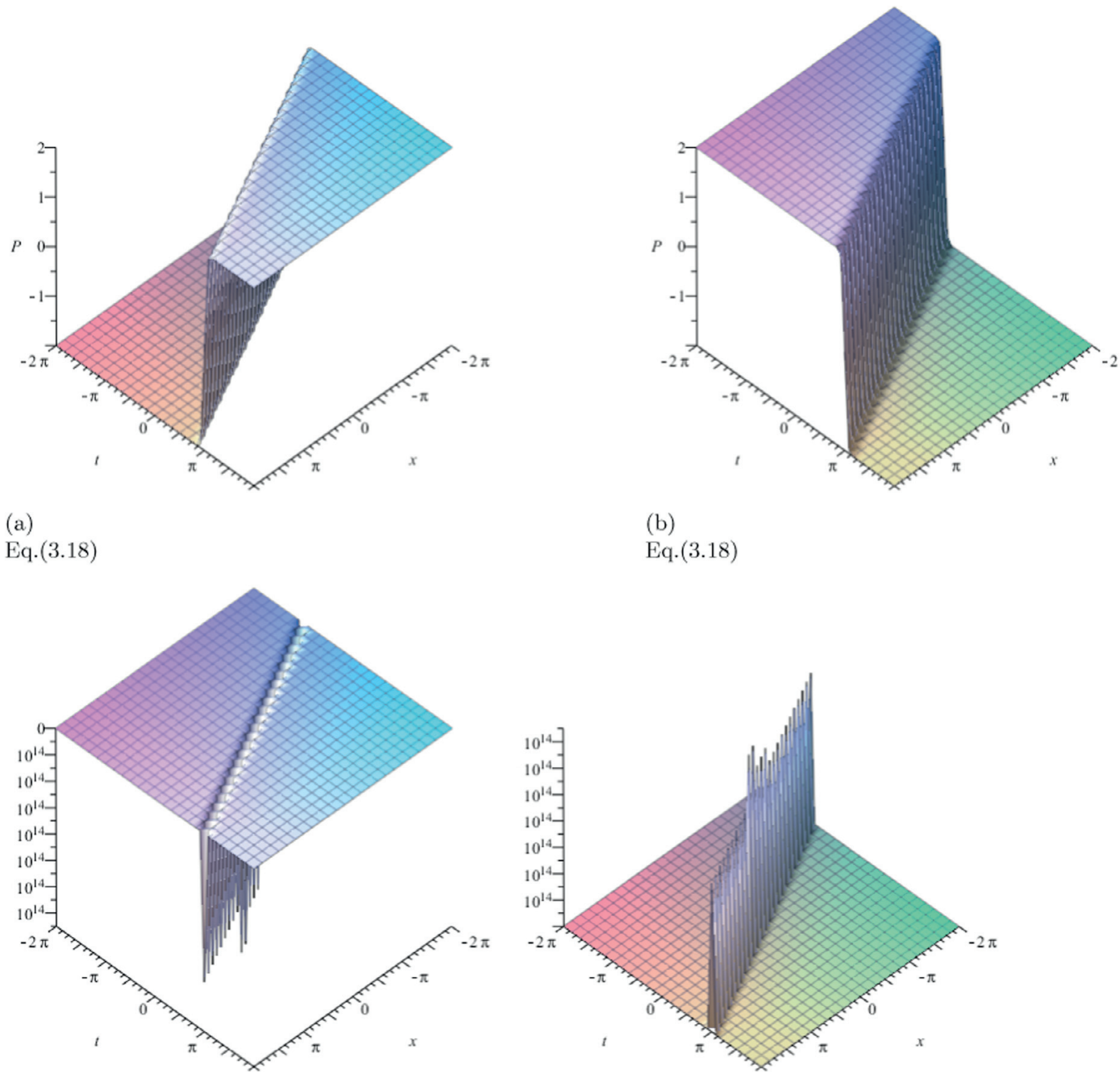

(a) Eq.(3.18)

(b) Eq.(3.18)

(c) Eq.(3.19)

(d) Eq.(3.19)

Fig. 1 The Solitary wave solution of Eqs. (3.18) and (3.19).

[23–26], the modified simple equation method [27–32] and so on.

The objective of this paper is to apply the modified simple equation method for finding the exact traveling wave solution of the generalized Hirota–Satsuma coupled KdV system [33], which play an important role in mathematical physics.

The rest of this paper is organized as follows: In Section 2, we give the description of the modified simple equation method. In Section 3, we use this method to find the exact solutions of the nonlinear evolution equations pointed out above. In Section 5, conclusions are given.

2. Description of the modified simple equation method

Consider the following nonlinear evolution equation

$$F(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{yy} \ldots.) = 0, \tag{2.1}$$

where F is a polynomial in $u(x, t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method:

Step 1. We use the wave transformation

$$u(x, y, t) = u(\xi), \qquad \xi = (x + y - c\,t), \tag{2.2}$$

where c is a nonzero constant, to reduce Eq. (2.1) to the following ODE:

$$P(u, u', u'', u''', \ldots..) = 0, \tag{2.3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while $' = \frac{d}{d\xi}$.

Step 2. Suppose that the solution of Eq. (2.3) has the formal solution:

$$u(\xi) = \sum_{k=0}^{N} A_k \left[\frac{\psi'(\xi)}{\psi(\xi)}\right]^k, \tag{2.4}$$

where A_k are arbitrary constants to be determined, such that $A_N \neq 0$, while the function $\psi(\xi)$ is an unknown function to be determined later, such that $\psi' \neq 0$.

Step 3. Determined the positive integer N in Eq. (2.4) by considering the homogenous balance between the highest order derivatives and the nonlinear terms in Eq. (2.3). Moreover precisely, we define the degree of $u(\xi)$ as $D(u(\xi)) = m$, which gives rise to degree of other expression as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = n + q,\ D\left(u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right) = np + s(n+q).$$

Step 4. Substitute Eq. (2.4) into Eq. (2.3), we calculate all the necessary derivative u', u'', ... of the function $u(\xi)$ and we account the function $\psi(\xi)$. As a result of this substitution, we get a polynomial of $\psi^{-j}(j = 0, 1, 2, ...)$. In this polynomial, we gather all terms of the same power of $\psi^{-j}(j = 0, 1, 2, ...)$, and we equate with zero all coefficient of this polynomial. This operation yields a system of equations which can be solved to find A_k and $\psi(\xi)$. Consequently, we can get the exact solution of Eq. (2.1).

3. Application

Here, we will apply the modified simple equation method described in Section 2 to find the exact traveling wave solutions and the solitary wave solutions of the generalized Hirota–Satsuma coupled KdV system [33]. We consider the generalized Hirota–Satsuma couple KdV system

$$\begin{cases} u_t = \frac{1}{4}u_{xxx} + 3uu_x + 3\left(-v^2 + w\right)_x, \\ v_t = -\frac{1}{2}v_{xxx} - 3uv_x, \\ w_t = -\frac{1}{2}w_{xxx} - 3uw_x. \end{cases} \tag{3.1}$$

When $w = 0$, Eq. (3.1) reduce to be the well known Hirota–Satsuma couple KdV equation. Using the wave transformation $u(x,t) = u(\xi),\ v(x,t) = v(\xi),\ w(x,t) = w(\xi),\ \xi = k(x - \lambda_1 t)$ carries the partial differential equation (3.1) into the ordinary differential equation

$$\begin{cases} -\lambda_1 k u' = \frac{1}{4}k^3 u''' + 3kuu' + 3k\left(-v^2 + w\right)', \\ -\lambda_1 k v' = -\frac{1}{2}k^3 v''' - 3kuv', \\ -\lambda_1 k w' = -\frac{1}{2}k^3 w''' - 3kuw'. \end{cases} \tag{3.2}$$

Suppose we have the relations between (u and v) and (w and v) $\Rightarrow\left(u = \alpha v^2 + \beta v + \gamma\right)$ and $(w = Av + B)$ where α, β, γ, A and B are arbitrary constants. Substituting this relations into second and third equations of Eq. (3.2) and integrating them, we get the same equation and integrate it once again we obtain

$$k^2 v'^2 = -2\alpha v^4 - 2\beta v^3 + 2(\lambda_1 - 3\gamma)v^2 + 2c_1 v + c_2, \tag{3.3}$$

where c_1 and c_2 is the arbitrary constants of integration, and hence, we obtain

$$\begin{aligned} k^2 u'' &= 2\alpha k^2 v'^2 + k^2(2\alpha v + \beta)v'' \\ &= 2\alpha\left[-\alpha v^4 - 2\beta v^3 + 2(\lambda_1 - 3\gamma)v^2 + 2c_1 v + c_2\right] \\ &\quad + (2\alpha v + \beta)\left[-2\alpha v^3 - 3\beta v^2 + 2(\lambda_1 - 3\gamma)v + c_1\right]. \end{aligned} \tag{3.4}$$

So that, we have

$$P'' + lP - mP^3 = 0. \tag{3.5}$$

Where

$$c_1 = \frac{1}{2\alpha^2\left(\beta^2 + 2\lambda_1\alpha\beta - 6\alpha\beta\gamma\right)},\quad v(\xi) = aP(\xi) - \frac{\beta}{2\alpha},$$

$$\alpha = \frac{\beta^2 - 4}{4(\gamma - \lambda_1)},\quad A = \frac{4\beta(\lambda_1 - \gamma)}{\beta^2 - 4},$$

$$\begin{aligned} B = \frac{1}{6(-\gamma + \lambda_1)\left(\beta^2 - 4\right)^2}&(16c_3\lambda_1\beta^2 - 2c_3\lambda_1\beta^4 - 16c_3\gamma\beta^2 \\ &+ 3c_3\gamma\beta^4 + 56\lambda_1^2\gamma\beta^2 \\ &- 48\gamma^2\lambda_1\beta^2 - 16c_2 + c_2\beta^6 - 12c_2\beta^4 + 12c_2\beta^2 - 16\gamma^2\lambda_1 \\ &- 32\lambda_1^2\gamma - 8\lambda_1^3\beta^2 + \beta^4\gamma^3 \\ &- 2\beta^4\lambda_1^3 + 32c_3\gamma - 32c_3\lambda_1 + 48\gamma^3 + \beta^4\gamma^2\lambda_1), \end{aligned}$$

$$l = \frac{-a}{k^2}\left(\frac{3\beta^2}{2\alpha} + 2\lambda_1 - 6\gamma\right),\quad m = \frac{-2\alpha a^3}{k^2}.$$

Balancing between the highest order derivatives and nonlinear terms appearing in P'' and $P^3 \Rightarrow (N + 2 = 3N) \Rightarrow (N = 1)$. So that, by using Eq. (2.4) we get the formal solution of Eq. (3.5)

$$P(\xi) = A_0 + A_1\left(\frac{\psi'}{\psi}\right). \tag{3.6}$$

Substituting Eq. (3.6) and its derivative into Eq. (3.5) and collecting all term with the same power of ψ^{-3}, ψ^{-2}, ψ^{-1}, ψ^0 we obtained:

$$\psi^{-3} : A_1\psi'^3[2 + mA_1^2], \tag{3.7}$$

$$\psi^{-2} : 3A_1\psi'[-\psi'' + mA_0A_1\psi'] = 0, \tag{3.8}$$

$$\psi^{-1} : A_1[\psi''' + \psi'(l + 3mA_0^2)] = 0, \tag{3.9}$$

$$\psi^0 : A_0[l + mA_0^2] = 0. \tag{3.10}$$

From Eqs. (3.7) and (3.10) we obtain

$$A_1 = \pm\sqrt{\frac{-2}{m}},\ A_0 = 0\ or\ A_0 = \pm\sqrt{\frac{-l}{m}},\ where (m < 0),\ (l > 0).$$

Let us discuss the following cases.

Case 1. When $A_0 \neq 0$

we deduce from Eqs. (3.8) and (3.9) that :

$$\psi' = \frac{1}{mA_0A_1}\psi'', \tag{3.11}$$

and

$$\psi' = \frac{-1}{l + 3mA_0^2}\psi''' \tag{3.12}$$

Eqs. (3.11) and (3.12) yield

$$\frac{\psi'''}{\psi''} = E_0, \tag{3.13}$$

where $E_0 = \frac{-(l+3mA_0^2)}{mA_0A_1}$. Integrating Eq. (3.13) and using (3.11), we deduce that

$$\psi' = \frac{c_1}{mA_0A_1} exp(E_0\,\xi), \tag{3.14}$$

and consequently, we get

$$\psi = \frac{c_1}{mE_0A_0A_1} exp(E_0\,\xi) + c_2, \tag{3.15}$$

where c_1 and c_2 are arbitrary constants. Substituting (3.14) and (3.15) into Eq. (3.6), we have the exact traveling wave solution:

$$P(\xi) = \pm\sqrt{\frac{-l}{m}} \pm \sqrt{\frac{-2}{m}}\left(\frac{\frac{c_1}{mA_0A_1} exp(E_0\,\xi)}{\frac{c_1}{mE_0A_0A_1} exp(E_0\,\xi) + c_2}\right), \tag{3.16}$$

when $c_1 = mE_0A_0A_1$. So that, Eq. (3.16) can be written in the form

$$P(\xi) = \pm\sqrt{\frac{-l}{m}} \pm E_0\sqrt{\frac{-2}{m}}\left(\frac{exp(E_0\,\xi)}{exp(E_0\,\xi) + c_2}\right). \tag{3.17}$$

So that, we get the solitary wave solutions

- **Case 1.** When $E_0 < 0$ and $c_2 = 1$.

$$P_{(1,2)}(\xi) = \pm\sqrt{\frac{-l}{m}} \pm \frac{E_0}{2}\sqrt{\frac{-2}{m}}\left(1 - tanh\left(\frac{E_0}{2}\xi\right)\right), \tag{3.18}$$

- **Case 2.** When $E_0 < 0$ and $c_2 = -1$.

$$P_{(3,4)}(\xi) = \pm\sqrt{\frac{-l}{m}} \pm \frac{E_0}{2}\sqrt{\frac{-2}{m}}\left(1 - coth\left(\frac{E_0}{2}\xi\right)\right), \tag{3.19}$$

Finally, we did not study the case when ($E_0 > 0$) and the reason of this:

Under the conditions of values of the parameters l and $m \Rightarrow E_0$ cannot be equal negative value.

Case 2. When $A_0 = 0$

In this case, we deduce from Eqs. (3.8) and (3.9) that $\psi' = 0$, and hence this case will be rejected.

- **Note that:**
  All the obtained results have been checked with Maple 16 by putting them back into the original equation and found correct.

4. Physical meaning of each solutions:

Here, we explain the physical meaning of solution since we cannot that:

Eqs. (3.18) and (3.19) depend on some parameters like l, m, E_0, k, λ_1 when this parameters take special values we can draw the solutions and explain what is the mean of this figures.

For example : when ($m = -2,\ l = 8,\ k = 1,\ \lambda_1 = 2,$) we obtain ($A_0 = 2,\ A_1 = 1$) $\Rightarrow$ So that Eq. (3.18) has two figures which represent kink soliton solutions.

Also, when ($m = -2,\ l = 8,\ k = 1,\ \lambda_1 = 2,$) we obtain ($A_0 = 2,\ A_1 = 1$) $\Rightarrow$ So that Eq. (3.19) has two figures which represent anti kink soliton solutions.

5. Conclusion

The modified simple equation method has been successfully used to find the exact traveling wave solutions of some nonlinear evolution equations. As an application, the traveling wave solutions for the generalized Hirota-Satsuma couple KdV system which have been constructed using the modified simple equation method. Let us compare between our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: Our results of the generalized Hirota–Satsuma couple KdV system are new and different from those obtained in [33–36] and also we draw the figures of solutions and explain the physical meaning of each one. It can be concluded that this method is reliable and propose a variety of exact solutions NPDEs. The performance of this method is effective and can be applied to many other nonlinear evolution equations. The solutions represent the solitary traveling wave solution for the generalized Hirota–Satsuma couple KdV system.

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