ADDITIONAL EXTREME DISTRIBUTION FOR MODELING EXTREME VALUE DATA

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Received 17/2/2017

Revised 29/7/2017

Accepted 3/8/2017

Abstract

In this article an additional extreme value distribution (AEVD) is introduced by Box-Cox transformation. Both generalized Pareto distribution under power normalization (GPDP) and generalized extreme value distribution under power normalization (GEVP) can be obtained from AEVD as special cases. The AEVD can be applied in a wide variety of application fields especially for modelling extreme data. A system of nonlinear equations is solved numerically to get maximum likelihood estimation (MLE) for the distribution parameters. The proposed distribution is applied for modelling daily maximum air pollution from Barking Dagenham station. Various criteria reveal that AEVD improves the fitting of this data compared with other distributions.

Subject Classifications: 62G32, 60G70, 62F40.

Key words: Generalized extreme value distribution; Box-Cox-GEVP distribution; Air Pollution.

1 Introduction

Probabilistic extreme value theory is a curious and enchanting blend of an tremendous variety of applications including natural phenomena such as floods, rainfall, air pollution, corrosion and wind gusts. First of all, probabilistic extreme value theory deals with the asymptotic behaviour of the maximum and the minimum order statistics of identically distribution (iid) random variables (rv's). The distributional properties of extremes and intermediate order statistics, and exceedances over (below) high (low) thresholds are strongly depend on the upper and lower tails of the underlying distribution.

Extreme value distributions (EVDs) originally appeared as three possible limiting distributions of suitably linear normalization of a sequence of iid rv's. EVDs are the foundation of modelling and analyzing rare values, especially in environmental data e.g. daily maximum ozone measurements, maximum wind speed and and maximum water level. Generally, two well known methods, Block Maxima (BM) and Peak Over Threshold (POT) are used to analyze such data. The BM method modeled by generalized extreme value distribution under linear normalization (GEVL) was introduced by [1]. On the other hand, Peak Over Threshold (POT) modeled by generalized Pareto distribution under linear normalization (GPD) was proposed by [2]. A more general extreme value distribution named Box-Cox-GEVL was introduced by [3] and it is applied for stock data. However, some of previous distributions are not adequate in many areas. Two alternative models are built via some of the possible limiting distributions of power normalized maxima obtained by Pantcheva. The first probability model in this context, is GEVP which was discussed in [4] while the second one was introduced by [6] and it is known as GPDP. [5] applied BM method to analyze air pollutants SO₂, PM₁₀ and Ozone data using GEVP. Recently, [7] suggested and studied a new model, linear-power model to overcome some drawback of the power model In this paper, we give a simple generalization of BM and POT methods through the application of AEVD to model air pollutant data from Barking Dagenham station during the period 1-1-2010 to 31-12-2015, for daily maximum nitrogen oxides, nitrogen dioxide and sulfur dioxide concentrations.

2 Preliminaries

2.1 Box-Cox distribution under linear normalization

Let $Y_1, Y_2, \dots, Y_n$ be iidrv's with distribution function (df) $F(y) = P(Y \leq y)$ and assume $Y_{1:n} < Y_{2:n} < \dots < Y_{n:n}$ be the corresponding order statistics. [1] re-parameterized the Extreme type Theorem for modeling the rare event. We can summarize this theorem as: for any constant in sequences $c_n > 0$ and d_n if $P(Y_{n:n} \leq c_n y + d_n) = F^n(c_n y + d_n)$ weakly converged to a nondegenerate df $L(x)$, then L ought to be of a similar type of the GEVL

$$L_\gamma(y; \mu, \sigma) = \exp \left[- \left[1 + \gamma \left(\frac{y - \mu}{\sigma} \right) \right]^{-\frac{1}{\gamma}} \right] \quad 1 + \gamma(y - \mu)/\sigma > 0, \quad (2.1)$$

which is a unique model for the extreme value distribution, where μ and $\sigma > 0$ are location and scale parameters. The GEVL produces three different distributions, according to $\gamma < 0$, $\gamma > 0$, and $\gamma = 0$ ($\gamma \rightarrow 0$), which are known as Weibull, Frechet and Gumbel families of df's, respectively. So, many statistical method either parametric or nonparametric derived from the theory of estimation can be used to estimate the parameters γ , σ and μ .

The BM approach is applied when the data is independent sample maxima. In fact, some of these blocks contain small or nil values whereas others may contain extremely large values. So, many important data are lost, and consequently the BM method becomes unsuitable for the few data. The above reason restricts the utilization of the BM method. Therefore, the Peak Over Threshold (POT) method is considered an another trend for extreme data modeling through asymptotic distribution, initiated by [2]. The GPDL is the limit distribution of scaled peaks exceeding high thresholds $y > \mu$, which can be written as,

$$H_\gamma(y; \mu, \sigma) = 1 - \left[1 + \gamma \left(\frac{y - \mu}{\sigma} \right) \right]^{-\frac{1}{\gamma}}. \quad (2.2)$$

When $\gamma < 0$ there is an upper end point, so $\mu < y \leq \mu - \frac{\sigma}{\gamma}$. The GPDL family implies the exponential, uniform or Pareto distributions based on γ value. Evidently, in the statistical modeling of POT method all data is used, rather than using the BM method.

[3] used the Box-Cox transformation (see also, [8]) to composite GEVL and GPDL in the following distribution

$$W_\gamma(x; \mu, \sigma, \lambda) = \left[\frac{\exp \left[-\lambda \left[1 + \gamma \left(\frac{y - \mu}{\sigma} \right) \right]^{-\frac{1}{\gamma}} \right] - 1}{\lambda} \right] + 1, \quad (2.3)$$

with probability density function

$$w_\gamma(x; \mu, \sigma, \lambda) = \frac{1}{\sigma} \exp \left[-\lambda \left[1 + \gamma \left(\frac{y - \mu}{\sigma} \right) \right]^{-\frac{1}{\gamma}} \right] \left[1 + \gamma \left(\frac{x - \mu}{\sigma} \right) \right]^{-\frac{1}{\gamma} - 1},$$

and inverse distribution

$$W_\gamma^{-1}(x; \mu, \sigma, \lambda) = \mu - \frac{\sigma}{\gamma} + \frac{\sigma}{\gamma} \left(-\frac{1}{\lambda} \log(\lambda(q - 1) + 1) \right)^{-\gamma}.$$

The distribution introduced in (2.3) is called Box-Cox-GEVL, which tends to the GPDL and GEVL distribution. Specifically, when $\lambda = 1$, $W_\gamma(y) \rightarrow L_\gamma(y)$, and when $\lambda = 0$, $W_\gamma(y) \rightarrow H_\gamma(y)$ by L'Hopital's Method. To get the support of distribution defined in (2.3), first when $\lambda \neq 0$ and $\gamma > 0$ we get unbounded upper tail. To find the Lower bound let $R(y) = \exp(-\lambda \left(1 + \gamma \left(\frac{y - \mu}{\sigma} \right) \right)^{-\frac{1}{\gamma}})$. Then

$$\begin{aligned} \frac{R(y) - 1}{\lambda} + 1 &\geq 0 \text{ i.e.,} \\ \exp(-\lambda \left(1 + \gamma \left(\frac{y - \mu}{\sigma} \right) \right)^{-\frac{1}{\gamma}}) &\geq 1 - \lambda, \text{ i.e.,} \\ y &\geq \mu + \frac{\sigma}{\gamma} \left(\left(\frac{-\lambda}{\log(1 - \lambda)} \right)^\gamma - 1 \right). \end{aligned}$$

Therefore, when $\lambda = 0$, we get the lower support $x > \mu$

On the other hand, when $\lambda \neq 0$ and $\gamma < 0$, to find the upper bound let $-\gamma = \gamma^*$. Then $\frac{R(y)-1}{\lambda} + 1 \leq 1$, which implies that $y \leq \mu - \frac{\sigma}{\gamma^*}$. For the lower bound, for which $\frac{R(y)-1}{\lambda} + 1 \geq 0$, then $x \geq \mu + \frac{\sigma}{\gamma^*} \left(1 - \left(\frac{-\log(1-\lambda)}{\lambda}\right)^{\gamma^*}\right)$, when $\lambda = 0$ we get the boundary of GPDL as $\mu < x \leq \mu - \frac{\sigma}{\gamma}$.

2.2 The Box-Cox- under Power Normlization Distribution

In 1984 Pancheva began to develop the theory of extreme values under power normalization and then [9] showed that the p -max stable dfs attract more dfs than linear max stable dfs, which practically means that the linear model may fail to model the data, while the power model succeed to do that, but not the vice versa. [10] considered the power normalization and derived all the possible limit df's H of $Y_{n:n} = \max(Y_1, \dots, Y_n)$ subjected to this normalization. Namely, [10] determined all the possible limit types of H , for which

$$F^n(c_n |y|^{d_n} \mathcal{S}(x)) = P(C_n^{-1}(Y_{n:n}) \leq x) = P \left[\left| \frac{Y_{n:n}}{a_n} \right|^{\frac{1}{b_n}} \mathcal{S}(X_{n:n}) \leq x \right] \xrightarrow{w} H(y), \quad (3.1)$$

where $a_n > 0$ and $b_n > 0$ are reasonable power normalizing constants and $\mathcal{S}(y) = -1, 0, 1$, if $y < 0, y = 0, y > 0$, respectively. These limit types are usually called the power max stable df's (p -max stable df's). The p -max stable df's of six p -types of df's are $V_{1;\xi}(y) = \exp(-(\log y)^{-\xi}), y > 1$; $V_{2;\xi}(y) = \exp(-(-\log y)^{\xi}), 0 < y \leq 1$; $V_{3;\xi}(y) = \exp(-(-\log(-y))^{-\xi}), -1 < y \leq 0$; $V_{4;\xi}(y) = \exp(-(\log(-y))^{\xi}), y \leq -1$; $V_5(y) = \exp(-\frac{1}{y}), y > 0$; and $V_6(y) = \exp(-|y|), y \leq 0$.

Moreover, [11] stated that these p -types attract more than linear stable df's. Therefore, using the power normalization allow to obtain a wider class of limit df's, which can be used in solving approximation problems. Consequently, one can basically expand the area of applications of the extreme value model. A merged approach to the results of [9] and [11] has been obtained by [12]. As in the case of nonlinear normalization, [4] has summarized these types by the following von Mises type representations

$$P_{1;\gamma}(y; a, b) = \exp \left[-(1 + \gamma \log ay^b)^{-\frac{1}{\gamma}} \right], \quad y > 0, \quad 1 + \gamma \log ay^b > 0, \quad (3.2)$$

and

$$P_{2;\gamma}(y; a, b) = \exp \left[-(1 - \gamma \log a(-y)^b)^{-\frac{1}{\gamma}} \right], \quad y < 0, \quad 1 - \gamma \log a(-y)^b < 0. \quad (3.3)$$

[6] introduced GPDP for each of the models (3.2) and (3.3), by $Q_{1;\gamma}(y) = 1 + \log P_{1;\gamma}(y; a, b)$ and $Q_{2;\gamma}(y) = 1 + \log P_{2;\gamma}(y; a, b)$, respectively. It can be shown that (c.f., Theorem 2.2, [6]) each of the GPDP satisfies the POT stability property, i.e., the left truncated GPDP yields a GPDP again.

The block method are modeled by equation (3.2) for positive data and the corresponding model for POT method can be written as the following

$$Q_{1;\gamma}(y; a, b) = 1 - (1 + \gamma \log ay^b)^{-\frac{1}{\gamma}}, \quad y \geq \left(\frac{1}{a}\right)^{\frac{1}{b}}. \quad (3.4)$$

When $\gamma < 0$ there is a finite support $\left(\frac{1}{a}\right)^{\frac{1}{b}} \leq x \leq \left(\frac{1}{a} \exp\left(\frac{1}{|\gamma|}\right)\right)^{\frac{1}{b}}$. We propose more general extreme value distribution under nonlinear normlization constant using Box and Cox transformation in the following distribution

$$F_{1,\gamma}(y; a, b, \lambda) = \left[\frac{\exp \left[-\lambda (1 + \gamma \log ay^b)^{-\frac{1}{\gamma}} \right] - 1}{\lambda} \right] + 1 \quad (3.5),$$

with probability density function

$$f_{1,\gamma}(y; a, b, \lambda) = \frac{b}{y} \exp \left[-\lambda (1 + \gamma \log ay^b)^{-\frac{1}{\gamma}} \right] (1 + \gamma \log ay^b)^{-\frac{1}{\gamma}-1},$$

and inverse function

$$F_{1,\gamma}^{-1}(y; a, b, \lambda) = \exp \left[\left(-\frac{1}{\lambda} \log(\lambda(q-1) + 1) \right)^{-\gamma} - \frac{1}{b\gamma} - \frac{\log(a)}{b} \right].$$

The Box-Cox-GEVP1 distribution introduced in Eq. (3.5) tend to the GPDP of [6] in Eq. (3.4) and the GEVP distribution of [4] for positive data. Specifically, when $\lambda = 1$ the Box-Cox-GEVP1 reduces to the GEVP distribution.

0 equals zero, the Box.Cox.GEV converges to the GPDP for positive data by L'Hopital's method. To get the support defined of distribution defined in (3.5), first when $\lambda \neq 0$ and $\gamma > 0$, we get unbounded upper tail. To find the lower bound, let $J(y) = \exp(-\lambda(1 + \gamma \log ay^b)^{-\frac{1}{\gamma}})$ then the lower bounded is given from the inequality $\frac{J(y)-1}{\lambda} + 1 \geq 0$, i.e. $y \geq \left(\frac{1}{a} \exp(\frac{(-\frac{\log(1-\lambda)}{\lambda})^\gamma}{\gamma} - \frac{1}{\gamma})\right)^{\frac{1}{b}}$, when $\lambda = 0$ we get the boundary of GPDP as $y \geq (\frac{1}{a})^{(\frac{1}{b})}$. Second, when $\lambda \neq 0$ and $\gamma < 0$, to find the upper bound, let $-\gamma = \gamma^*$ and $\frac{J(y)-1}{\lambda} + 1 \leq 1$ then $y \leq \left(\frac{1}{a} \exp(\frac{1}{\gamma^*})\right)^{\frac{1}{b}}$. For lower bound let $\frac{J(y)-1}{\lambda} + 1 \geq 0$ then $y \geq \left(\frac{1}{a} \exp(\frac{(-\frac{\log(1-\lambda)}{\lambda})^\gamma}{\gamma^*} + \frac{1}{\gamma^*})\right)^{\frac{1}{b}}$ when $\lambda = 0$ we tend to the boundary of GPDP $(\frac{1}{a})^{(\frac{1}{b})} \leq y \leq (\frac{1}{a} \exp(\frac{1}{|\gamma|}))^{\frac{1}{b}}$. Also, we can get Box-Cox-GEVP2 for negative data by BMP method by (3.3) and POTP method by

$$Q_{2,\gamma}(y; a, b) = 1 - (1 - \gamma \log a(-y)^b)^{-\frac{1}{\gamma}}, \quad (3.6)$$

When $\gamma > 0$ there is upper and lower bounded, so $(\frac{1}{a})^{(\frac{1}{b})} \leq -y \leq \left(\frac{1}{a} \exp(\frac{1}{\gamma})\right)^{(\frac{1}{b})}$, Similarly when $\gamma < 0$, let $-\gamma = \gamma^*$, $\left(\frac{1}{a} \exp(\frac{1}{\gamma^*})\right)^{(\frac{1}{b})} \leq -y \leq (\frac{1}{a})^{(\frac{1}{b})}$.

We can summarize the two equation (3.3) and (3.6) in the following distribution

$$F_{2,\gamma}(y; a, b, \lambda) = \left[\frac{\exp\left[-\lambda(1 - \gamma \log a(-y)^b)^{-\frac{1}{\gamma}}\right] - 1}{\lambda} \right] + 1. \quad (3.7)$$

We can also get the probability density function and inverse function for this distribution as before. This distribution nested to GEVP in Eq. (3.3), when $\lambda = 1$ and GPDP in Eq. (3.6) when $\lambda = 0$. For application model (3.7) for real data we must known the upper and lower bounded of this model, which is given by: when, $\lambda \neq 0$ and $\gamma > 0$ thus,

$$\left(\frac{\exp(\frac{(-\frac{\log(1-\lambda)}{\lambda})^\gamma}{\gamma})}{a}\right)^{\frac{1}{b}} \leq -y \leq \left(\frac{\exp(\frac{1}{\gamma})}{a}\right)^{\frac{1}{b}} \text{ and when } \lambda \neq 0 \text{ and } \gamma < 0 \text{ consequently, let } -\gamma = \gamma^*, \text{ we get } \left(\frac{\exp(\frac{1}{\gamma^*})}{a}\right)^{\frac{1}{b}} \leq -y \leq \left(\frac{\exp(\frac{(-\frac{\log(1-\lambda)}{\lambda})^\gamma}{\gamma^*} - 1)}{a}\right)^{\frac{1}{b}}$$

3 Air Pollution Application

In this section we apply the new suggested models on a data set for air pollution from the London Air Quality Network (LAQN). This network is unified resource for air pollution measurements which are fundamental to support air quality administration. The LAQN was framed in 1993 to arrange and enhance air pollution checking in London. People living in London city can check the updates of the level of air pollution data through the web site www.londonair.org.uk. The greater part of London's 33 districts supply estimations to the system and also these information are progressively being supplemented by estimations from neighborhood specialists encompassing London, in this manner giving a general point of view of air pollution in London and the Home Counties. [13] clear that, All measurements under go quality control and assurance procedures with each measurement traceable to national metrological standards provided by the National Physical Laboratory. [14] developed empirical trend to predict the annual mean of NO_2 levels in London using observation data. [15] analyzed the relationship between air contaminations and ischemic stroke subtypes. In this study data was taken from site Barking Dagenham at Rush Green square, that monitors Nitrogen oxides, sulphur dioxide, PM10 and climatology data. The daily maximum level of the nitric oxide (NO), nitrogen dioxide (NO_2) and sulphur dioxide (SO_2) were recorded every hour. So a total of 53000 readings have been obtained in the period from 1-1-2010 to 31-12-2015. This data can be downloaded by any researcher in the form of a report every half hour, every hour or every day according to the type of study from the following site www.londonair.org.uk/london/asp/datadownload.asp. The daily maximum of these data was used for application of these models. The summary statistics is given in the Table 1.

In this case study we deal with this data as follows: first we estimate the parameter of block data (daily maximum) by GEVL model and estimate the peak over mean of the data using GPDL model by MLE method. The log-likelihood of

Table 1: Descriptive statistics for maximum data for air pollution

	n	minimum	maximum	median	mean	SD	skewness	kurtosis
<i>NO</i>	2006	0.2	597.7	10.7	31.29	51.67776	3.65	19.570
<i>NO₂</i>	2003	2.30	229.1	45.1	47.417	22.18	0.935	2.814
<i>SO₂</i>	2077	0.1	67.3	4.1	5.013	4.16	7.57	89.02

Box-Cox GEVL distribution is:

$$\ell_{\gamma}(y; \mu, \sigma, \lambda) = -k \log \sigma - \lambda \sum_{i=1}^k \left[1 + \gamma \left(\frac{y_i - \mu}{\sigma} \right) \right]^{-\frac{1}{\gamma}} - \left(1 + \frac{1}{\gamma} \right) \sum_{i=1}^k \log \left[1 + \gamma \left(\frac{y_i - \mu}{\sigma} \right) \right], \quad (3.8)$$

Eq (3.8) gives the log-likelihood for GEVL distribution if $\lambda = 1$, and GPDl distribution if $\lambda = 0$. Analytically we cannot maximize the log-likelihood function in Eq (3.8), so we get the negative log-likelihood for minimization by using *FindMinimum* function in mathematica Package. Table 2 summarizes the MLE results, including parameter estimates, Akaike Information Criteria (AIC) and Bayesian Akaike Criteria (BIC), for comparison the models, which have small values of AIC and BIC are more favorable. Also, we used Kolmogorov-Smirnov (K-S) test, where in this test we have four parameters $[h, p, K, c.v]$. Namely, h is equal to 0 or 1, p is the p -value, K is the maximum difference between fitting curve and the empirical distribution and $c.v$ is a critical value. Therefore, the null hypotheses is rejected when $h = 1$ and $K > c.v$ and accepted when $h = 0$, and $K \leq c.v$

Table 2: The MLE results, including parameter estimates, (AIC) and (BIC)

Estimation parameter for maximum data by using GEVL model					
Pollutant	γ	σ	μ	AIC	BIC
<i>NO</i>	1.075	7.907	6.856	8518.99	17044.8
<i>NO₂</i>	-0.0396	18.3	37.53	8944.9	17900
<i>SO₂</i>	0.1806	1.6008	3.634	4473.51	8957.94
Estimation parameter for peak over threshold data by GPDl model					
Pollutant	γ	σ	μ (thrushold)	AIC	BIC
<i>NO</i>	0.1634	50.32	31.227	2761.08	5529.04
<i>NO₂</i>	-0.07077	20.34	47.4170	3677.32	7363.14
<i>SO₂</i>	0.396	1.996	5.0138	1351.04	2709.48

Table 3: K-S test

Fitting parameter for GEVL					
Pollutant	h	p	K	$c.v$	Decision
<i>NO</i>	1	0.06	0.032	0.0272	reject model
<i>NO₂</i>	0	0.1662	0.0211	0.0273	accepted model
<i>SO₂</i>	1	7.99×10^{-7}	0.0245	0.0636	reject model
Fitting parameter for GPDl					
Pollutant	h	p	K	$c.v$	Decision
<i>NO</i>	1	0	0.2916	0.0272	reject model
<i>NO₂</i>	1	0	0.5372	0.0273	reject model
<i>SO₂</i>	1	0	0.6909	0.0268	reject model

From Table 3 we reject the GEVL model for *NO*, *SO₂* and accept this model for *NO₂* pollutant (see Figure 1). Also, we reject the GPDl model for all pollutant, so we applied the Box-Cox GEVL for block maximum data and peak over threshold data. The estimate parameter, AIC, BIC and Kolmogorov-Smirnov test for the Box-GEVL model are summarized in Table 4 and 5. Table 5 show that this model is accepted for extreme data and peak over threshold data (see Figure 2-7).

Figure 1: The GEVL model for daily maximum of NO_2 concentration for Barking Dagenham station

Table 4: The MLE results, including parameter estimates, (AIC) and (BIC)

Estimation parameter for maximum data by using Box-Cox GEVL.						
Pollutant	γ	σ	μ	λ	AIC	BIC
NO	0.13	6.6	6.63	0.985	1005.6	20021.92
NO_2	-0.04	19	38	0.999	8952.09	17908.6
SO_2	0.187	1.79	2.7	0.996	3790.51	8595.58
Estimation parameter for peak over threshold data by using Box-Cox GEVL						
Pollutant	γ	σ	$\mu(\text{thrushold})$	λ	AIC	BIC
NO	0.19	51.6	31.2	0.0004	2743.74	5516.65
NO_2	-0.066	20	47.417	0.01	3584.06	7319.45
SO_2	0.1	0.826	5.1	0.0002	1241.53	2692

Table 5: K-S test

Fitting parameter for Box-Cox GEVL model for maximum data					
Pollutant	h	p	K	$c.v$	Decision
NO	0	0.7966	0.025	0.0272	accepted model
NO_2	0	0.0892	0.0245	0.0273	accepted model
SO_2	0	0.977	0.0023	0.0268	accepted model
Fitting parameter for Box-Cox GEVL model for peak over thrushold (average).					
Pollutant	h	p	K	$c.v$	Decision
NO	0	0.2795	0.0340	0.0523	accepted model
NO_2	0	0.2746	0.0262	0.0486	accepted model
SO_2	0	0.51030	0.0226	0.0480	accepted model

Figure 2: *The Box-Cox GEVL for daily maximum of NO concentration for Barking Dagenham station*

Figure 3: *The Box-Cox GEVL for daily maximum of NO₂ concentration for Barking Dagenham station*

Figure 4: *The Box-Cox GEVL for daily maximum of SO₂ concentration for Barking Dagenham station*

Figure 5: *The Box-Cox GEVL for peak over average of NO concentration for Barking Dagenham station*

Figure 6: *The Box-Cox GEVL for peak over average of NO₂ concentration for Barking Dagenham station*

Figure 7: *The Box-Cox GEVL for peak over average of SO₂ concentration for Barking Dagenham station*

In the second trend, we applied The generalized extreme value distribution under power normalization (GEVP) in Eq. 3.2 for daily maximum data and GPDP model for data peak over threshold (mean). As before the log-likelihood of Box-Cox GEVP is

$$\ell_{\gamma}^*(y; a, b, \lambda) = k \log b - \sum_{i=1}^k \log y_i - \lambda \sum_{i=1}^k (1 + \gamma \log ay^b)^{-\frac{1}{\gamma}} - \left(1 + \frac{1}{\gamma}\right) \sum_{i=1}^k \log (1 + \gamma \log ax^b), \quad (3.9)$$

Eq (3.9) gives the log-likelihood for GEVP distribution if $\lambda = 1$, and GPDP distribution if $\lambda = 0$.

Table 6: The MLE results, including parameter estimates, (AIC) and (BIC)

Estimation parameter for maximum data by using GEVP model					
Pollutant	γ	a	b	AIC	BIC
<i>NO</i>	-0.1935	0.2022	0.7993	6150.95	12320.4
<i>NO</i> ₂	-0.2897	0.001	1.8201	9255.05	18520.4
<i>SO</i> ₂	-0.1744	0.0985	1.834	4690.51	9404.53
Estimation parameter for peak over threshold data					
Pollutant	γ	a	b	AIC	BIC
<i>NO</i>	-0.4453	0.2338	0.5749	3090.73	6188
<i>NO</i> ₂	-0.2624	0.00027	2.188	3970.95	7820.68
<i>SO</i> ₂	-0.157	0.0284	2.437	1679.19	3365.77

The estimation parameter AIC, BIC, and K-S test given in Table 6 and 7, indicates that both GEVP for maximum data and GPDP for peak over threshold data failed to fit this data. Hence, we applied the suggested model Box-Cox-GEVP for maximum data and truncated data. Results for AIC, BIC and parameter estimate summarized in Table 8. Moreover, the result of the K-S test is given in Table 9, which indicate that the suggested model fits the given data, except *NO*₂ for maximum data (see, Figure 8-12).

Table 7: K-S test

Fitting parameter for GEVL					
Pollutant	<i>h</i>	<i>p</i>	<i>K</i>	<i>c.v</i>	Decision
<i>NO</i>	1	0.00046	0.043	0.0272	reject model
<i>NO</i> ₂	1	0	0.0656	0.0273	reject model
<i>SO</i> ₂	1	0	0.0245	0.1063	reject model
Fitting parameter for GPDL					
Pollutant	<i>h</i>	<i>p</i>	<i>K</i>	<i>c.v</i>	Decision
<i>NO</i>	1	0	0.5454	0.0272	reject model
<i>NO</i> ₂	1	0	0.4613	0.0273	reject model
<i>SO</i> ₂	1	0	0.5505	0.0268	reject model

4 Discussion and conclusion

We proposed additional GEV distribution referred to as the Box-Cox GEV distribution and we studied its mathematical properties. We have estimated the parameters of the Box-Cox GEV distribution by the method of maximum likelihood and gave explicit expressions for log likelihood function. We applied two types of models for air pollution in one station of London air quality network called Barking Dagenham. The pollutants *NO*, *NO*₂ and *SO*₂ are modeled by GEVL for daily maximum and GPDL for peak over threshold data which are rejected except the pollutant *NO*₂ for maximum data with AIC=8952.09 and BIC=17908.6. The Box-Cox GEV are applied for maximum data and Peak over threshold data. This model was accepted for *NO*, *NO*₂ and *SO*₂ for maximum data with AIC=1000.5, 8952.09, 3790.51 and BIC=20021.92, 17908.6, 8595.58 respectively, also accepted for threshold data over mean with AIC=2743.73, 3584.06, 1241.5 and BIC=5516.65, 7319.45, 2692. The Box-Cox GEV is more favorable to model the given data of the pollutant except for *NO*₂ which is accepted in Box-Cox GEV and GEVL for maximum data. However, *NO*₂ had smaller values of AIC and BIC in GEVL

Figure 8: *The Box-Cox GEVP model for daily maximum of NO concentration for Barking Dagenham station*

Figure 9: *The Box-Cox GEVP model for daily maximum of SO₂ concentration for Barking Dagenham station*

Figure 10: *The Box-Cox GEVP model for peak over average of NO concentration for Barking Dagenham station*

Figure 11: *The Box-Cox GEVP model for peak over average of NO_2 concentration for Barking Dagenham station*

Figure 12: *The Box-Cox GEVP model for peak over average of SO_2 concentration for Barking Dagenham station*

Table 8: The MLE results, including parameter estimates, (AIC) and (BIC)

Estimation parameter for maximum data by using Box-Cox GEVP.						
Pollutant	γ	a	b	λ	AIC	BIC
<i>NO</i>	-0.142	0.385	1.25	0.999999	5330.96	10676.5
<i>NO₂</i>	-0.29	0.001429	1.84	0.999999999999	9041.58	18094
<i>SO₂</i>	-0.14	0.15	1.9	9.9×10^9	3796.51	78854.2
Estimation parameter for peak over threshold data by using Box-Cox GEVP						
Pollutant	γ	a	b	λ	AIC	BIC
<i>NO</i>	-0.4	0.077	0.77	0.001	2609.32	5427
<i>NO₂</i>	-0.23	0.00029	2.2	0.012	3026.37	7064.08
<i>SO₂</i>	-0.011	0.04	2.11	0.009	1180.55	2570.95

Table 9: K-S test

Fitting parameter for Box-Cox GEVP model for maximum data					
Pollutant	<i>h</i>	<i>p</i>	<i>K</i>	<i>c.v</i>	Decision
<i>NO</i>	0	0.9931	0.0012	0.0272	accepted model
<i>NO₂</i>	1	2.7×10^{-6}	0.0564	0.0273	rejected model
<i>SO₂</i>	0	0.9406	0.0038	0.0268	accepted model
Fitting parameter for Box-Cox GEVP model for peak over threshold (average).					
Pollutant	<i>h</i>	<i>p</i>	<i>K</i>	<i>c.v</i>	Decision
<i>NO</i>	0	0.3349	0.0315	0.0523	accepted model
<i>NO₂</i>	0	0.0657	0.0381	0.040	accepted model
<i>SO₂</i>	0	0.1136	0.0409	0.0480	accepted model

which is better. The second trend of study includes the application of the model of GEVP and GPDP for data as before, which are rejected (see Table 7). Therefore, a suggested model Boc-Cox GEVP was applied. This new model was accepted for *NO* and *SO₂* with AIC= 5330.96, 3796.51 and BIC=10676.5, 73854.2 respectively and it was rejected for *NO₂* for maximum data. Moreover, it was accepted for *NO*, *NO₂* and *SO₂* with AIC= 2609.32, 3026.37, 1180.55 and BIC=5427, 7064.08, 2570.959 for peak over threshold data . It could be concluded that in this study the models from Boc-Cox GEVP is better than Boc-Cox GEVL based on AIC and BIC criteria for pollutant *NO* and *SO₂*, while, GEVL model is the best for pollutant *NO₂*.

Acknowledgments

The authors would like to thank the Editor and the referee for careful reading and comments which greatly improved the paper.

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