THE EXISTENCE OF SOLUTIONS FOR A NONLOCAL PROBLEM OF AN IMPLICIT FRACTIONAL-ORDER DIFFERENTIAL EQUATION

Fatma M. Gaafar

Faculty of Science, Damanhour University, Damanhour, Egypt *

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Abstract

In this paper, we discuss existence of at least one integral solution and also uniqueness for an implicit functional differential equation of fractional order with Riemann-Liouville fractional derivative. The existence will be demonstrated by means of Schauder fixed point theorem and Banach contraction principle. Some examples are also discussed to illustrate the main results.

Key words: Riemann-Liouville fractional derivative; Weighted space; Initial value problem; Existence of solutions. Mathematics Subject Classification: 26A33, 34A08, 34A12

1 Introduction

Fractional calculus which involves integro differential singular operators has lately caught interested and attention mathematicians owing to their extensive applications in a multiplicity of fields such as dynamical systems, solid mechanics, viscoelasticity, control etc., see for instance the monographs by: Baleanu et al. [1], Coimbra et al. [2], Coimbra [3], Dalir and Bashour [4], Diethelm [5], Glockle and Nonnenmacher [6], Hilfer [7], Ingman and Suzdalnitsky [8], Kilbas et al. [9], Machado et al. [10], Metzler et al. [11] Rossikhin and Shitikova [12], Sabatier et al. [13], Sanko and Marichev [14], Sweilam and AL-Mrawm [15], Yajima and Yamasaki [16] and the references in that.

Numerous researchers form mathematics group fond on investigating existence, stability, uniqueness and additional properties for implicit fractional differential problems (IFDPs) by assorted formulas of fractional differential equations with different formulae of fractional derivative operators. The researchers investigate the case of implicit functions, they considered the nonlinear function $f$ depends on the fractional derivative of the unknown function, see for example, Abbas et al. [17], Benavides [18], Benchohra et al. [19]- [23], El-Sayed and Bin-Taher [24]- [26], Guezane-Lakoud and Khaldi [27], Nieto et al. [28], Vityuk and Mykhailenko [29] and references therein. The papers on integrable solutions for fractional differential equations is extremely constrained, see papers by: Benchohra et al. [19, 20, 22, 23], El-Sayed and Abd El-Salam [30-32], El-Sayed and Hashem [33] and references therein.

Motivated by the above works, In this paper, we study the existence of at least one integrable solution for the implicit fractional order differential problem (IFDP):

$$D^\alpha u(t) = f(t, u(t), D^\alpha u(t), \int_0^T k(t,s)u(s)ds) \quad a.e. \ t \in (0,T], \ T < \infty,$$

(1.1)

$$I^{1-\alpha} u(t)|_{t=0} = b\Gamma(\alpha), \ b \in R,$$

(1.2)

where $f : [0,T] \times R^3 \to R$, $k : [0,T] \times [0,T] \to R$ are given functions and $D^\alpha$ is the Riemann-Liouville fractional-order derivative of order $\alpha \in (0,1)$. Moreover, we will examine the uniqueness of the solution in $L^1(J,R)$, $J = (0,T]$ space and in the weighted space $C_{1-\alpha}(J,R)$. We provide examples to clarify our acquired outcomes.

* E-mail adress: fatmagaafar2@yahoo.com
2 Preliminaries

Let \( L_1(J,R) \) denote the space of all Lebesgue integrable functions on the interval \( J = (0,T] \) with the standard norm \( \|u\|_{L_1} = \int_0^T |u(t)|dt \).

Let \( C(J,R) = \{ u: u(t) \text{ is continuous on } J : \|u\| = \max_{t \in J} |u(t)| \} \),
\( C_{1-\alpha}(J,R) = \{ u: t^{1-\alpha} u(t) \text{ is continuous on } J = [0,T] \text{ with the weighted norm} \|u\|_{C_{1-\alpha}} = \max_{t \in J} t^{1-\alpha} |u(t)| \} \).

**Definition 2.1.** ([1,14]) The Riemann-Liouville fractional integral of the function \( f \in L_1(J,R) \) is known as
\[
I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \, ds, \quad \alpha \in R^+,
\]

**Definition 2.2.** ([1,14]) The Riemann-Liouville fractional derivative of \( f \in L_1(J,R) \) is known as
\[
D^\alpha f(t) = \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f(s) \, ds, \quad \alpha \in (0,1).
\]

The following theorems will be needed.

**Theorem 2.1.** (Kolmogorov compactness criterion [34]) Assume that \( \Omega \subseteq L^p(J,R), 1 \leq p < \infty \). If \( \Omega \) is bounded in \( L^p(J,R) \) and \( \lim_{h \to 0} \|u_h - u\|_{L^p} = 0 \) uniformly for \( u \in \Omega \) where
\[
u_h(t) = \frac{1}{h} \int_t^{t+h} u(s) \, ds,
\]
then in \( L_p(J,R) \), \( \Omega \) is relative compact.

The superposition operator generated by \( f \) is identified as follows:

**Definition 2.3.** [35] Let \( f : J \times R \to R \) be a Carathéodory function. The superposition operator generated by \( f(t,x) \) is the operator \( F_x \) which assigns to each real measurable function on \( J \) the real function \( (Fx)(t) = f(t,x(t)), t \in J \).

**Theorem 2.2.** (Krasnosel’skii [36]) The superposition operator \( F \) created by the function \( f \) maps the space \( L_1(J,R) \) continuously into itself if and only if
\[
|f(t,x)| \leq |a(t)| + b|x|, \quad \text{for all } t \in J \text{ and } x \in R,
\]
where \( a(t) \) is a function in \( L_1(J,R) \) and \( b \) is a nonnegative constant.

3 Existence of solutions

**Definition 3.1.** A function \( u \in L_1(J,R) \) is said to be a solution of the IFDP (1.1)-(1.2) if \( u \) satisfies (1.1)-(1.2).

Consider the following assumptions

(h1) the function \( f : J \times R^3 \to R \) is Carathéodory, i.e. measurable in \( t \in J \) for each \( (u,v,w) \in R^3 \) and continuous in \( (u,v,w) \in R^3 \) for almost all \( t \in J \).
(h2) there is function \( a \in L_1(J,R) \) and constants \( b_i \geq 0, i = 1,2,3 \) such that
\[
|f(t,u,v,w)| \leq |a(t)| + b_1|u| + b_2|v| + b_3|w|, \quad u,v,w \in R \text{ and } t \in J.
\]
(h3) \( k(t,s) \) is continuous for all \( (t,s) \in [0,T] \times [0,T] \) and there is positive constant \( K \) such that \( \max_{t,s \in [0,T]} |k(t,s)| \leq K \).

For the existence of a solution to problem (1.1)-(1.2), we need to the following lemma.

**Lemma 3.1.** The problem
\[
D^\alpha u(t) = y(t),
\]
under the condition \( I^{1-\alpha} u(t)|_{t=0} = b \Gamma(\alpha) \),
Conversely, let $\alpha > 0$. Differentiating both sides, we get

\[ u(t) = b t^{\alpha - 1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds. \quad (3.1) \]

**Proof.** Letting $D^\alpha u(t) = y(t)$, then $D^{\alpha-1} u(t) = y(t)$, integrating both sides from 0 to $t$, we get

\[ I^{\alpha-1} u(t) - I^{\alpha-1} u(t)|_{t=0} = \int_0^t y(s) \, ds, \]

\[ I^{\alpha-1} u(t) = b \Gamma(\alpha) + \int_0^t y(s) \, ds, \]

operating by $I^\alpha$ on both sides, we get

\[ I u(t) = \frac{b t^\alpha}{\alpha} + I^{\alpha+1} y(t), \]

differentiating both sides, we get (3.1).

Conversely, let $u(t)$ be a solution of (3.1), operating by $I^{\alpha-1}$ on it, then

\[ I^{\alpha-1} u(t) = I^{\alpha-1} b t^{\alpha-1} + I^{\alpha-1} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds, \]

\[ = b \Gamma(\alpha) + \int_0^t y(s) \, ds, \]

and

\[ I^{\alpha-1} u(t)|_{t=0} = b \Gamma(\alpha). \]

**Lemma 3.2.** The solution of the IFDP (1.1)-(1.2) if it exists, it has the integral form

\[ u(t) = b t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds, \quad (3.2) \]

where $y(t)$ is the solution of the functional integral equation

\[ y(t) = f(t, b t^{\alpha-1} + I^\alpha y(t), y(t), b \int_0^T s^{\alpha-1} k(t,s) \, ds + \int_0^T k(t,s) \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} y(\xi) \, d\xi \, ds). \quad (3.3) \]

**Proof.** Letting $y(t)$ be such that $y(t) = D^\alpha u(t)$, substituting in equation (1.1), then we have

\[ y(t) = f(t, u(t), y(t), \int_0^T k(t,s) u(s) \, ds), \quad (3.4) \]

using Lemma 3.1,

\[ u(t) = b t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds, \]

substitute by the estimation of $u(t)$ in (3.4), then we get (3.3).

**Theorem 3.1.** Let the assumptions $(h_1)$-$\bigl(h_3\bigr)$ are satisfied.

If

\[ \frac{b_1 T^\alpha}{\Gamma(\alpha + 1)} + b_2 + \frac{b_3 K T^{\alpha+1}}{\Gamma(\alpha + 1)} < 1, \quad (3.5) \]

then IFDP (1.1)-(1.2) has at least one solution $u \in L_1(J, R)$.

**Proof.** Convert functional integral equation (3.3) into a fixed point problem. Consider the following operator $F$

\[ F : L_1(J, R) \to L_1(J, R) \]

defined by

\[ (F y)(t) = f(t, b t^{\alpha-1} + I^\alpha y(t), y(t), b \int_0^T s^{\alpha-1} k(t,s) \, ds + \int_0^T k(t,s) \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} y(\xi) \, d\xi \, ds). \quad (3.6) \]
Let \( B_r = \{ y \in L_1(J,R) : \| y \|_{L_1} \leq r \} \subset L_1(J,R) \), where

\[
r \geq \frac{\| a \|_{L_1} + \| b \|_{T^\alpha} + \| b \|_{KT^{\alpha+1}}}{1 - \left( \frac{b_1 T^{\alpha}}{\Gamma(\alpha+1)} + b_2 + \frac{b_3 KT^{\alpha+1}}{\Gamma(\alpha+1)} \right)}.
\]

Evidently, \( B_r \) is closed, bounded and also convex.

Now, we will demonstrate that \( FB_r \subset B_r \); actually, from (3.5), (3.6) and from the assumptions \((h_2)-(h_3)\), let \( y \) be an arbitrary element in \( B_r \), we have

\[
\| Fy \|_{L_1} = \int_0^T |(Fy)(t)| dt \\
= \int_0^T \left| f(t, b^\alpha y(t), y(t), b \int_0^T s^{\alpha-1}k(t,s)ds + \int_0^T k(t,s) \int_0^s (s - \xi)^{\alpha-1} \frac{y(\xi)}{\Gamma(\alpha)} d\xi ds \right| dt,
\]

\[
\leq \int_0^T \left( |a(t)| + b_1 |b| t^{\alpha-1} + b_1 \int_0^T \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |y(s)| ds + b_2 |y(t)| + b_3 |b| \int_0^T s^{\alpha-1} |k(t,s)| ds \\
+ b_3 \int_0^T |k(t,s)| \int_0^s \frac{(s - \xi)^{\alpha-1}}{\Gamma(\alpha)} |y(\xi)| d\xi ds \right) dt,
\]

\[
\leq \| a \|_{L_1} + b_1 |b| T^{\alpha} + b_1 \int_0^T \frac{1}{\Gamma(\alpha)} |y(s)| ds dt + b_2 \| y \|_{L_1} \\
+ b_3 |b| K \int_0^T T^{\alpha} dt + b_3 K \int_0^T dt \int_0^T \frac{1}{\Gamma(\alpha)} |y(\xi)| d\xi \\
\leq \| a \|_{L_1} + b_1 |b| T^{\alpha} + b_1 \frac{T^{\alpha}}{\Gamma(\alpha+1)} |y|_{L_1} + b_2 \| y \|_{L_1} + b_3 |b| KT^{\alpha+1} \\
+ b_3 \frac{KT^{\alpha+1}}{\Gamma(\alpha+1)} \| y \|_{L_1},
\]

which signifies that the operator \( F \) maps \( B_r \) into itself. Assumption \((h_1)\) guarantees that \( F \) is continuous.

Now, we will demonstrate that \( F \) is compact, that is \( FB_r \) is relative compact. Let \( \Omega \) be a bounded subset of \( B_r \), then \( F(\Omega) \) is bounded in \( L_1(J,R) \), i.e., condition (i) of Kolmogorov compactness criterion \([34]\) is satisfied. It remain to demonstrate that \( (Fy)_h \to Fy \) in \( L_1(J,R) \) as \( h \to 0 \) for each \( y \in B_r \).

Let \( y \in B_r \), then

\[
\| (Fy)_h - (Fy) \|_{L_1} = \int_0^T \left| (Fy)_h(t) - (Fy)(t) \right| dt,
\]

\[
= \int_0^T \frac{1}{h} \int_t^{t+h} (Fy)(\eta) d\eta - (Fy)(t) \left| dt,
\]

\[
\leq \int_0^T \left( \frac{1}{h} \int_t^{t+h} |(Fy)(\eta) - (Fy)(t)| d\eta \right) dt,
\]

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\[
\begin{align*}
&\leq \int_0^T \frac{1}{h} \int_t^{t+h} \left| f(\eta, b\eta^{\alpha-1} + I^\alpha y(\eta), y(\eta), b \int_0^T s^{\alpha-1} k(\eta, s)ds + \int_0^T k(\eta, s) \int_0^s \frac{(s - \xi)^{\alpha-1}}{\Gamma(\alpha)} y(\xi)d\xi ds \right| d\eta dt

\end{align*}
\]

from assumptions \((h_1) - (h_2)\) and Theorem 2.2 we get \( f \in L_1(J, R) \), it follows that [34]

\[
\frac{1}{h} \int_t^{t+h} \left| f(\eta, b\eta^{\alpha-1} + I^\alpha y(\eta), y(\eta), b \int_0^T s^{\alpha-1} k(\eta, s)ds + \int_0^T k(\eta, s) \int_0^s \frac{(s - \xi)^{\alpha-1}}{\Gamma(\alpha)} y(\xi)d\xi ds \right| d\eta
\]

\[
\rightarrow 0 \quad \text{as} \quad h \rightarrow 0, \text{ a.e } t \in (0, T].
\]

Hence, \((Fy)_h \rightarrow (Fy)\) uniformly as \( h \rightarrow 0 \).

Then, by Theorem 2.1, we have got that \( F(\Omega) \) is relative compact, i.e., \( F \) is a compact operator.

As a consequence of Schauder’s fixed point theorem [37], the operator \( F \) has a fixed point in \( B_r \), which demonstrates the existence of at least one solution \( y \in B_r \subset L_1(J, R) \) of the functional integral equation (3.3), consequently from (3.2) we have \( u(t) \) has at least one solution, therefore IFDP (1.1)-(1.2) has at least one solution in \( B_r \).

Now, for uniqueness of solution to IFDP (1.1)-(1.2). Consider the following assumptions:

\((h_4)\) there exist constants \( k_i \geq 0, \quad i = 1, 2, 3 \) such that

\[
|f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2)| \leq k_1 |u_1 - u_2| + k_2 |v_1 - v_2| + k_3 |w_1 - w_2|,
\]

for any \( u_j, v_j, w_j \in R, \quad j = 1, 2 \) and \( t \in J \).

\((h_5)\) \( f(t, 0, 0, 0) \in L_1(J, R) \)

**Theorem 3.2.** Assume that the conditions \((h_1), (h_3) - (h_5)\) are satisfied. If

\[
A = \frac{k_1 T^\alpha}{\Gamma(\alpha + 1)} + k_2 + k_3 \frac{K T^{\alpha+1}}{\Gamma(\alpha + 1)} < 1,
\]

then the IFDP (1.1)-(1.2) has a unique solution \( u \in L_1(J, R) \).

**Proof.** From condition \((h_4)\) we can obtain,

\[
|f(t, u, v, w)| \leq |f(t, 0, 0, 0)| + k_1 |u| + k_2 |v| + k_3 |w|,
\]

this show that the assumptions of Theorem 3.1 are satisfied.

Let \( y_1, y_2 \in L_1(J, R) \) be two solutions of the functional integral equation (3.3), then

\[
|y_1(t) - y_2(t)| = \left| f(t, b^{t^{\alpha-1}} + I^\alpha y_1(t), y_1(t), b \int_0^T s^{\alpha-1} k(t, s)ds + \int_0^T k(t, s) \int_0^s \frac{(s - \xi)^{\alpha-1}}{\Gamma(\alpha)} y_1(\xi)d\xi ds \right|
\]

\[
\rightarrow \int_0^T \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} |y_1(s) - y_2(s)| ds + k_2 |y_1(t) - y_2(t)|
\]

\[
+ k_3 K \int_0^T \frac{(s - \xi)^{\alpha-1}}{\Gamma(\alpha)} |y_1(\xi) - y_2(\xi)|d\xi ds,
\]

\[
\leq k_1 \int_0^T \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} |y_1(s) - y_2(s)| ds + k_2 |y_1(t) - y_2(t)| + k_3 K \int_0^T \frac{(T - \xi)^{\alpha}}{\Gamma(\alpha + 1)} |y_1(\xi) - y_2(\xi)|d\xi,
\]

\[
\leq k_1 \int_0^T \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} |y_1(s) - y_2(s)| ds + k_2 |y_1(t) - y_2(t)| + k_3 K T^{\alpha} \frac{K T^{\alpha+1}}{\Gamma(\alpha + 1)} |y_1 - y_2|_{L_1}.
\]

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Thus

\[ \|y_1 - y_2\|_{L_1} \leq k_1 \int_0^T \int_0^t \left( \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right) |y_1(s) - y_2(s)| ds \, dt + k_2 \int_0^T |y_1(t) - y_2(t)| dt + \frac{k_3 KT^{\alpha+1}}{\Gamma(\alpha+1)} \|y_1 - y_2\|_{L_1}, \]

\[ \leq k_1 \frac{T^\alpha}{\Gamma(\alpha+1)} \|y_1 - y_2\|_{L_1}, \]

then, \((1-A) \|y_1 - y_2\|_{L_1} \leq 0\) which signifies that \(\|y_1 - y_2\|_{L_1} = 0\), and we have \(y_1 = y_2\), then there exists a unique integrable solution of the problem (1.1)-(1.2).

**Example 3.1.** Consider the following IFDP:

\[ D^+ u(t) = h(t) \left( \frac{(t-3)^2}{1+(f_0^t e^{-t}u(s) ds)^2} + \frac{\cos t}{10} \frac{u^2(t) + e^{-t}(D^+ u(t))}{1+|u|+|v|} + \frac{1}{6} \sin(f_0^t e^{-t}u(s) ds) \right), \quad a.e. \ t \in (0,1], \]

\[ I^+ u(t)|_{t=0} = b \Gamma(\frac{1}{2}), \quad b \in R. \]

where \(h(t) = \begin{cases} t, & 0 < t \leq \frac{1}{2}, \\ t^2, & \frac{1}{2} < t \leq 1. \end{cases} \)

Set \(f(t, u, v, w) = h(t) \left( \frac{(t-3)^2}{1+w^2} + \frac{\cos t}{10} \frac{u^2 + e^{-t}v}{1+|u|+|v|} + \frac{1}{6} \sin w \right)\), then we have

\[ |f(t, u, v, w)| \leq (t-3)^2 + \frac{1}{10} \left( \frac{|u|^2}{1+|u|+|v|} + \frac{e^{-t} |v|}{1+|u|+|v|} \right) + \frac{1}{6} |\sin w| \]

and the condition \(\frac{b_1 T^\alpha}{\Gamma(\alpha+1)} + b_2 + \frac{b_3 KT^{\alpha+1}}{\Gamma(\alpha+1)} = 0.72263935 < 1\), is satisfied with \(\alpha = \frac{1}{2}, b_1 = \frac{1}{10}, b_2 = \frac{1}{10}, b_3 = \frac{1}{6}, K = e, T = 1\).

It follows from Theorem 3.1 that the problem has at least one integrable solution on (0,1].

**Example 3.2.** Consider the following IFDP:

\[ D^+ u(t) = \frac{e^{-t}}{1+4e^t} \frac{1}{1+|\ln(t+s)| u(s) ds |u(t)|} \]

\[ I^+ u(t)|_{t=0} = b \Gamma(\frac{1}{2}), \quad b \in R. \]

Set \(f(t, u, v) = \frac{e^{-t}}{1+|u|+|v|} + \frac{1}{10} \cos v\), then we have

\[ |f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2)| \leq \frac{e^{-t}}{1+4e^t} \frac{1}{1+|u_1|+|v_1|} \left| \frac{1}{1+|w_1|+|u_1|} - \frac{1}{1+|w_2|+|u_2|} \right| \]

\[ + \frac{1}{10} \left| |\cos v_1 - \cos v_2| \right| \]

\[ \leq \frac{1}{5} \left| |u_1 - u_2| + |w_1 - w_2| \right| \]

\[ \leq \frac{1}{5} \left( |u_1 - u_2| + |w_1 - w_2| + \frac{1}{10} |v_1 - v_2| \right), \]

and \(f(t, 0, 0, 0) = \frac{e^{-t}}{1+4e^t} + \frac{1}{10} \in L_1(0,1]\), also \(\frac{k_1 T^\alpha}{\Gamma(\alpha+1)} + k_2 + \frac{k_3 KT^{\alpha+1}}{\Gamma(\alpha+1)} = 0.4688095 < 1\) is satisfied with \(\alpha = \frac{1}{5}, k_1 = \frac{1}{5}, k_2 = \frac{1}{10}, k_3 = \frac{1}{5}, K = \ln 2, T = 1\). It follows from Theorem 3.2 that the problem has a unique integrable solution in (0,1].

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4 Unique solution in the weighted space $C_{1-\alpha}(J, R)$

Here we examine the existence of a unique solution of IFDP (1.1)-(1.2) in the weighted space $C_{1-\alpha}(J, R)$.

**Definition 4.1.** By a solution of IFDP (1.1)-(1.2) we mean a function \(\{u : t^{1-\alpha}u(t)\text{ is continuous on the interval } J\}\) and this function satisfies (1.1)-(1.2).

Consider the following assumption:

\[C\]

This function satisfies (1.1)-(1.2).

\[
\frac{(2\alpha k_1 + k_2 KT\alpha)\Gamma(\alpha) T\alpha}{\Gamma(2\alpha + 1)} + k_2 < 1,
\]

then IFDP (1.1)-(1.2) has a unique solution \(u \in C_{1-\alpha}(J, R)\).

**Proof.** Define the operator \(F\) by (3.6), the operator \(F\) maps \(C_{1-\alpha}(J, R)\) into itself, for this let \(y \in C_{1-\alpha}(J, R)\), \(t_1, t_2 \in J\), \(t_1 < t_2\) such that \(|t_2 - t_1| < \delta\), we have

\[
|t_2^{1-\alpha}(Fy)(t_2) - t_1^{1-\alpha}(Fy)(t_1)|
= |t_2^{1-\alpha} f(t_2, bt_2^{\alpha-1} + I^{\alpha}y(t_2), y(t_2), b \int_0^T s^{\alpha-1}k(t_2, s)ds + \int_0^T k(t_2, s) \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}y(\xi)d\xi ds)
- t_1^{1-\alpha} f(t_2, bt_2^{\alpha-1} + I^{\alpha}y(t_2), y(t_2), b \int_0^T s^{\alpha-1}k(t_2, s)ds + \int_0^T k(t_2, s) \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}y(\xi)d\xi ds)
+ t_1^{1-\alpha} f(t_1, bt_1^{\alpha-1} + I^{\alpha}y(t_1), y(t_1), b \int_0^T s^{\alpha-1}k(t_1, s)ds + \int_0^T k(t_1, s) \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}y(\xi)d\xi ds)
- t_1^{1-\alpha} f(t_1, bt_1^{\alpha-1} + I^{\alpha}y(t_1), y(t_1), b \int_0^T s^{\alpha-1}k(t_1, s)ds + \int_0^T k(t_1, s) \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}y(\xi)d\xi ds)
\leq |t_2^{1-\alpha} - t_1^{1-\alpha}|
\]

\[
\times |f(t_2, bt_2^{\alpha-1} + I^{\alpha}y(t_2), y(t_2), b \int_0^T s^{\alpha-1}k(t_2, s)ds + \int_0^T k(t_2, s) \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}y(\xi)d\xi ds)
+ t_1^{1-\alpha} f(t_2, bt_2^{\alpha-1} + I^{\alpha}y(t_2), y(t_2), b \int_0^T s^{\alpha-1}k(t_2, s)ds + \int_0^T k(t_2, s) \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}y(\xi)d\xi ds)
\]

Firstly, we evaluate \(f\) in the first term of the last inequality, from condition \((h_4)\) we have

\[
|f(t_2, bt_2^{\alpha-1} + I^{\alpha}y(t_2), y(t_2), b \int_0^T s^{\alpha-1}k(t_2, s)ds + \int_0^T k(t_2, s) \int_0^s \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}y(\xi)d\xi ds)
\]
Secondly, we evaluate the second term in the inequality (4.1), we have

\[
\leq |f(t, 0, 0, 0)| + \left| f\left( t_2, b\tau_2^\alpha + I^\alpha y(t_2), y(t_2), b \int_0^T s^{\alpha-1} k(t_2, s) \, ds \right) \right|
\]

\[
+ \int_0^T k(t_2, s) \int_0^s \frac{(s - \xi)^{\alpha-1}}{\Gamma(\alpha)} y(\xi) d\xi \, ds
\]

\[
\leq f_0 + k_1 |b| t_2^\alpha - 1 + k_1 \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} |y(s)| ds + k_2 t_2^\alpha - t_1^\alpha - 1 |y(t_2)|
\]

\[
+ k_3 |b| \int_0^T s^{\alpha-1} |k(t_2, s)| ds + k_3 \int_0^T |k(t_2, s)| \int_0^s \frac{(s - \xi)^{\alpha-1}}{\Gamma(\alpha)} \xi^{\alpha-1} |y(\xi)| d\xi ds,
\]

\[
\leq f_0 + k_1 |b| t_2^\alpha - 1 + k_1 |y||C_{1-\alpha} + k_2 t_2^\alpha - 1 ||y||C_{1-\alpha} + \frac{k_3 |b| KT^\alpha}{\alpha}
\]

\[
+ \frac{k_3 KT^\alpha}{\alpha(2\alpha + 1)} ||y||C_{1-\alpha}.
\]
Now we return to equation (4.1), substituting from (4.2) and (4.3) into (4.1), we obtain

\[ \leq k_1|b| |t_2^{\alpha-1} - t_1^{\alpha-1}| + k_1||y||_{C_{1-\alpha}} \int_t^t \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} s^{\alpha-1} ds + k_2|t_2^{\alpha-1} - t_1^{\alpha-1}| |y(t_2)| \]

\[ + k_2 |t_2^{\alpha-1} - t_1^{\alpha-1}| |y(t_2)| + k_3|b| \int_0^T s^{\alpha-1} |k(t_2, s) - k(t_1, s)| ds \]

\[ + k_4 |t_2^{\alpha-1} - t_1^{\alpha-1}| |y(t_2) - t_1^{\alpha-1} y(t_1)| \]

\[ + k_5 |t_2^{\alpha-1} - t_1^{\alpha-1}| |y(t_2)| + k_6 |t_2^{\alpha-1} - t_1^{\alpha-1}| |y(t_1)| \]

\[ + k_7 |b| \int_0^T s^{\alpha-1} |k(t_2, s) - k(t_1, s)| ds + \frac{k_3 T^{2\alpha-1} \Gamma(\alpha)||y||_{C_{1-\alpha}}}{\Gamma(2\alpha)} \int_0^T |k(t_2, s) - k(t_1, s)| ds \] (4.3)

Now we return to equation (4.1), substituting from (4.2) and (4.3) into (4.1), we obtain

\[ |t_2^{\alpha-1} - t_1^{\alpha-1}| (f_0 + k_1|b| t_2^{\alpha-1} + \frac{k_1 T^{2\alpha-1} \Gamma(\alpha)}{\Gamma(2\alpha)} ||y||_{C_{1-\alpha}} + k_2 t_2^{\alpha-1} ||y||_{C_{1-\alpha}} + \frac{k_3 |b| KT^{\alpha}}{\alpha} \]

\[ + \frac{k_3 KT^{2\alpha} \Gamma(\alpha)}{\Gamma(2\alpha + 1)} ||y||_{C_{1-\alpha}} + k_4 |t_2^{\alpha-1} - t_1^{\alpha-1}| |y(t_2)| + k_6 |t_2^{\alpha-1} - t_1^{\alpha-1}| |y(t_2)| \]

\[ + k_7 |b| \int_0^T s^{\alpha-1} |k(t_2, s) - k(t_1, s)| ds + \frac{k_3 T^{2\alpha-1} \Gamma(\alpha)||y||_{C_{1-\alpha}}}{\Gamma(2\alpha)} \int_0^T |k(t_2, s) - k(t_1, s)| ds \]

As \( t_2 \to t_1 \), the right side of the above inequality tends to zero.
And therefore the operator \( F : C_{1-\alpha}(J, R) \to C_{1-\alpha}(J, R) \).
Now to prove $F$ is a contraction mapping, let $y, z \in C_{1-\alpha}(\bar{J}, R)$, then

$$|t^{1-\alpha}(Fy)(t) - t^{1-\alpha}(Fz)(t)|$$

$$= t^{1-\alpha} \left| \int_0^t f(t, bt^{\alpha-1} + \Gamma(t)y(t), y(t), b \int_0^T s^{\alpha-1}k(s, t)ds + \int_0^T k(t, s) \left( \int_0^s (\xi - \mu)^{\alpha-1}y(\xi)d\xi \right)ds \right| - f(t, bt^{\alpha-1} + \Gamma(t)z(t), z(t), b \int_0^T s^{\alpha-1}k(s, t)ds + \int_0^T k(t, s) \left( \int_0^s (\xi - \mu)^{\alpha-1}z(\xi)d\xi \right)ds \right| ,$$

$$\leq k_1 t^{1-\alpha} \int_0^t (t-s)^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} |y(s) - z(s)|ds + k_2 t^{1-\alpha}|y(t) - z(t)|$$

$$+ k_3 t^{1-\alpha} \int_0^t |k(t, s)| \left( \int_0^s (\xi - \mu)^{\alpha-1}y(\xi) - z(\xi)d\xi \right)ds,$$

$$\leq k_1 T^{1-\alpha} ||y - z||c_{1-\alpha} \int_0^t (t-s)^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} s^{\alpha-1}ds + k_2 ||y - z||c_{1-\alpha}$$

$$+ k_3 K T^{1-\alpha} ||y - z||c_{1-\alpha} \int_0^T \left( \int_0^s (\xi - \mu)^{\alpha-1} \xi^{\alpha-1}d\xi \right)ds,$$

$$\leq \left[ \frac{k_1 \Gamma(\alpha) T^{\alpha}}{\Gamma(2\alpha)} + k_2 + \frac{k_3 K \Gamma(\alpha) T^{2\alpha}}{\Gamma(2\alpha + 1)} \right] ||y - z||c_{1-\alpha},$$

this implies that

$$||Fy - Fz||c_{1-\alpha} \leq \left[ \frac{2\alpha k_1 + k_3 K T^{\alpha}}{\Gamma(2\alpha + 1)} \right] ||y - z||c_{1-\alpha}.$$
REFERENCES


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