PERMUTATION GROUPS AND PERIODICITY OF SYSTEMS OF DIFFERENCE EQUATIONS

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Abstract

Let $k \in \mathbb{N}$, $\mathbb{Z}_k = \{1, 2, \dots, k\}$ and S_k be the group of all permutations on $\mathbb{Z}_k$. Let $\pi \in S_k$ be of order l and f_i be a function from a nonempty set X into itself, $i = 1, \dots, k$. In this paper, we show that a sufficient condition for a system of difference equations

$$x_{n+1}^{(1)} = f_1(x_{n-s}^{(\pi(1))}), \quad x_{n+1}^{(2)} = f_2(x_{n-s}^{(\pi(2))}), \dots, x_{n+1}^{(k)} = f_k(x_{n-s}^{(\pi(k))}), \quad n \in \mathbb{Z}^{\geq 0},$$

to be periodic with a period d is that each difference equation

$$y_{n+l(s+1)-s} = g_i(y_{n-s}), \quad n \in \mathbb{Z}^{\geq 0},$$

is periodic, $i = 1, \dots, k$, with a period d_i that divides d . Here, g_i is defined by

$$g_i = f_i f_{\pi(i)} \dots f_{\pi^{l-1}(i)}, \quad i = 1, \dots, k.$$

Finally, the periodicity of many systems of rational difference equations is established.

Keywords: Systems of difference equations, permutation groups, periodicity.

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1 Introduction and preliminaries

The theory of permutation groups is important to diverse area of mathematics such as Galois theory, invariant theory, the representation theory of Lie groups, and combinatorics. See for instance [1]. In general, the theory of abstract groups plays an important part in present day mathematics and science. Groups arise in a bewildering number of apparently unconnected subjects. Thus they appear in algebra and analysis, in geometry and topology, in crystallography and quantum mechanics, in physics and chemistry, and even in biology. See [2], [3]. In this paper, as another application of the theory of permutation groups, we investigate the periodicity of systems of difference equations. Recently, there has been a great interest in difference equations, because they describe naturally many real-life problems in biology, ecology, genetics, psychology, sociology, and so forth. The authors in [4] investigated the periodicity of the system

$$x_{n+1}^{(1)} = \frac{x_n^{(2)}}{x_n^{(2)} - 1}, \quad x_{n+1}^{(2)} = \frac{x_n^{(3)}}{x_n^{(3)} - 1}, \dots, x_{n+1}^{(k)} = \frac{x_n^{(1)}}{x_n^{(1)} - 1}, \quad n \in \mathbb{Z}^{\geq 0}, \quad (1.1)$$

as well as the periodicity of the system

$$x_{n+1}^{(1)} = \frac{x_n^{(k)}}{x_n^{(k)} - 1}, \quad x_{n+1}^{(2)} = \frac{x_n^{(1)}}{x_n^{(1)} - 1}, \dots, x_{n+1}^{(k)} = \frac{x_n^{(k-1)}}{x_n^{(k-1)} - 1}, n \in \mathbb{Z}^{\geq 0}. \quad (1.2)$$

For some related results about periodicity, we refer the reader to the interesting papers [5-9]. Motivated by systems (1.1) and (1.2), we investigate the periodic character of the general system

$$x_{n+1}^{(1)} = f_1(x_{n-s}^{(\pi(1))}), \quad x_{n+1}^{(2)} = f_2(x_{n-s}^{(\pi(2))}), \dots, x_{n+1}^{(k)} = f_k(x_{n-s}^{(\pi(k))}). \quad (1.3)$$

Here, $\pi \in S_k$ and f_i is a certain function from a non-empty set into itself, $i = 1, \dots, k$. At the end of this paper, the periodicity of many systems of rational difference equations is investigated. It is well-known that for any permutation $\pi \in S_k$, there is a natural number l such that the property $\pi^l = I$ holds, where I is the identity permutation and π^l is the composition of π with itself l -times. The smallest l for which this property holds is called the order of π .

Definition 1.1. A system (1.3) is called periodic with period d if every solution $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})$ satisfies

$$x_{n+d}^{(i)} = x_n^{(i)}, n \in \mathbb{Z}^{\geq 0}, i \in \mathbb{Z}_k.$$

Definition 1.2. A function $f : X \rightarrow X$ is called involution (resp. idempotent) if $f^2 = I$ (resp. $f^2 = f$). Here, I is the identity function.

2 Main results

Let $\pi \in S_k$ be a permutation of order l , X be any non-empty set and $f_i : X \rightarrow X, i = 1, \dots, k$. In this section we investigate the periodicity of the system

$$x_{n+1}^{(1)} = f_1(x_{n-s}^{(\pi(1))}), \quad x_{n+1}^{(2)} = f_2(x_{n-s}^{(\pi(2))}), \dots, x_{n+1}^{(k)} = f_k(x_{n-s}^{(\pi(k))}) \quad (2.1)$$

in terms of the periodicity of each of the difference equations

$$y_{n+l(s+1)-s} = g_i(y_{n-s}), n \geq 0 \quad (2.2.i)$$

where g_i is defined by

$$g_i = f_i f_{\pi(i)} \dots f_{\pi^{l-1}(i)}, i = 1, \dots, k. \quad (2.2)$$

We need the following lemma to prove our main result.

Lemma 2.1. (i) If a k -tuple $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})$ is a solution of system (2.1), then it satisfies the following equations

$$x_{n+l(s+1)-s}^{(i)} = f_i f_{\pi(i)} \dots f_{\pi^r(i)} (x_{n+l(s+1)-s-(r+1)(s+1)}^{(\pi^{r+1}(i))}), i \in \mathbb{Z}_k, \quad (2.3)$$

for every $r \in \mathbb{Z}^{\geq 0}$.

(ii) Assume that l is even. If a k -tuple $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})$ is a solution of system (2.1), then it satisfies the following equations

$$x_{n+\frac{l}{2}(s+1)-s}^{(i)} = f_i f_{\pi(i)} \dots f_{\pi^r(i)} (x_{n+\frac{l}{2}(s+1)-s-(r+1)(s+1)}^{(\pi^{r+2}(i))}), i \in \mathbb{Z}_k, \quad (2.4)$$

for every $r \in \mathbb{Z}^{\geq 0}$.

Proof. (i) Relation (2.3) is true at $r = 0$. Assume that relation (2.3) is true for a fixed r . We have

$$\begin{aligned} x_{n+l(s+1)-s}^{(i)} &= f_i f_{\pi(i)} \dots f_{\pi^r(i)} (x_{n+l(s+1)-s-(r+1)(s+1)}^{(\pi^{r+1}(i))}) \\ &= f_i f_{\pi(i)} \dots f_{\pi^r(i)} f_{\pi^{r+1}(i)} (x_{n+l(s+1)-s-(r+1)(s+1)-(s+1)}^{(\pi^{r+2}(i))}) \\ &= f_i f_{\pi(i)} \dots f_{\pi^{r+1}(i)} (x_{n+l(s+1)-s-(r+2)(s+1)}^{(\pi^{r+2}(i))}), n \geq 0. \end{aligned}$$

(ii) can be shown similarly. □

As a direct consequence, putting $r = l - 1$, we get the following result

Corollary 2.2. •

(i) If a k -tuple $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})$ is a solution of system (2.1), then x_n^i satisfies the equation

$$y_{n+l(s+1)-s} = g_i(y_{n-s}), n \geq 0, \quad (2.5.i)$$

for each i .

(ii) Assume that l is even. If a k -tuple $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})$ is a solution of system (2.1), then x_n^i satisfies the equation

$$y_{n+\frac{l}{2}(s+1)-s} = g_i(y_{n-\frac{l}{2}(s+1)-s}), n \geq 0, \quad (2.6.i)$$

for each i .

To arrive at our main result, we need to check the following Lemma

Lemma 2.3. Let $g : X \rightarrow X$.

(i) A solution of the equation

$$y_{n+l(s+1)-s} = g(y_{n-s}), n \in \mathbb{Z}^{\geq 0} \quad (2.5)$$

is given by

$$y_{(m+1)(s+1)l+r-s} = g^{m+1}(y_{r-s}), r = 0, \dots, (s+1)l, m \in \mathbb{Z}^{\geq 0}. \quad (2.6)$$

(ii) Assume that l is even. A solution of the equation

$$y_{n+\frac{l}{2}(s+1)-s} = g(y_{n-\frac{l}{2}(s+1)-s}), n \in \mathbb{Z}^{\geq 0} \quad (2.7)$$

is given by

$$y_{m(s+1)\frac{l}{2}+r-s} = \begin{cases} g^{\frac{m}{2}}(y_{r-s}), & m \text{ upiseven}, \\ g^{\frac{m+1}{2}}(y_{r-s-(s+1)\frac{l}{2}}), & m \text{ upisodd}. \end{cases} \quad (2.8)$$

Proof. (i) Relation (2.6) is true when $m = 0$. Assume it is true for a fixed m . Simple calculations show that

$$\begin{aligned} y_{(m+2)(s+1)l+r-s} &= g(y_{(m+2)(s+1)l+r-s-(s+1)l}) \\ &= g(y_{(m+1)(s+1)l+r-s}) \\ &= g(g^{m+1}(y_{r-s})) \\ &= g^{m+2}(y_{r-s}). \end{aligned}$$

(ii) Relation (2.8) is true at $m = 0$ and $m = 1$. Let m be a fixed natural number and the relation is true for every natural number less than m . For the case where m is even, we have

$$\begin{aligned} y_{m(s+1)\frac{l}{2}+r-s} &= g(y_{m(s+1)\frac{l}{2}+r-s-(s+1)l}) \\ &= g(y_{(m-2)(s+1)\frac{l}{2}+r-s}) \\ &= g(g^{\frac{m-2}{2}}(y_{r-s})) \\ &= g^{\frac{m}{2}}(y_{r-s}). \end{aligned}$$

That is relation (2.8) is true for m , when m is even. The case where m is odd can be treated similarly. □

Now, we are ready to prove our main results.

Theorem 2.4. *A sufficient condition for the system (2.1) to be periodic with a period d is that each difference equation (2.5.i) (resp. (2.6.i) when l is even) is periodic, with a period d_i such that $d = \text{l.c.m.}(d_1, \dots, d_k)$, the least common multiple of $d_1, \dots, d_k$.*

Proof. Let $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})$ be a solution of system (2.1). Then $x_n^{(i)}$ is a solution of (2.5.i) (resp. (2.6.i) when l is even). The periodicity of $x_n^{(i)}$ with period $d_i, i = 1, \dots, k$ imply the periodicity of system (2.1) with period $d = \text{l.c.m.}(d_1, \dots, d_k)$, the least common multiple of $d_1, \dots, d_k$. $\square$

Theorem 2.5. *If f_i is an involution on X for each i such that $f_i f_j = f_j f_i, i, j \in \mathbb{Z}_k$, then system (2.1) is periodic with period $2l(s+1)$.*

Proof. Let $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})$ be a solution of system (2.1). Then $x_n^{(i)}$ is a solution of equation (2.5.i) for each i . By Lemma 2.3, this solution is given by

$$y_{(m+1)(s+1)l+r-s} = g_i^{m+1}(y_{r-s}), r = 0, \dots, (s+1)l, m \in \mathbb{Z}^{\geq 0}.$$

The assumptions of the theorem imply that g_i^2 is the identity function for each i . We deduce that each equation (2.5.i) is periodic with period $2l(s+1)$. Indeed, we conclude that

$$\begin{aligned} x_{(m+3)(s+1)l+r-s}^{(i)} &= g_i^{m+3}(x_{r-s}^{(i)}) \\ &= g_i^{m+1}(x_{r-s}^{(i)}) \\ &= x_{(m+1)(s+1)l+r-s}^{(i)} \end{aligned}$$

$\square$

Theorem 2.6. *If $f_1 = \dots = f_k = f$ is an involution on X and l is even, then system (2.1) is periodic with period $l(s+1)$.*

Proof. The hypothesis implies that $g_i = f^l = I, i \in \mathbb{Z}_k$ and equation (2.5.i) is periodic with period $l(s+1)$ for each i . $\square$

Theorem 2.7. *Assume that f_i is an idempotent on X for each i such that $f_i f_j = f_j f_i, i, j \in \mathbb{Z}_k$. Then system (2.1) is periodic with period $l(s+1)$.*

Proof. Let $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(k)})$ be a solution of system (2.1). Then $x_n^{(i)}$ is a solution of equation (2.5.i) for each i . The assumptions of the theorem imply that $g_i^2 = g_i$ for each i . By lemma 2.3, we have

$$\begin{aligned} x_{(m+2)(s+1)l+r-s}^{(i)} &= g_i^{m+2}(x_{r-s}^{(i)}) \\ &= g_i^{m+1}(x_{r-s}^{(i)}) \\ &= x_{(m+1)(s+1)l+r-s}^{(i)}. \end{aligned}$$

Consequently, system (2.1) is periodic with period $l(s+1)$. $\square$

Theorem 2.8. *Let α and a be complex numbers. The system*

$$x_{n+1}^{(1)} = \frac{x_{n-s}^{(\pi(1))} + a}{\alpha x_{n-s}^{(\pi(1))} - 1}, x_{n+1}^{(2)} = \frac{x_{n-s}^{(\pi(2))} + a}{\alpha x_{n-s}^{(\pi(2))} - 1}, \dots, x_{n+1}^{(k)} = \frac{x_{n-s}^{(\pi(k))} + a}{\alpha x_{n-s}^{(\pi(k))} - 1}, \quad (2.9)$$

is periodic with period $l(s+1)$.

Proof. Set $f_i(x) = \frac{x+a}{\alpha x-1}, x \in \mathbb{C} \setminus \{\frac{1}{\alpha}\}, i \in \mathbb{Z}_k$. We can see that $f_i : \mathbb{C} \setminus \{\frac{1}{\alpha}\} \rightarrow \mathbb{C} \setminus \{\frac{1}{\alpha}\}$ is an involution for every i . By theorem 2.6, we insure that system (2.9) is periodic with period $l(s+1)$. $\square$

Theorem 2.9. *The coupled system*

$$x_{n+1} = \frac{y_{n-s} + a}{\alpha_1 y_{n-s} - 1}, \quad y_{n+1} = \frac{x_{n-s} + a}{\alpha_2 x_{n-s} - 1}, \quad n \in \mathbb{Z}^{\geq 0}, \quad (2.10)$$

where $\alpha_2 = -(\frac{2}{a} + \alpha_1)$ and $a \neq 0$, is periodic with period $4(s+1)$.

Proof. One can see that $f_i(x) = \frac{x+a}{\alpha_i x - 1}$ is a function from $\mathbb{C} \setminus \{\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, -\frac{a}{2}\}$ onto itself, $f_1 f_2 = f_2 f_1 = \frac{-xa}{2x+a}$ and f_i is involution for every i . Applying theorem 2.5, we arrive at the periodicity of the system with period $4(s+1)$. $\square$

3 Illustrative examples

(i) The system

$$x_{n+1} = \frac{y_{n-s} + a}{\alpha y_{n-s} - 1}, y_{n+1} = \frac{z_{n-s} + a}{\alpha z_{n-s} - 1}, z_{n+1} = \frac{x_{n-s} + a}{\alpha x_{n-s} - 1}, \quad (3.1)$$

is periodic with period $6(s+1)$. Indeed, the permutation which corresponds to this system is $\pi = (1 \ 2 \ 3)$. So, its order is 3. This implies that the system is periodic with period $6(s+1)$.

(ii) The system

$$x_{n+1} = \frac{z_{n-s} + a}{\alpha z_{n-s} - 1}, y_{n+1} = \frac{x_{n-s} + a}{\alpha x_{n-s} - 1}, z_{n+1} = \frac{y_{n-s} + a}{\alpha y_{n-s} - 1}, \quad (3.2)$$

is periodic with period $6(s+1)$. The permutation which corresponds to this system is $\pi = (1 \ 3 \ 2)$. Again its order is 3. This implies that the system is periodic with period $6(s+1)$.

(iii) The system

$$x_{n+1} = \frac{z_{n-s} + a}{\alpha z_{n-s} - 1}, y_{n+1} = \frac{w_{n-s} + a}{\alpha w_{n-s} - 1}, z_{n+1} = \frac{y_{n-s} + a}{\alpha y_{n-s} - 1}, w_{n+1} = \frac{x_{n-s} + a}{\alpha x_{n-s} - 1} \quad (3.3)$$

is periodic with period $4(s+1)$. Indeed, The permutation which describes this system is $\pi = (1 \ 3 \ 2 \ 4)$ with order 4. Then the periodicity of the system is $4(s+1)$.

(iv) The system

$$x_{n+1} = \frac{y_{n-s} + a}{\alpha y_{n-s} - 1}, y_{n+1} = \frac{x_{n-s} + a}{\alpha x_{n-s} - 1}, z_{n+1} = \frac{u_{n-s} + a}{\alpha u_{n-s} - 1}, u_{n+1} = \frac{z_{n-s} + a}{\alpha z_{n-s} - 1} \quad (3.4)$$

is periodic with period $2(s+1)$. The permutation here is $\pi = (1 \ 2)(3 \ 4)$. So its order is 2 and thereby the system is periodic with period $2(s+1)$.

(v) The system

$$x_{n+1} = \frac{z_n + a}{\alpha z_n - 1}, y_{n+1} = \frac{x_n + a}{\alpha x_n - 1}, z_{n+1} = \frac{u_n + a}{\alpha u_n - 1}, u_{n+1} = \frac{y_n + a}{\alpha y_n - 1} \quad (3.5)$$

is periodic with period 4.

(vi) The following matrices

$$A_1 = \begin{pmatrix} -a & \frac{1-a^2}{b} \\ b & a \end{pmatrix}, b \neq 0$$

and

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are involution matrices. Therefore the system

$$x_{n+1}^{(1)} = Ax_{n-s}^{(\pi(1))}, x_{n+1}^{(2)} = Ax_{n-s}^{(\pi(2))}, \dots, x_{n+1}^{(k)} = Ax_{n-s}^{(\pi(k))}, \quad (3.6)$$

is periodic with period $l(s+1)$ (resp. $2l(s+1)$) when $l \equiv 0 \pmod{2}$ (resp. $l \equiv 1 \pmod{2}$), where π is a permutation of order l and $A \in \{A_1, A_2\}$.

(vii) The following matrices

$$A_1 = \frac{1}{2} \begin{pmatrix} 1 - \cos\theta & \sin\theta \\ \sin\theta & 1 + \cos\theta \end{pmatrix}, \theta \in \mathbb{R}$$

and

$$A_2 = \begin{pmatrix} a & 1-a & 0 \\ a & 1-a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are idempotent matrices. Therefore the system

$$x_{n+1}^{(1)} = Ax_{n-s}^{(\pi(1))}, x_{n+1}^{(2)} = Ax_{n-s}^{(\pi(2))}, \dots, x_{n+1}^{(k)} = Ax_{n-s}^{(\pi(k))}, \quad (3.7)$$

is periodic with period $l(s+1)$ (resp. $2l(s+1)$) when $l \equiv 0 \pmod{2}$ (resp. $l \equiv 1 \pmod{2}$), where π is a permutation of order l and $A \in \{A_1, A_2\}$.

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A NEW RESULT ON THE STABILITY OF A STOCHASTIC DIFFERENTIAL EQUATION OF THIRD-ORDER WITH A TIME-LAG

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Abstract

Here, stochastic asymptotic stability (SAS) of the zero solution of a stochastic delay differential equation (SDDE) of third order such as

$$\ddot{x}(t) + f(\dot{x}(t))\ddot{x}(t) + g(\dot{x}(t - \tau)) + h(x(t - \tau)) + \sigma x(t)\dot{\omega}(t) = 0$$

is discussed. To arrive at the aim of this paper, a suitable Lyapunov functional (LF) is defined and then used to find conditions that guaranteeing the (SAS) of solutions. We give an example to verify the analysis made in this paper.

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1 Introduction

A stochastic differential equation (SDE) involves a variable which indicates random white noise calculated as the derivative of Brownian motion or the Wiener process. However, other types of random behaviors are possible, for example, jump processes. There are two dominating versions of stochastic calculus (SC), the Itô (SC) and the Stratonovich (SC). Both of these (SC) have advantages and disadvantages (see, for example, [1]) and conveniently, one can readily convert an Itô (SDE) to an equivalent Stratonovich (SDE) and back again.

The theory of (SDEs) has attracted considerable attention of many scholars in the last years. Therefore, it has extensively been studied in the relevant literature and there exists a large number of books, which provides full details of the background of probability theory and (SC), for example, see the books of [1], [2], [3], [4], [5], [6] and the references of these sources.

Since then the number of contributions to statistics, numerics and control theory of (SDEs) have been rapidly increasing. It is well known that stochastic models (SMs) have important roles in science and industry, where many authors can encounter with (SDEs).

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Therefore, (SMs) can have many important roles in various scientific areas such as biology, economy, medicine, engineering and so on (see, [7], [8], [9]). By this time, numerous kind of (LFs) have been used as basic tools to study qualitative problems in many deterministic/(SDEs) and (DDEs).

In addition, (SDDEs) represent a relatively new field of the qualitative theory of (DEs) and (DDEs). The significance of (SDDEs) has become more evident in recent years due to a great variety of their applications in modeling real world phenomena. Unfortunately, it is in general not possible to give or obtain analytic expressions for solutions of (SDEs) and (SDDEs). Therefore, most of the papers are interested in to characterize the qualitatively behaviors of solutions (QBSs) without solving (SDEs) and (SDDEs). To the best of our knowledge, up to now in the relative literature, the Lyapunov's theory is the most powerful tool for qualitative analysis of (SDEs) and (SDDEs) of higher order without solving that equations.

Hence, many papers dealt with the (QBSs) of higher order (DDEs) with (a or multiple) time lag(s) and obtained many interesting results by means of this theory, see, for example, the papers given in [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21] and the references cited therein to mention few). In deed, the many authors focused on (QBSs) of different models of (SDEs) and they got numerous interesting results over the last few years (see, for example, [22], [23], [24], [25], [26], [27]) and the references therein.

With respect to our observation, up to this moment, the corresponding problem for the stability (S) and boundedness (B) of solutions of higher order (SDDEs) have been studied in the literature by only a few researches, see, for instance, [28], [29], [30], [31], [32] and [33]. Therefore, it is worth discussing the (QBSs) of (SDDEs). In 2015, the authors of [30] discussed the (AS) of the zero solution of (SDDEs) such as

$$\ddot{x}(t) + a_1\dot{x}(t) + g_1(\dot{x}(t - r_1(t))) + f_1(x(t)) + \sigma_1 x(t)\dot{\omega}(t) = 0$$

and

$$\ddot{x}(t) + a_2\dot{x}(t) + f_2(x(t))\dot{x}(t) + f_3(\dot{x}(t - r_2(t))) + \sigma x(t - h(t))\dot{\omega}(t) = 0.$$

In 2017, the author of [32] studied (S) and (B) of solutions of the (SDDEs) in the following form

$$\ddot{x}(t) + a\ddot{x}(t) + b\dot{x}(t) + h(x(t - \tau)) + \sigma x(t)\dot{\omega}(t) = p(t, x(t), \dot{x}(t), \ddot{x}(t)).$$

It should be noted that throughout the references of this papers, (LFs) have been used as a basic tool to investigate the (QBSs) mathematical modes therein.

Here, we take into consideration the (SDDE) of third order

$$\ddot{x}(t) + f(\dot{x}(t))\ddot{x}(t) + g(\dot{x}(t - \tau)) + h(x(t - \tau)) + \sigma x(t)\dot{\omega}(t) = 0. \quad (1.1)$$

(SDDE) (1.1) can be written as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -f(y)z - g(y) - h(x) + \int_{t-\tau}^t g'(y(s))z(s)ds + \int_{t-\tau}^t h'(x(s))y(s)ds - \sigma x(t)\dot{\omega}(t), \end{aligned} \quad (1.2)$$

where σ and τ are two positive constants; $f(\dot{x})$, $g(\dot{x})$ and $h(x)$ are continuous functions, $g(0) = f(0) = h(0) = 0$; $\omega(t) \in \mathbb{R}^m$ is a m -dimensional standard Brownian motion or Wiener process defined on the probability space, a stochastic process representing the noise.

In this work, we aim to find new conditions for the (SAS) of the zero solution of (SDDE) (1.1) by defining an appropriate (LF).

Remark 1.1. *We have the following observations:*

- (1) If $\sigma = 0$ in (1.1), then we have a general (DDE) of third-order, which has been investigated extensively in relevant literature.
- (2) Whenever $f(\dot{x}(t)) = a$ and $g(\dot{x}(t - \tau)) = b\dot{x}(t)$, where a and b are two positives constants, then (SDDE) (1.1) reduces to a (SDDE) of third-order discussed in [32]. Therefore Theorem 3.1 includes and extends the stability result discussed in [32].

- (3) The obtained results in [28], [29], [31] and [33] are on (SDDEs) of second-order, but the obtained results in [30] and [32] are only on (SDDEs) of third order in the previous literature.

Motivated by references of this paper and the papers and books can found in the literature, we aim to do a contribution to the literature by obtaining a new result on the (SAS) of solutions of a new (SDDE) model, which has not been discussed in the literature yet. This fact is the novelty and originality of this article.

2 Preliminary results

Let $\omega(t) = (\omega_1(t), \dots, \omega_m(t))$ be an m -dimensional Brownian motion. Let us consider an n -dimensional (SDE)

$$dx(t) = F(t, x(t))dt + G(t, x(t))d\omega(t) \quad \text{on } t \geq 0, \quad (2.1)$$

with initial condition $x(0) = x_0 \in \mathbb{R}^n$. We suppose that the functions $F : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $G : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ are continuous and satisfy the Lipschitz condition (see, for example, [22], [26]). Hence, the existence and uniqueness of solutions of (SDE) (2.1) are guaranteed on $t \geq 0$, say $x(t; x_0)$. In addition, we assume that $f(t, 0) = 0$ and $g(t, 0) = 0$.

We will show family of non-negative and differentiable functions $V(t, x_t)$ by $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$. That is, those functions once and twice continuously differentiable in t and x , respectively.

Let us define an operator $\mathcal{L}$ on $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ by

$$\mathcal{L}V(t, x_t) = V_t(t, x_t) + V_x(t, x_t) \cdot f(t, x) + \frac{1}{2} \text{trace}[g^T(t, x) V_{xx}(t, x_t) g(t, x)], \quad (2.2)$$

where

$$V_x = (V_{x_1}, \dots, V_{x_n}), \quad V_t(t, x_t) = \frac{\partial V(t, x_t)}{\partial t}, \quad V_x(t, x_t) = \left(\frac{\partial V(t, x_t)}{\partial x_1}, \dots, \frac{\partial V(t, x_t)}{\partial x_n} \right).$$

Furthermore,

$$V_{xx} = (V_{x_i x_j})_{n \times n} = \left(\frac{\partial^2 V(t, x_t)}{\partial x_i \partial x_j} \right)_{n \times n}, \quad i, j = 1, \dots, n.$$

Further, let $\mathcal{K}$ be the family of all nondecreasing and continuous functions $\vartheta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\vartheta(0) = 0$ and $\vartheta(r) > 0$, if $r > 0$.

Theorem 2.1. ([2], [4]) *Assume that there exist $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ and $\vartheta \in \mathcal{K}$ such that*

- (i) $V(t, 0) = 0$, $\vartheta(|x|) \leq V(t, x_t)$, and
- (ii) $\mathcal{L}V(t, x_t) \leq 0$, for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then the zero solution of the (SDE) is (SS).

Theorem 2.2. ([2], [4]) *Assume that there exist $V \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ and $\vartheta_1, \vartheta_2, \vartheta_3 \in \mathcal{K}$ such that*

- (i) $\vartheta_1(|x|) \leq V(t, x_t) \leq \vartheta_2(|x|)$, and
- (ii) $\mathcal{L}V(t, x_t) \leq -\vartheta_3(|x|)$, for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Then the zero solution of the (SDE) is stochastically asymptotic stable in the large.

3 Main result

We now state the main (S) result for (SDDE) (1.1).

Theorem 3.1. *Let*

$\alpha, \beta, \gamma, a_1, a_2, L$ and M be positive constants such that

- (i) $h(0) = 0$, $\frac{h(x)}{x} \geq \alpha > 0$ and $h(x) \text{sgn} x > 0$, for all $x \neq 0$ and $\sup\{h'(x)\} = \frac{\gamma}{2}$.

(ii) $g(0) = 0$, $\frac{g(y)}{y} \geq \beta > 0$, for all $y \neq 0$.

(iii) $0 \leq f(y) - a_1 \leq a_2$, for all y .

(iv) $|h'(x)| \leq L$ and $|g'(y)| \leq M$, for all x, y .

(v) $2\alpha - 2 - a_1 - \beta > \sigma^2$.

(vi) $a_1\beta - \gamma > 2\beta + 6$.

Then the zero solution of (SDDE) (1.1) is stochastically asymptotic stable if

$$\tau < \min \left\{ \frac{2\alpha - 2 - a_1 - \beta - \sigma^2}{2(L + M)}, \frac{a_1\beta - \gamma - 2\beta - 6}{4\mu(L + M) + 4L(\mu + 2)}, \frac{a_1\beta - \gamma - 2\beta}{4\beta(L + M) + 4\beta M(\mu + 2)} \right\},$$

where

$$\mu = \frac{a_1\beta + \gamma}{4\beta}.$$

Proof of Theorem 3.1. Define the (LF) $V(\cdot) = V(t, x_t, y_t, z_t)$ as

$$\begin{aligned} V(\cdot) = & \mu \int_0^x h(\xi) d\xi + h(x)y + \mu \int_0^y f(\eta) \eta d\eta + \int_0^y g(\eta) d\eta + \mu yz \\ & + \frac{1}{2}z^2 + xz + x^2 + \lambda \int_{-\tau}^0 \int_{t+s}^t y^2(\theta) d\theta ds + \delta \int_{-\tau}^0 \int_{t+s}^t z^2(\theta) d\theta ds, \end{aligned} \quad (3.1)$$

where $x_t = x(t + s)$, $s \leq 0$ and λ and δ are positive constants, and we determine them later.

Next, we have to show that $V(t, x_t, y_t, z_t)$ satisfies the assumptions of Theorem 2.2.

Moreover, by applying Itô formula in (3.1) using system (1.2), we find that

$$\begin{aligned} \mathcal{L}V(\cdot) = & h'(x)y^2 + 2xy + yz + \mu z^2 - \mu g(y)y - f(y)z^2 - f(y)xz - xg(y) - xh(x) \\ & + (\mu y + x + z) \left\{ \int_{t-\tau}^t g'(y(s))z(s)ds + \int_{t-\tau}^t h'(x(s))y(s)ds \right\} \\ & + \lambda r y^2 - \lambda \int_{t-\tau}^t y^2(s)ds + \delta r z^2 - \delta \int_{t-\tau}^t z^2(s)ds + \frac{1}{2}\sigma^2 x^2. \end{aligned}$$

Using assumptions (i) – (iii) of Theorem 3.1, we get

$$\begin{aligned} \mathcal{L}V(\cdot) \leq & \frac{\gamma}{2}y^2 + 2xy + yz + \mu z^2 - \mu\beta y^2 - a_1 z^2 - a_1 xz - \beta xy - \alpha x^2 \\ & + (\mu y + x + z) \left\{ \int_{t-\tau}^t g'(y(s))z(s)ds + \int_{t-\tau}^t h'(x(s))y(s)ds \right\} \\ & + \lambda \tau y^2 - \lambda \int_{t-\tau}^t y^2(s)ds + \delta \tau z^2 - \delta \int_{t-\tau}^t z^2(s)ds + \frac{1}{2}\sigma^2 x^2. \end{aligned} \quad (3.2)$$

Since $|h'(x)| \leq L$, $|g'(y)| \leq M$ and by using the inequality $2uv \leq u^2 + v^2$, we have

$$\begin{aligned} (\mu y + x + z) \int_{t-\tau}^t h'(x(s))y(s)ds & \leq \frac{1}{2}\mu L \tau y^2 + \frac{1}{2}L \tau x^2 + \frac{1}{2}L \tau z^2 + (\frac{1}{2}\mu + 1)L \int_{t-\tau}^t y^2(s)ds, \\ (\mu y + x + z) \int_{t-\tau}^t g'(y(s))z(s)ds & \leq \frac{1}{2}\mu M \tau y^2 + \frac{1}{2}M \tau x^2 + \frac{1}{2}M \tau z^2 + (\frac{1}{2}\mu + 1)M \int_{t-\tau}^t z^2(s)ds. \end{aligned}$$

Then, by substituting in (3.2), we find

$$\mathcal{L}V(\cdot) \leq - \left(\frac{2\alpha - 2 - a_1 - \beta - \sigma^2}{2} - \frac{L + M}{2}\tau \right) x^2$$

$$\begin{aligned}
& - \left\{ \mu\beta - \frac{\beta + \gamma + 3}{2} - \frac{\mu(L + M)}{2}\tau - \lambda\tau \right\} y^2 \\
& - \left(-\mu + \frac{a_1}{2} - \frac{1}{2} - \frac{L + M}{2}\tau - \delta r \right) z^2 \\
& - \left(\lambda - \frac{\mu L}{2} - L \right) \int_{t-\tau}^t y^2(s) ds - \left(\delta - \frac{\mu M}{2} - M \right) \int_{t-\tau}^t z^2(s) ds.
\end{aligned}$$

If we take

$$\lambda = \frac{L(\mu + 2)}{2} > 0 \quad \text{and} \quad \delta = \frac{M(\mu + 2)}{2} > 0,$$

then it is easy to see that

$$\begin{aligned}
\mathcal{L}V(\cdot) \leq & - \left(\frac{2\alpha - 2 - a_1 - \beta - \sigma^2}{2} - \frac{L + M}{2}\tau \right) x^2 \\
& - \left\{ \frac{a_1\beta - \gamma - 2\beta - 6}{4} - \frac{\mu(L + M) + L(\mu + 2)}{2}\tau \right\} y^2 \\
& - \left(\frac{a_1\beta - \gamma - 2\beta}{4\beta} - \frac{L + M + M(\mu + 2)}{2}\tau \right) z^2.
\end{aligned}$$

Thus, if

$$\tau < \min \left\{ \frac{2\alpha - 2 - a_1 - \beta - \sigma^2}{2(L + M)}, \frac{a_1\beta - \gamma - 2\beta - 6}{4\mu(L + M) + 4L(\mu + 2)}, \frac{a_1\beta - \gamma - 2\beta}{4\beta(L + M) + 4\beta M(\mu + 2)} \right\}.$$

Hence, we can obtain such that

$$\mathcal{L}V(\cdot) \leq -\rho_1(x^2 + y^2 + z^2), \quad \text{for some constant } \rho_1 > 0. \quad (3.3)$$

It is clear from assumptions $f(y) \geq a_1$ and $g(y) \geq \beta y$, and from conditions (ii) and (iii) of Theorem 3.1 that

$$V(\cdot) \geq \mu \int_0^x h(\xi) d\xi + h(x)y + \frac{1}{2}\mu a_1 y^2 + \frac{1}{2}\beta y^2 + \mu yz + \frac{1}{2}z^2 + xz + x^2.$$

It follows that

$$\begin{aligned}
V(\cdot) & \geq \mu \int_0^x h(\xi) d\xi + \frac{1}{2\beta}(\beta y + h(x))^2 - \frac{1}{2\beta}h^2(x) + \frac{1}{2}\mu a_1 y^2 + \mu yz \\
& \quad + (x + \frac{z}{2})^2 + \frac{1}{4}z^2 \\
& = \frac{1}{2\beta}(\beta y + h(x))^2 + (\mu y + \frac{z}{2})^2 + \frac{1}{2}\mu(a_1 - 2\mu)y^2 + (x + \frac{z}{2})^2 \\
& \quad + \frac{1}{2\beta y^2} \left[4 \int_0^x h(\xi) \left\{ \int_0^y (\mu\beta - h'(x))\eta d\eta \right\} d\xi \right]; \quad y \neq 0.
\end{aligned} \quad (3.4)$$

Now we recall that

$$\begin{aligned}
a_1 - 2\mu & = a_1 - 2\left(\frac{a_1\beta + \gamma}{4\beta}\right) = \frac{a_1\beta - \gamma}{2\beta} > 0, \\
\mu\beta - h'(x) & \geq \frac{a_1\beta + \gamma}{4\beta} - \frac{\gamma}{2} = \frac{a_1\beta - \gamma}{4} > 0.
\end{aligned}$$

Therefore we find

$$\begin{aligned}
& \frac{1}{2\beta y^2} \left[4 \int_0^x h(\xi) \left\{ \int_0^y (\mu\beta - h'(x))\eta d\eta \right\} d\xi \right] \\
& \geq \frac{1}{2\beta y^2} \left[4 \int_0^x h(\xi) \left\{ \int_0^y \left(\frac{a_1\beta - \gamma}{4}\right)\eta d\eta \right\} d\xi \right] = \frac{a_1\beta - \gamma}{4\beta} \int_0^x h(\xi) d\xi,
\end{aligned}$$

which together with (3.4), implies the following inequality

$$V(\cdot) \geq \frac{1}{2\beta}(\beta y + h(x))^2 + (\mu y + \frac{z}{2})^2 + \frac{1}{2}\mu(\frac{a_1\beta - \gamma}{2\beta})y^2 + (x + \frac{z}{2})^2 \\ + \frac{a_1\beta - \gamma}{4\beta} \int_0^x h(\xi)d\xi.$$

Then

$$V(\cdot) \geq \rho_2(x^2 + y^2 + z^2), \quad \text{for some constant } \rho_2 > 0. \quad (3.5)$$

Now since $f(y) \leq a_1 + a_2$, $|h'(x)| \leq L$ and $|g'(y)| \leq M$, then, by using the mean-value theorem for the derivatives, we can find $h(x) \leq Lx$ and $g(y) \leq My$. Therefore we can re-arrange (3.1) such as the following form

$$V(\cdot) \leq \frac{1}{2}\mu Lx^2 + Lxy + \frac{1}{2}\mu(a_1 + a_2)y^2 + \frac{1}{2}My^2 + \mu yz + \frac{1}{2}z^2 \\ + xz + x^2 + \lambda \int_{-\tau}^0 \int_{t+s}^t y^2(\theta)d\theta ds + \delta \int_{-r}^0 \int_{t+s}^t z^2(\theta)d\theta ds. \quad (3.6)$$

On the other hand, it is obvious that

$$\int_{-\tau}^0 \int_{t+s}^t y^2(\theta)d\theta ds \leq \|y\|^2 \int_{t-\tau}^t (\theta - t + \tau)d\theta = \frac{\tau^2}{2}\|y\|^2, \\ \int_{-\tau}^0 \int_{t+s}^t z^2(\theta)d\theta ds \leq \|z\|^2 \int_{t-\tau}^t (\theta - t + \tau)d\theta = \frac{\tau^2}{2}\|z\|^2.$$

So that by using the pervious inequalities in (3.6) and the estimate $uv \leq \frac{1}{2}(u^2 + v^2)$, we have

$$V(\cdot) \leq \frac{1}{2}\left\{(\mu + 1)L + 3\right\}\|x\|^2 + \frac{1}{2}\left\{(a_1 + a_2 + 1)\mu + M + L + \lambda\tau^2\right\}\|y\|^2 \\ + \frac{1}{2}\left(\mu + 2 + \delta\tau^2\right)\|z\|^2.$$

Then there exists a positive constant ρ_3 satisfying the following

$$V(\cdot) \leq \rho_3(x^2 + y^2 + z^2), \quad \text{for some } \rho_3 > 0. \quad (3.7)$$

Therefore, from the results (3.3), (3.5) and (3.7), we see that all the conditions of Theorem 2.2 hold, and hence the zero solution of (1.1) is stochastically asymptotic stable.

This completes the proof of the last theorem.

4 Example

In this section we provide an example to validate the theorem proved in the former section.

Example 1: Consider the below (SDDE) of third order as follows:

$$\ddot{x} + \left(8 + \frac{1}{1 + \dot{x}^2}\right)\ddot{x} + 8\dot{x}(t - \tau) + \sin(\dot{x}(t - \tau)) + 12x(t - \tau) + \frac{x(t - \tau)}{1 + x^2(t - \tau)} + 2x(t)\dot{\omega}(t) = 0. \quad (4.1)$$

Its equivalent system is given as

$$\dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -\left(8 + \frac{1}{1 + y^2}\right)z - (8y + \sin y) + \int_{t-\tau}^t \{8 + \cos y(s)\}z(s)ds \\ - \left(12x + \frac{x}{1 + x^2}\right) + \int_{t-\tau}^t \left\{12 + \frac{1 - x^2(s)}{(1 + x^2(s))^2}\right\}y(s)ds - 2x(t)\dot{\omega}(t). \quad (4.2)$$

It follows that (SDDE) (4.1) is a special case of (SDDE) (1.1), and when we compare (SDDE) (4.1) with (SDDE)(1.1) we obtain the following relations:

$$f(y) = 8 + \frac{1}{1+y^2}, \quad \text{since } 0 \leq \frac{1}{1+y^2} \leq 1,$$

it tends to $a_1 = 8$, and $a_2 = 1$. Also since

$$g(y) = 8y + \sin y, \quad \frac{g(y)}{y} - 8 = \frac{\sin y}{y},$$

therefore

$$-1 \leq \frac{g(y)}{y} - 8 \leq 1;$$

$$g'(y) = 8 + \cos y, \quad \text{for } |8 + \cos y| \leq 9.$$

Then we can take $\beta = 7$ and $M = 9$. Since

$$h(x) = 12x + \frac{x}{1+x^2}, \quad \text{it follows that } \frac{h(x)}{x} \geq 12.$$

Then, we get

$$|h'(t) - 12| \leq 1, \quad \text{then } |h'(t)| \leq 13,$$

hence we can take $\alpha = 12$ and $L = 14$.

Next, we can note that $\sup\{|h'(x)|\} = 13$, then we can take $\gamma = 26$.

By noting the former discussion, we see that

$$\mu = \frac{a_1\beta + \gamma}{4\beta} \cong 2.93, \quad a_1\beta - \gamma - 2\beta - 6 = 10; \quad 2\alpha - 2 - a_1 - \beta = 7 > \sigma^2 = 4.$$

Thus, we can note that all assumptions (i) – (iv) of Theorem 3.1 hold true.

This result implies that the zero solution of (SDDE) (4.1) is stochastically asymptotic stable if $\tau = 0.008$.

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