SHIFTED GEGENBAUER OPERATIONAL MATRIX AND ITS APPLICATIONS FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

This paper introduces a new numerical mechanism for solving multi- order fractional differential equations (MOFDEs) and systems of fractional differential equations, in which the fractional derivatives are expressed in Riemman- liouville (RL) sense. A new shifted ultraspherical (Gegenbauer) operational matrix (SGOM) of fractional integration of arbitrary order is induced. By using this matrix jointly with the Tau method, the solution of fractional differential equation (FDE) is decreased to the solution of a system of algebraic equations (AEs). Helpful problems are built-in to show the powerful and validity of the proposed technique.

Keywords: Riemman- liouville fractional integral; Differential equation of fractional- order; Operational matrix; Gegenbauer polynomials; Tau method.

MSC: 33A08; 33C45; 65M70

1 Introduction

Recently, the recurrent appearance of ordinary and partial fractional differential equations have attracted the attentions of numerous studies in different fields like fluid mechanics, viscoelasticity, biology, physics and engineering, etc [1]; this is because the fractional- order models are more accurate than integer- order models [2–4]. It is difficult to find exact solutions for several FDEs, so approximate and numerical mechanisms are introduced [5]. Different analytical and numerical methods such as Adomian decomposition method [6–8], variational iteration method [9–11], homotopy perturbation method [12–14], homotopy analysis method [15, 16], collocation method [17], Galerkin method [18], spectral methods [17, 18] and other methods are investigated.

For more decades, spectral methods have obtained a great interest in solving differential equations. These methods are characterized by their precision for any number of unknowns. There are three main spectral images, they are the Galerkin, collocation and Tau methods [19].

In the spectral methods, the explicit formula for operational matrices of fractional integration and fractional differentiation for classical orthogonal polynomials are needed [19]. The mechanism of the proposed method depends on transforming the FDEs into a system of AEs which is very easy to solve. Recently, some types of orthogonal polynomials have been introduced as basis functions of the operational matrices of fractional derivatives and integrals which are used to solve ordinary and partial fractional differential equations [21, 22].

In this paper we investigate the operational matrix of the fractional integral of the shifted Gegenbauer polynomials and use it with the Tau method to present numerical solutions to MOFDEs and systems of fractional differential equations. The Gegenbauer polynomials have many useful properties. The most important characteristic of them is achieving rapid rates of convergence, for more details see [23–26].

The paper is organized as follows. In section 2, we review necessary definitions and properties of fractional calculus and ultraspherical (Gegenbauer) polynomials. In section 3, the SGOM of fractional integration is proved. In section 4, the proposed mechanism of applying SGOM of fractional integration for solving linear MOFDEs and systems of fractional differential equations is discussed. In section 5, Some applications of the proposed method are given. Finally a conclusion is given in section 6.

2 Preliminaries and Definitions:

2.1 Fractional Calculus Definitions

Definition 1. One of the popular definition of fractionl integral is the RL, which get from the relation

$$\begin{aligned} I^\nu f(x) &= \frac{1}{\Gamma(\nu)} \int_0^x (x-\xi)^{\nu-1} f(\xi) d\xi, \quad m-1 < \nu < m, \quad m \in N, \quad \nu > 0, \quad x > 0, \\ I^0 f(x) &= f(x). \end{aligned} \quad (2.1)$$

For more properties of I^ν see [27], we just recall the next property

$$I^\nu x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\nu+\beta+1)} x^{\nu+\beta}. \quad (2.2)$$

Definition 2.

D^ν is the RL fractional derivative of order ν is defined by

$$D^\nu f(x) = \frac{d^m}{dx^m} (I^{m-\nu} f(x)), \quad m-1 < \nu \leq m, \quad m \in N, \quad \nu \in R, \quad (2.3)$$

where m is the smallest integer order greater than ν .

Lemma 1.

If $m-1 < \nu \leq m$, $m \in N$, then

$$\begin{aligned} D^\nu I^\nu f(x) &= f(x), \\ I^\nu D^\nu f(x) &= f(x) - \sum_{i=0}^{m-1} f^{(i)}(0^+) \frac{x^i}{i!}, \quad x > 0. \end{aligned} \quad (2.4)$$

2.2 Shifted ultraspherical (Gegenbauer) polynomials and some properties

The ultraspherical (Gegenbauer) polynomials $C_j^{(\alpha)}(x)$, of degree $j \in \mathbb{Z}^+$, and associated with the parameter $\alpha > \frac{-1}{2}$ are a sequence of real polynomials in the finite domain $[-1, 1]$. They are a family of orthogonal polynomials which has many applications.

Definition 1.

The Gegenbauer polynomials are the Jacobi polynomials, $P_j^{(\alpha, \beta)}$, with $\alpha = \beta = \alpha - \frac{1}{2}$ so that

$$C_j^{(\alpha)}(x) = \frac{\Gamma(\alpha + \frac{1}{2})\Gamma(j+2\alpha)}{\Gamma(2\alpha)\Gamma(j+\alpha+\frac{1}{2})} P_j^{(\alpha-\frac{1}{2}, \alpha-\frac{1}{2})}(x), \quad j = 0, 1, 2, \dots$$

- There are useful relations between the Chebyshev polynomial of the first and second kind and the Legendre polynomial with the Gegenbaure polynomials as follows

$$T_j(x) \equiv \frac{j}{2} \lim_{\alpha \rightarrow 0} \alpha^{-1} C_j^{(\alpha)}(x), \quad j \geq 1,$$

$$C_j^{(1)}(x) \equiv \frac{1}{j+1} U_j(x),$$

$$L_j(x) \equiv C_j^{(\frac{1}{2})}(x).$$

- The Gegenbauer polynomials can be created from the next recurrence equation

$$(j + 2\alpha)C_{j+1}^{(\alpha)}(x) = 2(j + \alpha)x C_j^{(\alpha)}(x) - j C_{j-1}^{(\alpha)}(x), \quad j = 1, 2, \dots$$

with

$$C_0^{(\alpha)}(x) = 1, \quad C_1^{(\alpha)}(x) = x.$$

- The orthogonality relation of the Gegenbauer polynomials is given by the weighted inner product

$$\langle C_i^{(\alpha)}(x), C_j^{(\alpha)}(x) \rangle = \int_{-1}^1 C_i^{(\alpha)}(x) C_j^{(\alpha)}(x) \omega^{(\alpha)}(x) dx = \lambda_j^{(\alpha)} \delta_{i,j},$$

where $\omega^{(\alpha)}(x)$ is the weight function, it is an even function given from relation

$$\omega^{(\alpha)}(x) = (1 - x^2)^{\alpha - \frac{1}{2}},$$

and

$$\lambda_j^{(\alpha)} = \|C_j^{(\alpha)}(x)\|^2 = \frac{2^{1-2\alpha} \pi \Gamma(j + 2\alpha)}{j! (j + \alpha) \Gamma^2(\alpha)},$$

is the normalization factor and $\delta_{i,j}$ is the Kronecker delta function.

- The shifted Gegenbauer polynomials are formed by replacing the variable x with $\frac{2x}{L} - 1$, $0 \leq x \leq L$. So, we can write shifted Gegenbauer polynomials as

$$C_{S,j}^{(\alpha)}(x) = C_j^{(\alpha)}\left(\frac{2x}{L} - 1\right).$$

- The analytical form of the shifted Gegenbauer polynomial is given from

$$\begin{aligned} C_{S,j}^{(\alpha)}(x) &= \sum_{k=0}^j (-1)^{j-k} \frac{\Gamma(\alpha + \frac{1}{2}) \Gamma(j + k + 2\alpha)}{\Gamma(k + \alpha + \frac{1}{2}) \Gamma(2\alpha) (j - k)! k! L^k} x^k, \\ C_{S,j}^{(\alpha)}(0) &= (-1)^j \frac{\Gamma(j + 2\alpha)}{\Gamma(2\alpha) j!}. \end{aligned} \tag{2.5}$$

- The orthogonal relation of shifted Gegenbauer polynomials is getting from

$$\langle C_{S,i}^{(\alpha)}(x), C_{S,j}^{(\alpha)}(x) \rangle = \int_0^L C_{S,i}^{(\alpha)}(x) C_{S,j}^{(\alpha)}(x) \omega_S^{(\alpha)}(x) dx = \lambda_{S,j}^{(\alpha)} \delta_{i,j}, \tag{2.6}$$

where $\omega_S^{(\alpha)}(x)$ is the weight function, it is even function given from the relation

$$\omega_S^{(\alpha)}(x) = (xL - x^2)^{\alpha - \frac{1}{2}},$$

and

$$\lambda_{S,j}^{(\alpha)} = \left(\frac{L}{2}\right)^{2\alpha} \lambda_j^{(\alpha)}.$$

- This polynomial recover the shifted Chebyshev polynomial of the first kind $T_{S,j}(x) \equiv C_{S,j}^{(0)}(x)$, the shifted Legendre polynomial $L_{S,j}(x) \equiv C_{S,j}^{(\frac{1}{2})}(x)$, and the shifted Chebyshev polynomial of the second kind $C_{S,j}^{(1)}(x) \equiv \frac{1}{j+1} U_{S,j}(x)$.

- The q times repeated derivative of Gegenbauer polynomials given from the relation

$$D^q C_{S,N,j}^{(\alpha)}(x) = \frac{2^{2q}(\alpha + q - 1)!}{(\alpha - 1)!} C_{S,N,j-q}^{(\alpha+q)}(x), \quad (2.7)$$

by substituting $x = 0$ at (2.7), we get

$$D^q C_{S,N,j}^{(\alpha)}(0) = \frac{(-1)^{j-q} 2^{2q} (\alpha + q - 1)! \Gamma(j + q + 2\alpha)}{(\alpha - 1)! \Gamma(2\alpha + 2q) (j - q)!}. \quad (2.8)$$

- The square integrable function $y(x)$ in $[0, L]$ can be approximated by shifted Gegenbauer polynomials as:

$$y(x) = \sum_{j=0}^N \tilde{y}_{S,N,j} C_{S,N,j}^{(\alpha)}(x),$$

where the coefficients $\tilde{y}_{S,N,j}$ are getting from

$$\tilde{y}_{S,N,j} = (\lambda_{S,j}^{(\alpha)})^{-1} \int_0^L y(x) \omega_S^{(\alpha)}(x) C_{S,j}^{(\alpha)}(x) dx. \quad (2.9)$$

- The approximation of a function $y(x)$ can be written in the vector form as

$$y(x) = Y^T \phi(x), \quad (2.10)$$

where $Y^T = [Y_0, Y_1, \dots, Y_N]$ is the shifted Gegenbauer coefficient vector, and

$$\phi(x) = [C_{S,N,0}^{(\alpha)}(x), C_{S,N,1}^{(\alpha)}(x), \dots, C_{S,N,N}^{(\alpha)}(x)]^T \quad (2.11)$$

is the shifted Gegenbauer vector.

- Consider $f(x)$, $g(x)$ be integrable functions in $[0, L]$. These functions can be approximated by shifted Gegenbauer polynomials as:

$$f(x) = \sum_{i=0}^N \tilde{f}_{S,N,i} C_{S,N,i}^{(\alpha)}(x) = F^T \phi(x),$$

$$g(x) = \sum_{j=0}^N \tilde{g}_{S,N,j} C_{S,N,j}^{(\alpha)}(x) = G^T \phi(x),$$

where $\tilde{f}_{S,N,i}$, $\tilde{g}_{S,N,j}$ are obtained from relation (2.9). The approximation of the product $f(x)g(x)$ can be obtained from the following relation

$$f(x)g(x) = \sum_{k=0}^N \tilde{v}_{S,N,k} C_{S,N,k}^{(\alpha)}(x) = \Upsilon^T \phi(x), \quad (2.12)$$

where $\Upsilon^T = [\tilde{v}_{S,N,0}, \tilde{v}_{S,N,1}, \dots, \tilde{v}_{S,N,k}]$ is an unknown vector, its elements are calculated by

$$\tilde{v}_{S,N,k} = \frac{1}{\lambda_{S,k}^{(\alpha)}} \sum_{i=0}^N \sum_{j=0}^N \tilde{f}_{S,N,i} \times \tilde{g}_{S,N,j} \int_0^1 w_S^{(\alpha)}(x) C_{S,N,i}^{(\alpha)}(x) C_{S,N,j}^{(\alpha)}(x) C_{S,N,k}^{(\alpha)}(x) dx. \quad (2.13)$$

- The q times repeated integration of Gegenbauer vector is obtained by

$$I^q \phi(x) \simeq P^{(q)} \phi(x), \quad (2.14)$$

where $P^{(q)}$ is called the operational matrix (OM) of the integration of $\phi(x)$.

3 Fractional Shifted Gegenbauer Operational Matrix (SGOM).

At this section, the shifted Gegenbauer operational matrix (SGOM) of fractional integration in $[0, 1]$ will be proved.

Theorem(1) Let $\phi(x)$ be the shifted Gegenbauer vector and $\nu > 0$ then

$$I^\nu \phi(x) \simeq P^{(\nu)} \phi(x), \quad (3.1)$$

where $P^{(\nu)}$ is called OM of fractional integration of order ν in the RL sense, it is a square matrix of order $(N+1) \times (N+1)$ and is written as

$$P^{(\nu)} = \begin{pmatrix} \sum_{k=0}^0 \xi_{0,0,k} & \sum_{k=0}^0 \xi_{0,1,k} & \cdot & \cdot & \cdot & \sum_{k=0}^0 \xi_{0,N,k} \\ \sum_{k=0}^1 \xi_{1,0,k} & \sum_{k=0}^1 \xi_{1,1,k} & \cdot & \cdot & \cdot & \sum_{k=0}^1 \xi_{1,N,k} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sum_{k=0}^i \xi_{i,0,k} & \sum_{k=0}^i \xi_{i,1,k} & \cdot & \cdot & \cdot & \sum_{k=0}^i \xi_{i,N,k} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sum_{k=0}^N \xi_{N,0,k} & \sum_{k=0}^N \xi_{N,1,k} & \cdot & \cdot & \cdot & \sum_{k=0}^N \xi_{N,N,k} \end{pmatrix} \quad (3.2)$$

where $\xi_{i,j,k}$ is given by:

$$\xi_{i,j,k} = \Xi \times \Upsilon,$$

where

$$\begin{aligned} \Xi &= \sum_{k=0}^i (-1)^{i-k} \frac{\Gamma(\alpha + \frac{1}{2})\Gamma(i+k+2\alpha)}{\Gamma(k+\alpha+\frac{1}{2})\Gamma(2\alpha)\Gamma(k+\nu+1)(i-k)!}, \\ \Upsilon &= \sum_{f=0}^j (-1)^{j-f} \frac{j!(j+\alpha)\Gamma^2(\alpha)\Gamma^2(\alpha+\frac{1}{2})\Gamma(2\alpha+j+f)\Gamma(\nu+k+f+\alpha+\frac{1}{2})}{2^{(1-4\alpha)}\pi\Gamma(2\alpha+j)\Gamma(2\alpha)\Gamma(\alpha+f+\frac{1}{2})(j-f)!f!\Gamma(\nu+k+f+2\alpha+1)} \end{aligned} \quad (3.3)$$

Proof

From relation (2.5) and by using Eq. (2.2), we can write

$$\begin{aligned} I^\nu C_{S,N,i}^{(\alpha)}(x) &= \sum_{k=0}^i (-1)^{i-k} \frac{\Gamma(\alpha + \frac{1}{2})\Gamma(i+k+2\alpha)}{\Gamma(k+\alpha+\frac{1}{2})\Gamma(2\alpha)(i-k)!k!} I^\nu(x^k), \\ &= \sum_{k=0}^i (-1)^{i-k} \frac{\Gamma(\alpha + \frac{1}{2})\Gamma(i+k+2\alpha)}{\Gamma(k+\alpha+\frac{1}{2})\Gamma(2\alpha)(i-k)! \Gamma(\nu+k+1)} x^{k+\nu}, \quad i = 0, 1, 2, \dots, N. \end{aligned} \quad (3.4)$$

The function $x^{k+\nu}$ can be written as a series of $N+1$ terms of Gegenbauer polynomial,

$$x^{k+\nu} = \sum_{j=0}^N \tilde{x}_{S,N,j} C_{S,N,j}^{(\alpha)}(x), \quad (3.5)$$

Where

$$\tilde{x}_{S,N,j} = \sum_{f=0}^j (-1)^{j-f} \frac{j!(j+\alpha)\Gamma^2(\alpha)\Gamma^2(\alpha+\frac{1}{2})\Gamma(2\alpha+j+f)\Gamma(\nu+k+f+\alpha+\frac{1}{2})}{2^{(1-4\alpha)}\pi\Gamma(2\alpha+j)\Gamma(\alpha+f+\frac{1}{2})(j-f)!f!\Gamma(2\alpha)\Gamma(\nu+k+f+2\alpha+1)}. \quad (3.6)$$

Now, by employing equations (3.4)-(3.5) we obtain:

$$\begin{aligned} I^\nu C_{S,N,i}^{(\alpha)}(x) &= \sum_{k=0}^i \sum_{j=0}^N (-1)^{i-k} \frac{\Gamma(\alpha + \frac{1}{2})\Gamma(i+k+2\alpha)}{\Gamma(k+\alpha+\frac{1}{2})\Gamma(2\alpha)(i-k)!\Gamma(\nu+k+1)} \tilde{x}_{S,N,j} C_{S,N,j}^{(\alpha)}(x), \\ &= \sum_{j=0}^N \left(\sum_{k=0}^i \xi_{i,j,k} \right) C_{S,N,j}^{(\alpha)}(x), \quad i = 0, 1, \dots, N, \end{aligned} \quad (3.7)$$

where $\xi_{i,j,k}$ is given in Eq. (3.3).

Writing the last equation in a vector form gives

$$I^\nu C_{S,N,i}^{(\alpha)}(x) \simeq \left[\sum_{k=0}^i \xi_{i,0,k}, \sum_{k=0}^i \xi_{i,1,k}, \dots, \sum_{k=0}^i \xi_{i,N,k} \right] \phi(x), \quad i = 0, 1, \dots, N, \quad (3.8)$$

which finished our proof.

4 Procedure Solution of SGOM for Solving Fractional Differential Equations

4.1 Multi- order fractional differential equations

At this section, we apply the SGOM method to a MOFDE with RL fractional derivative. So, let ν be the highest fractional order of FDE. By using the properties of fractional integral, the proposed FDE is transformed into an integral equation which can be approximated by using SGOM. Consider the following MOFDE:

$$D^\nu y(x) = \sum_{i=1}^k \gamma_i D^{\beta_i} y(x) + \gamma_{k+1} y(x) + f(x), \text{ in } I = [0, 1], \quad (4.1)$$

with the initial conditions

$$y^{(i)}(x) = d_i, \quad i = 0, 1, \dots, m-1, \quad (4.2)$$

where $\gamma_i (i = 1, \dots, k+1)$ are real constant coefficients, $m-1 < \nu \leq m$, and $0 < \beta_1 < \beta_2 < \dots < \beta_k < \nu$.

Moreover $D^\nu y(x) \equiv y^{(\nu)}(x)$ refers to the RL fractional derivative of order ν .

The existence, uniqueness and continuous dependence of the solution of the problem (4.1- 4.2) are discussed in [28].

By applying the RL integral of order ν on Eq.(4.1) and use Eq.(2.4), we get

$$\begin{aligned} y(x) - \sum_{j=0}^{m-1} y^{(j)}(0^+) \frac{x^j}{j!} &= \sum_{i=1}^k \gamma_i I^{\nu-\beta_i} \left[y(x) - \sum_{j=0}^{m_i-1} y^{(j)}(0^+) \frac{x^j}{j!} \right] + \gamma_{k+1} I^\nu y(x) + I^\nu f(x), \\ y^{(i)}(0) &= d_i, \quad i = 0, 1, \dots, m-1, \end{aligned} \quad (4.3)$$

where $m_i - 1 < \beta_i \leq m_i, m_i \in N$, this implies that

$$y(x) = \sum_{i=1}^k \gamma_i I^{\nu-\beta_i} y(x) + \gamma_{k+1} I^\nu y(x) + g(x), \quad (4.4)$$

where

$$g(x) = I^\nu f(x) + \sum_{j=0}^{m-1} d_j \frac{x^j}{j!} - \sum_{i=1}^k \gamma_i I^{\nu-\beta_i} \left(\sum_{j=0}^{m_i-1} d_j \frac{x^j}{j!} \right).$$

The functions $y(x)$ and $g(x)$ are approximated by shifted Gegenbauer polynomials as

$$y_N(x) \simeq \sum_{i=0}^N \tilde{y}_{S,N,i}^{(\alpha)} C_{S,N,i}^{(\alpha)}(x) = Y^T \phi(x), \quad (4.5)$$

$$g_N(x) \simeq \sum_{i=0}^N \tilde{g}_{S,N,i}^{(\alpha)} C_{S,N,i}^{(\alpha)}(x) = G^T \phi(x), \quad (4.6)$$

where the vector $G = [g_0, g_1, \dots, g_N]^T$ is given and $Y = [Y_0, Y_1, \dots, Y_N]^T$ is an unknown vector. By using theorem (3) the RL integral of order ν and $(\nu - \beta_j)$ of Eq. (4.5), can be written as

$$I^\nu y_N(x) \simeq Y^T I^\nu \phi(x) \simeq Y^T P^{(\nu)} \phi(x), \quad (4.7)$$

and

$$I^{\nu-\beta_j} y_N(x) \simeq Y^T I^{\nu-\beta_j} \phi(x) \simeq Y^T P^{(\nu-\beta_j)} \phi(x), \quad j = 1, \dots, k, \quad (4.8)$$

where $P^{(\nu)}$ is the $(N+1) \times (N+1)$ square matrix of fractional integration of order ν . Using Eqs.(4.5- 4.8) the residual $R_N(x)$ for Eq. (4.4) can be written as

$$R_N(x) = \left(Y^T - Y^T \sum_{j=1}^k \gamma_j P^{(\nu-\beta_j)} - \gamma_{k+1} Y^T P^{(\nu)} - G^T \right) \phi(x). \quad (4.9)$$

By using Tau method, we generate $(N - m + 1)$ linear algebraic equations from

$$\langle R_N(x), C_{S,N,j}^{(\alpha)}(x) \rangle = \int_0^L R_N(x) C_{S,N,j}^{(\alpha)}(x) dx = 0, \quad j = 0, 1, \dots, N - m. \quad (4.10)$$

By using Eq. (4.5) in Eq. (4.2), we get

$$y^{(i)}(0) = \sum_{i=0}^N \tilde{y}_{S,N,i}^{(\alpha)} D^{(i)} C_{S,N,j}^{(\alpha)}(0) = d_i, \quad i = 0, 1, \dots, m - 1. \quad (4.11)$$

From Eqs.(4.10) and (4.11), $(N - m + 1)$ and m set of linear algebraic equations are generated. This linear system can be solved easily to get the unknown coefficients of the vector Y . So from Eq.(4.5) we can calculate $y_N(x)$ which is the solution of problem (4.1).

4.2 System of fractional linear differential equations with initial conditions

Consider the following system of FDEs

$$D^{(\nu)} y_i(x) = L_i \left(x, y_1(x), y_1^{(D_1)}(x), \dots, y_1^{(D_r)}(x), y_2(x), y_2^{(D_1)}(x), \dots, y_2^{(D_r)}(x), \dots, y_n(x), y_n^{(D_1)}(x), \dots, y_n^{(D_r)}(x) \right) + f_i(x), \quad (4.12)$$

with the initial conditions

$$y_i^{(j)}(0) = d_i^{(j)}, \quad j = 0, 1, \dots, m - 1, \quad i = 1, 2, \dots, n, \quad (4.13)$$

where $0 < D_1 < D_2 < \dots < D_r < \nu$, $m - 1 < \nu \leq m$, L_i are linear operators, $D^r y_i \equiv y_i^{(r)}$ and $f_i(i = 1, 2, \dots, n)$ are given as

source functions.

To solve this system, by applying the RL integral of order ν on Eq.(4.12) and use Eq.(2.4), we get

$$y_i(x) - \sum_{k=0}^{m-1} y_i^{(k)}(0) \frac{x^k}{k!} = L_i(I^{(\nu)}(x), I^{(\nu)}y_1(x), I^{(\nu-D_1)} \left[y_1(x) - \sum_{k=0}^{m_1-1} y_1^{(k)}(0) \frac{x^k}{k!} \right], \dots, I^{(\nu-D_r)} \left[y_1(x) - \sum_{k=0}^{m_r-1} y_1^{(k)}(0) \frac{x^k}{k!} \right], \\ I^{(\nu)}y_2(x), I^{(\nu-D_1)} \left[y_2(x) - \sum_{k=0}^{m_1-1} y_2^{(k)}(0) \frac{x^k}{k!} \right], \dots, I^{(\nu-D_r)} \left[y_2(x) - \sum_{k=0}^{m_r-1} y_2^{(k)}(0) \frac{x^k}{k!} \right], \dots, I^{(\nu)}y_n(x), \\ I^{(\nu-D_1)} \left[y_n(x) - \sum_{k=0}^{m_1-1} y_n^{(k)}(0) \frac{x^k}{k!} \right], \dots, I^{(\nu-D_r)} \left[y_n(x) - \sum_{k=0}^{m_r-1} y_n^{(k)}(0) \frac{x^k}{k!} \right]) + I^{(\nu)}f_i(x) \quad (4.14)$$

$$y_i^{(j)}(0) = d_i^{(j)}, \quad j = 0, 1, \dots, m-1, \quad i = 0, 1, \dots, n, \quad (4.15)$$

this implies that

$$y_i(x) = L_i(I^{(\nu)}y_1(x), I^{(\nu-D_1)}y_1(x), \dots, I^{(\nu-D_r)}y_1(x), I^{(\nu)}y_2(x), I^{(\nu-D_1)}y_2(x), \dots, I^{(\nu-D_r)}y_2(x), \dots, \\ I^{(\nu)}y_n(x), I^{(\nu-D_1)}y_n(x), \dots, I^{(\nu-D_r)}y_n(x)) + g_i(x), \quad (4.16)$$

where

$$g_i(x) = I^{(\nu)}(x) + I^{(\nu)}f_i(x) + \sum_{k=0}^{m-1} d_i^{(k)} \frac{x^k}{k!} - I^{(\nu-D_1)} \sum_{k=0}^{m_1-1} d_i^{(k)} \frac{x^k}{k!} - \dots - I^{(\nu-D_r)} \sum_{k=0}^{m_r-1} d_i^{(k)} \frac{x^k}{k!}, \quad i = 1, \dots, n, \quad (4.17)$$

The functions $y_i(x)$ and $g_i(x)$ are approximated by the shifted Gegenbauer polynomials as

$$y_{N,i}(x) \simeq \sum_{j=0}^N \tilde{y}_{S,N,i,j} C_{S,N,j}^{(\alpha)}(x) = Y_i^T \phi(x), \quad (4.18)$$

$$g_{N,i}(x) \simeq \sum_{j=0}^N \tilde{g}_{S,N,i,j} C_{S,N,j}^{(\alpha)}(x) = G_i^T \phi(x), \quad i = 1, \dots, n, \quad (4.19)$$

where the vector $G_i = [g_{i,0}, g_{i,1}, \dots, g_{i,N}]^T$ is given and $Y_i = [Y_{i,0}, Y_{i,1}, \dots, Y_{i,N}]^T$ is an unknown vector.

The RL integral of orders (ν) and $(\nu - D_r)$ of Eq. (4.18), can be written as

$$I^{(\nu)}y_{N,i}(x) \simeq Y_i^T I^{(\nu)}\phi(x) \simeq Y_i^T P^{(\nu)}\phi(x), \quad i = 1, \dots, n, \quad (4.20)$$

and

$$I^{(\nu-D_r)}y_{N,i}(x) \simeq Y_i^T I^{(\nu-D_r)}\phi(x) \simeq Y_i^T P^{(\nu-D_r)}\phi(x), \quad i = 1, \dots, n, \quad (4.21)$$

where $P^{(\nu)}$ is the $(N+1) \times (N+1)$ square matrix of fractional integration of order (ν) . By using Eq. (4.18- 4.21) the residual $R_{N,i}(x)$ for system (4.17) can be written as

$$R_{N,i}(x) = (Y_i^T - L_i(Y_1^T P^{(\nu-D_1)}, \dots, Y_1^T P^{(\nu-D_r)}, Y_2^T P^{(\nu-D_1)}, \dots, Y_2^T P^{(\nu-D_r)}, Y_r^T P^{(\nu-D_1)}, \dots, \\ Y_r^T P^{(\nu-D_r)}, Y_1^T P^{(\nu)}, \dots, Y_r^T P^{(\nu)}) - G_i^T)\phi(x), \quad i = 1, \dots, n. \quad (4.22)$$

By applying the Tau method, we generate $(N - m + 1)n$ linear algebraic equations

$$\langle R_{N,i}(t), C_{S,N,j}^{(\alpha)} \rangle = \int_0^L R_{N,i}(t) C_{S,N,j}^{(\alpha)}(t) dt = 0, \quad i = 1, \dots, n, \quad j = 0, 1, \dots, N - m. \quad (4.23)$$

Also by substituting Eqs. (2.8) and (4.18) into Eq. (4.13), we get

$$y_i^{(j)}(0) = \sum_{j=0}^N \tilde{y}_{S,N,i,j} C_{S,N,j}^{(\alpha)}(0) = d_i^{(j)}, \quad j = 0, 1, \dots, m-1, \quad i = 1, \dots, n. \quad (4.24)$$

From Eqs.(4.23) and (4.24), $(N-m+1)n$ and mn set of linear algebraic equations are generated. This linear system can be solved easily for the unknowns coefficient of the vector Y_i , $i = 1, \dots, n$. So from Eq. (4.18) we can calculate $y_{N,i}(t)$ which is the solution of problem (4.12).

4.3 Systems of fractional non-linear differential equations with initial conditions

Consider the following system of non- linear FDEs

$$D^\nu y_i(x) = F_i(x, y_1(x), y_2(x), \dots, y_r(x)), \quad i = 1, 2, \dots, n. \quad (4.25)$$

With the initial conditions

$$y_i^{(j)}(0) = d_i^{(j)}, \quad j = 0, 1, \dots, m-1, \quad i = 1, 2, \dots, n, \quad (4.26)$$

where $F_i(i = 1, 2, \dots, r)$ are non- linear functions.

To solve this system by using SGOM, we will follow the same procedure as discussed in the subsection (4.2) with the help of the Eqs. (2.12)- (2.13).

5 Illustrative Problems

Now, some problems are given to clarify the applicability and accuracy of the proposed mechanism.

Problem (1): Find the solution of the following initial value problem,

$$D^{\frac{3}{2}}y(x) + 3y(x) = 3x^3 + \frac{8}{\Gamma(0.5)}x^{1.5}, \quad y(0) = 0, \quad \dot{y}(0) = 0, \quad (5.1)$$

where the exact solution is $y(x) = x^3$ [19].

From Eq. (4.5) the approximate solution with $N = 3$, is written as

$$y(x) = \sum_{j=0}^3 \tilde{y}_{S,N,j} C_{S,N,j}^{(\alpha)}(x) = Y^T \phi(x), \quad g(x) = \sum_{j=0}^3 \tilde{g}_{S,N,j} C_{S,N,j}^{(\alpha)}(x) = G^T \phi(x).$$

$$I^{\frac{3}{2}}y(x) = Y^T I^{\frac{3}{2}}\phi(x) = Y^T P^{(\frac{3}{2})}\phi(x), \quad I^{\frac{3}{2}}g(x) = G^T I^{\frac{3}{2}}\phi(x) = G^T P^{(\frac{3}{2})}\phi(x).$$

From theorem (3), we have

$$P^{(\frac{3}{2})} = \begin{pmatrix} 0.300901 & 0.386873 & 0.0716431 & -0.00911822 \\ -0.128958 & -0.1003 & 0.0586171 & 0.0231462 \\ 0.0143286 & -0.0351703 & -0.0385771 & 0.0273546 \\ 0.0013026 & 0.00991982 & -0.019539 & -0.0222385 \end{pmatrix}, \quad G = \begin{pmatrix} 0.312525 \\ 0.579859 \\ 0.351002 \\ 0.091589 \end{pmatrix},$$

Making use of Eqs. (4.8) and (4.10) yields:

$$\tilde{y}_{S,3,0} + 0.818182\tilde{y}_{S,3,1} + 4.11423\tilde{y}_{S,3,2} - 0.272727\tilde{y}_{S,3,3} - 1.6331 = 0, \quad (5.2)$$

$$\tilde{y}_{S,3,0} - 2.53846\tilde{y}_{S,3,1} - 3\tilde{y}_{S,3,2} - 34.1179\tilde{y}_{S,3,3} + 3.34821 = 0. \quad (5.3)$$

And the initial conditions will be

$$\begin{aligned} Y^T \phi(0) &= \tilde{y}_{S,3,0} - \tilde{y}_{S,3,1} + \tilde{y}_{S,3,2} - \tilde{y}_{S,3,3} = 0, \\ Y^T D^{(1)}\phi(0) &= 2\tilde{y}_{S,3,1} - 6\tilde{y}_{S,3,2} + 12\tilde{y}_{S,3,3} = 0. \end{aligned} \quad (5.4)$$

Finally, by solving Eqs. ((5.2)- (5.4)) we obtain

$$\tilde{y}_{S,3,0} = 0.25, \quad \tilde{y}_{S,3,1} = 0.45, \quad \tilde{y}_{S,3,2} = 0.25, \quad \tilde{y}_{S,3,3} = 0.05.$$

The solution will be

$$y(x) = \sum_{j=0}^3 \tilde{y}_{S,3,j} C_{S,N,j}^{(\alpha)}(x) \simeq x^3.$$

Executing this problem takes 4.79 second using Mathematica 9 software on CPU Intel(R) Core(TM) i3 at $\nu = 1$, $N = 3$. Unfortunately, to the best of our knowledge, there is no available data in literature to compare this running time with other authors. For this purpose, we developed a mathematica code to run the same problem using the method shifted Jacobi operational matrix(SJOM) [19] on our computer, the run time found to be 4.73 second which is less than our method. The approximate and exact solutions of Problem (1) are displayed in Figure 1 while the absolute errors between the exact and approximate solution at N=3 are listed in Table 1.

Figure 1: Exact solution vs. proposed method, for Problem (1).

x	Absolute error
0	2.77556×10^{-17}
0.1	1.80476×10^{-15}
0.2	6.53817×10^{-15}
0.3	1.30763×10^{-14}
0.4	2.02754×10^{-14}
0.5	2.7145×10^{-14}
0.6	3.26128×10^{-14}
0.7	3.54161×10^{-14}
0.8	3.4639×10^{-14}
0.9	2.91989×10^{-14}
1	1.77636×10^{-14}

Table 1: Absolute error of proposed method for Problem (1).

Problem (2)

Find the solution of the following initial value problem,

$$D^2y(x) - 2Dy(x) + D^{\frac{1}{2}}y(x) + y(x) = x^3 - 6x^2 + 6x + \frac{16}{5\sqrt{\pi}}x^{2.5}, \quad y(0) = 0, \quad \dot{y}(0) = 0, \quad (5.5)$$

where the exact solution of this problem is $y(x) = x^3$ [19].

By using our proposed technique with $N = 3$, we obtain

$$\tilde{y}_{S,3,0} = 0.25, \quad \tilde{y}_{S,3,1} = 0.45, \quad \tilde{y}_{S,3,2} = 0.25, \quad \tilde{y}_{S,3,3} = 0.05.$$

Then the solution will be written as

$$y(x) = \sum_{j=0}^3 \tilde{y}_{S,3,j} C_{S,N,j}^{(\alpha)}(x) \simeq x^3.$$

Executing this problem takes 2.26 second using Mathematica 9 software on CPU Intel(R) Core(TM) i3 at $\nu = 1$, $N = 3$. Unfortunately, to the best of our knowledge, there is no available data in literature to compare this running time with other authors. For this purpose, we developed a mathematica code to run the same problem using the method shifted Jacobi operational matrix(SJOM) [19] on our computer, the run time found to be 2.18 second which is less than our method. Figure 2 show the approximate and exact solutions of Problem (2). The absolute errors between the exact and approximate solution at $N=3$ are tabulated in Table 2.

Figure 2: Exact solution vs. proposed method, for Problem (2).

x	Absolute error
0	0
0.1	1.33511×10^{-14}
0.2	5.09853×10^{-14}
0.3	1.09118×10^{-13}
0.4	1.8395×10^{-13}
0.5	2.71838×10^{-13}
0.6	3.6901×10^{-13}
0.7	4.71567×10^{-13}
0.8	5.75984×10^{-13}
0.9	6.78457×10^{-13}
1	7.75158×10^{-13}

Table 2: Absolute error of proposed method for Problem (2).

Problem (3) Find the solution of the following initial value problem,

$$D^2 y(x) + D^{\frac{3}{2}} y(x) + y(x) = 1 + x, \quad 0 < \nu < 2, \quad y(0) = 1, \quad \dot{y}(0) = 1, \quad (5.6)$$

where the exact solution of this problem is $y(x) = 1 + x$ [29]. Applying the mechanism described in section(4) with $N = 2$, we get

$$\tilde{y}_{S,2,0} = 1.5, \quad \tilde{y}_{S,2,1} = 0.5, \quad \tilde{y}_{S,2,2} = 5.62182 \times 10^{-16}.$$

So, the solution is

$$y(x) = \sum_{j=0}^2 \tilde{y}_{S,2,j} C_{S,N,j}^{(\alpha)}(x) \simeq 1 + x.$$

The executing of this problem takes 3.18 second using Mathematica 9 software on CPU Intel(R) Core(TM) i3 at $\nu = 1$, $N = 2$. Figure (3), displays a comparison between the approximate solution and exact solution. Table (3) illustrates the

Figure 3: Exact solution with the proposed method, for Problem (3).

absolute errors of the derived mechanism. The results show the efficiency and accuracy of the proposed method. **Problem (4)** Find the solution of the following linear initial value problem,

$$D^\nu y(x) + y(x) = 0, \quad 0 < \nu < 2, \quad y(0) = 1, \quad \dot{y}(0) = 0. \quad (5.7)$$

The second initial condition is for $\nu > 1$ only. The exact solution of this problem is [29]

$$y(x) = \sum_{k=0}^{\infty} \frac{(-x^\nu)^k}{\Gamma(\nu k + 1)}.$$

x	Absolute error
0	2.22045×10^{-16}
0.1	2.22045×10^{-16}
0.2	2.22045×10^{-16}
0.3	8.88178×10^{-16}
0.4	1.33227×10^{-15}
0.5	2.44249×10^{-15}
0.6	3.33067×10^{-15}
0.7	4.66294×10^{-15}
0.8	5.77316×10^{-15}
0.9	7.32747×10^{-15}
1	9.10383×10^{-15}

Table 3: Absolute error of $y(x)$ of Problem (3).

In this example, we studied two cases: **(a)** $0 < \nu < 1$ Using our proposed technique with $N = 5$, we obtain $\tilde{y}_{S,5,0} = 0.632121$, $\tilde{y}_{S,5,1} = -0.310915$, $\tilde{y}_{S,5,2} = 0.0514531$, $\tilde{y}_{S,5,3} = -0.00512503$, $\tilde{y}_{S,5,4} = 0.000365151$, $\tilde{y}_{S,5,5} = -0.0000202862$. The solution is

$$y(x) = \sum_{j=0}^5 \tilde{y}_{S,5,j} C_{S,N,j}^{(\alpha)}(x) \simeq \exp(-x).$$

The performance of the numerical solutions of this problem with distinct values of $\nu = 0.5, 0.75, 0.85, 0.95, 1$ and $N = 5$ are given in Fig. (4a). It's clear that, the solution obtained using the derived mechanism is in good harmony with the exact solution $y(x) = \exp(-x)$. Which illustrates the validity and accuracy of the proposed mechanism. The absolute error for $\nu = 1$ and $N=5$ are shown in table (4). Table (5) illustrate absolute error comparison of $y(x)$ for $\nu_1 = 0.85$ and different values of N ($N = 2, 5$) between our mechanism and the method mentioned in [29]. The executing of this problem takes 4.26 second using Mathematica 9 software on CPU Intel(R) Core(TM) i3 at $\nu = 1$, $N = 5$.

(b) $1 < \nu < 2$ By using our proposed technique with $N = 5$, $\tilde{y}_{S,5,0} = 0.84147$, $\tilde{y}_{S,5,1} = -0.233773$, $\tilde{y}_{S,5,2} = -0.0718349$, $\tilde{y}_{S,5,3} = 0.00394003$, $\tilde{y}_{S,5,4} = 0.00051646$, $\tilde{y}_{S,5,5} = -0.0000157022$. The solution is

$$y(x) = \sum_{j=0}^5 \tilde{y}_{S,5,j} C_{S,N,j}^{(\alpha)}(x) \simeq \cos(x).$$

The numerical solutions for $y(x)$ for $N = 5$ and $\nu = 1.5, 1.75, 1.85, 1.95, 2$ are plotted in Fig.(4b). The executing of this problem takes 0.95 second using Mathematica 9 software on CPU Intel(R) Core(TM) i3 at $\nu = 1$, $N = 5$.

Fig.(4a) The behavior of approximate solution for $N=5$, $\nu=0.5, 0.75, 0.85, 0.95, 1$, and the exact solution for Problem(4)

Fig.(4b) The behavior of approximate solution for $N=5$, $\nu=1.5, 1.75, 1.85, 1.95, 2$, and the exact solution for Problem(4)

$\nu = 1$				$\nu = 2$		
x	y exact	y approx.	Absolute error	y exact	y approx.	Absolute error
0	1	1	0	1	1	0
0.1	0.904837	0.904839	1.1893×10^{-6}	0.995004	0.995002	1.84354×10^{-6}
0.2	0.818731	0.818731	5.80334×10^{-7}	0.980067	0.980066	1.0705×10^{-6}
0.3	0.740818	0.740819	4.24608×10^{-7}	0.955336	0.955336	8.89051×10^{-7}
0.4	0.67032	0.670321	7.13037×10^{-7}	0.921061	0.92106	1.3405×10^{-6}
0.5	0.606531	0.606532	8.45652×10^{-7}	0.877583	0.877581	1.57876×10^{-6}
0.6	0.548812	0.548812	5.58632×10^{-7}	0.825336	0.825334	1.19684×10^{-6}
0.7	0.496585	0.496585	1.79343×10^{-7}	0.764842	0.764842	6.34569×10^{-7}
0.8	0.449329	0.449329	2.73325×10^{-7}	0.696707	0.696706	7.10935×10^{-7}
0.9	0.40657	0.40657	7.41096×10^{-7}	0.62161	0.621609	1.33314×10^{-6}
1	0.367879	0.367879	5.82876×10^{-7}	0.540302	0.540303	5.57251×10^{-7}

Table 4: Absolute error of $y(x)$ for $\nu = 1$ and $\nu = 2$ for Problem (4).

$N = 2$			$N = 5$	
x	LOM [29]	Present method	LOM [29]	Present method
0.1	2.6×10^{-2}	3.6×10^{-2}	2×10^{-3}	8×10^{-3}
0.2	2×10^{-2}	4×10^{-2}	3×10^{-3}	5.6×10^{-3}
0.3	8.2×10^{-3}	3.5×10^{-2}	6.2×10^{-4}	5.5×10^{-3}
0.4	4.8×10^{-3}	2.7×10^{-2}	2.9×10^{-3}	5.7×10^{-3}
0.5	1.5×10^{-2}	1.9×10^{-2}	2×10^{-3}	5.3×10^{-3}
0.6	2.2×10^{-2}	1.3×10^{-2}	7.2×10^{-4}	4.2×10^{-3}
0.7	2.4×10^{-2}	1×10^{-2}	2.5×10^{-3}	3.5×10^{-3}
0.8	1.9×10^{-2}	1.2×10^{-2}	1.3×10^{-3}	3.6×10^{-3}
0.9	8.3×10^{-3}	1.9×10^{-2}	1.5×10^{-3}	4×10^{-3}
1	1.1×10^{-2}	3.2×10^{-2}	5.5×10^{-4}	1.7×10^{-3}

Table 5: Absolute error comparison of $y(x)$ for $\nu = 0.85$ and different values of $N = 2, 5$ for Problem (4).

Problem (5) Find the solution of the following system of two linear FDEs [30]

$$\begin{aligned} D^{\nu_1} y_1(x) &= y_1(x) + y_2(x), \\ D^{\nu_2} y_2(x) &= -y_1(x) + y_2(x), \quad 0 < \nu_1, \nu_2 \leq 1, \end{aligned} \quad (5.8)$$

subject to the initial conditions $y_1(0) = 0$; $y_2(0) = 1$. The exact solution, when $\nu_1 = \nu_2 = 1$, is

$$\begin{aligned} y_1(x) &= e^x \sin(x), \\ y_2(x) &= e^x \cos(x). \end{aligned} \quad (5.9)$$

Using our proposed technique with $N = 5$; we obtain $\tilde{y}_{S,5,1,0} = 0.909316, \tilde{y}_{S,5,1,1} = 1.13408, \tilde{y}_{S,5,1,2} = 0.236329, \tilde{y}_{S,5,1,3} = 0.0099024, \tilde{y}_{S,5,1,4} = -0.00195987, \tilde{y}_{S,5,1,5} = -0.000297858,$
 $\tilde{y}_{S,5,2,0} = 1.37805, \tilde{y}_{S,5,2,1} = 0.27203, \tilde{y}_{S,5,2,2} = -0.140283, \tilde{y}_{S,5,2,3} = -0.0375811, \tilde{y}_{S,5,2,4} = -0.00340158, \tilde{y}_{S,5,2,5} = -0.0000800948.$
The solution will be

$$\begin{aligned} y_1(x) &= \sum_{j=0}^5 \tilde{y}_{S,5,1,j} C_{S,5,j}^{(\alpha)}(t) = e^x \sin(x), \\ y_2(x) &= \sum_{j=0}^5 \tilde{y}_{S,5,2,j} C_{S,5,j}^{(\alpha)}(t) = e^x \cos(x). \end{aligned} \quad (5.10)$$

Executing this problem takes 39.58 second using Mathematica 9 software on CPU Intel(R) Core(TM) i3 at $\nu = 1, N = 5$.

x	y_1 exact	y_1 approx.	Absolute error of y_1	y_2 exact	y_2 approx.	Absolute Error of y_2
0	0	3.81639×10^{-17}	3.25261×10^{-17}	1	1	0
0.1	0.110333	0.110308	2.4593×10^{-5}	1.09965	1.09967	1.67981×10^{-5}
0.2	0.242655	0.24264	1.53239×10^{-5}	1.19706	1.19707	1.48418×10^{-5}
0.3	0.398911	0.398897	1.30656×10^{-5}	1.28957	1.28959	1.64749×10^{-5}
0.4	0.580944	0.580925	1.90749×10^{-5}	1.37406	1.37409	2.37344×10^{-5}
0.5	0.790439	0.790417	2.24392×10^{-5}	1.44689	1.44692	3.09268×10^{-5}
0.6	1.02885	1.02883	1.72238×10^{-5}	1.50386	1.50389	3.39614×10^{-5}
0.7	1.2973	1.29729	8.64891×10^{-6}	1.5402	1.54024	3.49808×10^{-5}
0.8	1.59651	1.5965	7.72434×10^{-6}	1.55055	1.55059	4.0611×10^{-5}
0.9	1.92667	1.92666	1.3894×10^{-5}	1.52891	1.52897	5.19197×10^{-5}
1	2.28736	2.28737	1.4589×10^{-5}	1.46869	1.46874	4.39353×10^{-5}

Table 6: Absolute error of y_1 and y_2 for $N=5$ of Problem (5).

Problem (6): HIV Model Find the solution of the following non- linear system of FDEs [31]

$$\begin{aligned}
D^{\nu_1} T(x) &= q - \eta T(x) + r T(x) \left(1 - \frac{T(x) + I(x)}{T_{max}}\right) - k V(x) T(x), \\
D^{\nu_2} I(x) &= k V(x) T(x) - \beta I(x), \\
D^{\nu_3} V(x) &= \mu \beta I(x) - \gamma V(x), \quad 0 < \nu_1, \nu_2, \nu_3 \leq 1,
\end{aligned} \tag{5.11}$$

where $q = 0.1$; $\eta = 0.02$; $\beta = 0.3$; $r = 3$; $\gamma = 2.4$; $k = 0.0027$; $T_{max} = 1500$; $\mu = 10$; and with the initial conditions $T(0) = 0.1$; $I(0) = 0$; $V(0) = 0.1$.

By using the proposed technique with $N = 8$, and with the approximations: $T = \sum_{k=0}^8 \tilde{t}_{S,8,k} C_{S,8,k}^{(\alpha)}(x)$, $I = \sum_{k=0}^8 \tilde{i}_{S,8,k} C_{S,8,k}^{(\alpha)}(x)$, and $V = \sum_{k=0}^8 \tilde{v}_{S,8,k} C_{S,8,k}^{(\alpha)}(x)$, we obtain the following results

$$\begin{aligned}
\tilde{t}_{S,8,0} &= 0.803299, \quad \tilde{t}_{S,8,1} = 1.09378, \quad \tilde{t}_{S,8,2} = 0.511342, \quad \tilde{t}_{S,8,3} = 0.147005, \quad \tilde{t}_{S,8,4} = 0.0304938, \quad \tilde{t}_{S,8,5} = 0.00493337, \\
\tilde{t}_{S,8,6} &= 0.000650624, \quad \tilde{t}_{S,8,7} = 0.0000716641, \quad \tilde{t}_{S,8,8} = 6.66186 \times 10^{-6},
\end{aligned}$$

$$\begin{aligned}
\tilde{i}_{S,8,0} &= 0.0000180401, \quad \tilde{i}_{S,8,1} = 0.0000200831, \quad \tilde{i}_{S,8,2} = 1.94644 \times 10^{-6}, \quad \tilde{i}_{S,8,3} = -6.09932 \times 10^{-8}, \quad \tilde{i}_{S,8,4} = 3.18855 \times 10^{-8}, \\
\tilde{i}_{S,8,5} &= -3.2592 \times 10^{-9}, \quad \tilde{i}_{S,8,6} = 4.37941 \times 10^{-10}, \quad \tilde{i}_{S,8,7} = -3.0595 \times 10^{-11}, \quad \tilde{i}_{S,8,8} = 3.47443 \times 10^{-12},
\end{aligned}$$

$$\begin{aligned}
\tilde{v}_{S,8,0} &= 0.0378972, \quad \tilde{v}_{S,8,1} = -0.041608, \quad \tilde{v}_{S,8,2} = 0.0160092, \quad \tilde{v}_{S,8,3} = -0.00375697, \quad \tilde{v}_{S,8,4} = 0.000634897, \quad \tilde{v}_{S,8,5} = \\
&= -0.0000838099, \quad \tilde{v}_{S,8,6} = 9.07582 \times 10^{-6}, \quad \tilde{v}_{S,8,7} = -8.33014 \times 10^{-7}, \quad \tilde{v}_{S,8,8} = 6.66462 \times 10^{-8}.
\end{aligned}$$

x	MVIM [32]	VIM [32]	Bessel collocation [31]	Runge- Kutta [31]	Present method
0	0.1	0.1	0.1	0.1	0.1
0.2	0.2088080868	0.2088073214	0.2038616561	0.2088080833	0.208808304834729
0.4	0.4062407949	0.4061346587	0.3803309335	0.4062405393	0.406239718020324
0.6	0.7644287245	0.7624530350	0.6954623767	0.7644238890	0.764421068670511
0.8	1.4140941730	1.3978805880	1.2759624442	1.4140468310	1.414040883089124
1	0.2088080868	2.5067466690	2.3832277428	2.5915948020	2.591582119960000

Table 7: Numerical comparison of $T(x)$ for N=8 of problem (6).

x	MVIM [32]	VIM [32]	Bessel collocation [31]	Runge- Kutta [31]	Present method
0	$0.1e - 13$	0	0	0	$0.00000049910e - 14$
0.2	$0.60327016510e - 5$	$0.6032634366e - 5$	$0.6247872100e - 5$	$0.6032702150e - 5$	$0.6032719255e - 5$
0.4	$0.13158301670e - 4$	$0.1314878543e - 4$	$0.1293552225e - 4$	$0.1315834073e - 4$	$0.13158323604e - 4$
0.6	$0.21223310013e - 4$	$0.2101417193e - 4$	$0.2035267183e - 4$	$0.2122378506e - 4$	$0.21223733218e - 4$
0.8	$0.30174509323e - 4$	$0.2795130456e - 4$	$0.2837302120e - 4$	$0.3017741955e - 4$	$0.30177329588e - 4$
1	$0.40025404050e - 4$	$0.2431562317e - 4$	$0.3690842367e - 4$	$0.4003781468e - 4$	$0.40037683920e - 4$

Table 8: Numerical comparison of $I(x)$ for N=8 of problem (6).

x	MVIM [32]	VIM [32]	Bessel collocation [31]	Runge- Kutta [31]	Present method
0	0.1	0.1	0.1	0.1	0.1
0.2	0.06187990876	0.06187995314	0.06187991856	0.06187984331	0.061879872873767
0.4	0.03829595768	0.03830820126	0.03829493490	0.03829488788	0.038294891290728
0.6	0.02371029480	0.02392029257	0.02370431860	0.02370455014	0.023704537057040
0.8	0.01470041902	0.01621704553	0.01467956982	0.01468036377	0.014680339302072
1	0.009157238735	0.01608418711	0.02370431861	0.009100845043	0.009100826552200

Table 9: Numerical comparison of $V(x)$ for N=8 of problem (6).

Figure (6a): The approximate solution of $T(x)$ at $N=8$ for HIV model.

Figure (6b): The approximate solution of $I(x)$ at $N=8$ for HIV model

Figure (6c): The approximate solution of $V(x)$ at $N=8$ for HIV model.

The approximate solutions of $T(x)$, $I(x)$ and $V(x)$ for $\nu_1 = \nu_2 = \nu_3 = 1$ and $N = 8$ are given in Tables (7- 9) and compared with the solutions given by VIM [32], MVIM [32], Ring-kutta [31] and Bessel collocation method [31]. These numerical results are an evident for the efficiency of the proposed technique. Figures (6a-6c) display the approximate solutions of $T(x)$, $I(x)$ and $V(x)$ for different values of ν_1 , ν_2 and ν_3 . The executing of this problem takes 871.45 second using Mathematica 9 software on CPU Intel(R) Core(TM) i3 at $\nu = 1$, $N = 8$.

6 Conclusions

A general formulation of the SGOM of RL fractional integral has been developed. This method has been used to approximate the solutions of a set of MOFDEs and systems of FDEs. The proposed mechanism has been depended on the shifted Gegenbauer polynomials and the Tau method. The applicability, accuracy and rapidity by using few terms of the shifted Gegenbauer polynomials of the proposed mechanism are illustrated by numerical problems.

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