EXPONENTIATED PARETO DISTRIBUTION: A BAYES STUDY UTILIZING MCMC TECHNIQUE UNDER UNIFIED HYBRID CENSORING SCHEME

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Abstract: This article aims to study the problem of point and interval estimations of the exponentiated Pareto distribution utilizing unified hybrid censored scheme (HCS). We utilize three methods, including the maximum likelihood, parametric bootstrap and Bayes of estimating the unknown parameters, reliability, hazard rate functions and coefficient of variation. Furthermore, Markov Chain Monte Carlo samples utilizing importance sampling scheme are utilized to generate the Bayes estimates and the credible intervals for unknown quantities. The findings of Bayes method computed using balanced loss function. The suggested methods can be understood by analysing a set of real data.

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1 Introduction

In life-testing and reliability studies, it is preferred to use a censoring scheme that can balance the total time spent on the experiment, the number of units utilized in the experiment and the efficiency of statistical analysis under the findings of the experiment. Type-I (time) and Type-II (failure) censoring schemes are deemed to be two of the most known censoring schemes. In Type-I, the experimental time is fixed and the number of observed failures is random. On the other hand, in Type-II the number of observed failures is fixed and the experimental time is random. More above, the two types do not allow units removal at certain points, other than the experiment terminal point. In order to overcome the challenge of inflexibility in the traditional Type I and Type II censoring schemes, more comprehensive censoring schemes are proposed. Many authors presented several hybrid censoring schemes (HCS), including Type-I HCS (Epstein [1]), Type-II HCS (Childs et al. [2]) and generalized HCS, where it contains generalized Type-I and Type-II HCS (see, Chandrasekar et al. [3]). Generalized HCS have some drawbacks, for instance in generalized Type-I HCS the experiment terminated when the same time or before T , so we cannot ensure observing r failures. On the other hand, in the generalized Type-II HCS, we cannot observe any failure at all or observe only a few number of failures until the pre-fixed time T_2 . To overcome the problems of these schemes, Balakrishnan et al. [4] presented a unified HCS which can be specified as follows:

Fix integers $r, k \in \{1, \dots, n\}$ where $k < r < n$ and $T_1, T_2 \in (0, \infty)$ where $T_2 > T_1$. When k^{th} failure occurs before time T_1 , the experiment is terminated at $\min\{\max\{X_{r:n}, T_1\}, T_2\}$. When k^{th} failure occurs between T_1 and T_2 , the experiment is terminated at $\min\{X_{r:n}, T_2\}$ and when k^{th} failure occurs after time T_2 , the experiment is terminated at $X_{k:n}$. Therefore, depending on this censoring scheme, we can ensure that the experiment would be finished at most in time T_2 with at least k failures and if not, we can ensure exactly k failures. Thence, based on this unified HCS, we get the following six cases:

Case I: $0 < x_{k:n} < x_{r:n} < T_1 < T_2$, the experiment terminated when T_1 .

Case II: $0 < x_{k:n} < T_1 < x_{r:n} < T_2$, the experiment terminated when $x_{r:n}$.

Case III: $0 < x_{k:n} < T_1 < T_2 < x_{r:n}$, the experiment terminated when T_2 .

Case IV: $0 < T_1 < x_{k:n} < x_{r:n} < T_2$, the experiment terminated when $x_{r:n}$.

Case V: $0 < T_1 < x_{k:n} < T_2 < x_{r:n}$, the experiment terminated when T_2 .

Case VI: $0 < T_1 < T_2 < x_{k:n} < x_{r:n}$, the experiment terminated when $x_{k:n}$.

For more information about generalized HCS and unified HCS see, Epstein [1], Chandrasekar et al. [3], Balakrishnan et al. [4], Balakrishnan and Kundu [5], Panahi and Sayyareh [6], Shafay [7], Mahmoud and Ghazal [8] and Ghazal and Hasaballah [9].

Gupta et al. [10] proposed that the exponentiated Pareto distribution (EPD) can be utilized quite effectively in analyzing many lifetime data. Ali et al. [11, 12] studied several exponentiated distributions, including, EPD and discussed their properties. They indicated that the EPD is a good fit to the tail-distribution for NASDAQ data. Shawky and Abu-Zinadah [13] examined different estimators of the unknown parameters based on EPD. Afify [14] discussed estimation of the EPD under type I and type II censoring schemes. Finally, Mahmoud et al. [15] explored the estimation of parameters for the EPD depending on progressively type-II right censored data. The probability density function (PDF) and the cumulative distribution function (CDF) of the EPD with two shape parameters β and θ are given, respectively, by

$$f(x; \beta, \theta) = \beta \theta (1+x)^{-(\beta+1)} \left(1 - (1+x)^{-\beta}\right)^{\theta-1}, \quad x > 0, \quad \beta, \theta > 0, \quad (1.1)$$

and

$$F(x; \beta, \theta) = \left(1 - (1+x)^{-\beta}\right)^{\theta}. \quad (1.2)$$

Futher, the reliability $R(t)$ and hazard rate $H(t)$ functions of the EPD(β, θ) at time t are given, respectively, by

$$R(t) = 1 - \left(1 - (1+t)^{-\beta}\right)^{\theta}, \quad t > 0, \quad (1.3)$$

and

$$H(t) = \frac{\beta \theta (1+t)^{-(\beta+1)} \left(1 - (1+t)^{-\beta}\right)^{\theta-1}}{1 - \left(1 - (1+t)^{-\beta}\right)^{\theta}}, \quad t > 0. \quad (1.4)$$

Many scientific areas such as engineering, economics, biology, and psychology have utilized the coefficient of variation ($CV(X)$) in queueing and reliability theory (see, Sharma and Krishna [16]). $CV(X)$ of the EPD(β, θ) is

$$CV(X) = \frac{\sqrt{\text{Var}(X)}}{E(X)}, \quad E(X) > 0, \quad (1.5)$$

where $E(X)$ and $\text{Var}(X)$ are given, respectively, by

$$E(X) = \theta B\left(1 - \frac{1}{\beta}, \theta\right) - 1, \quad \beta > 1, \quad (1.6)$$

$$\text{Var}(X) = \theta B\left(1 - \frac{2}{\beta}, \theta\right) - \theta^2 \left(B\left(1 - \frac{1}{\beta}, \theta\right)\right)^2, \quad \beta > 2, \quad (1.7)$$

where $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is Beta function. According to Equation (1.5), the $CV(X)$ for the EPD(β, θ) is given by

$$CV(X) = \frac{\sqrt{\theta B\left(1 - \frac{2}{\beta}, \theta\right) - \theta^2 \left(B\left(1 - \frac{1}{\beta}, \theta\right)\right)^2}}{\theta B\left(1 - \frac{1}{\beta}, \theta\right) - 1}, \quad \beta > 2. \quad (1.8)$$

2 Estimation of the Parameters

In the present section, we estimate β and θ through considering the maximum likelihood estimators (MLEs) as well as $R(t)$, $H(t)$ and $CV(X)$. We calculate the observed Fisher information under the likelihood equations $I_{ij} = \frac{\partial^2 \ell(\underline{x}; \beta, \theta)}{\partial \phi_i \partial \phi_j}$ where $\phi_1 = \beta$, $\phi_2 = \theta$ for $i, j = 1, 2$. Also, we compute asymptotic confidence intervals (CIs) for β , θ , $R(t)$, $H(t)$ and $CV(X)$ utilizing the normality property of corresponding MLEs.

2.1 Maximum likelihood estimation

Assumeing that $X_{1:n} < X_{2:n} < \dots < X_{n:n}$ be a random sample of size n from $EPD(\beta, \theta)$, according to the unified HCS described in Section 1, the likelihood function takes the form

$$L(\underline{x}; \beta, \theta) = \frac{n!}{(n-D)!} \prod_{i=1}^D f(x_i) \left(1 - F(T^*)\right)^{n-D}, \quad (2.1)$$

where

$$(D, T^*) = \begin{cases} (D, T_1), & D_1 = D_2 = D = r, \dots, n, & \text{for Case I,} \\ (r, x_{r:n}), & D_1 = k, \dots, r-1; D_2 = r, & \text{for Case II,} \\ (D_2, T_2), & D_1 = k, \dots, r-1; D_2 = k, \dots, r-1; D_1 < D_2, & \text{for Case III,} \\ (r, x_{r:n}), & D_1 = 0, \dots, k-1; D_2 = r, & \text{for Case IV,} \\ (D_2, T_2), & D_1 = 0, \dots, k-1; D_2 = k, \dots, r-1, & \text{for Case V,} \\ (k, x_{k:n}), & D_2 = 0, \dots, k-1, & \text{for Case VI.} \end{cases} \quad (2.2)$$

Where, D_m indicates the number of failures till time T_m , $m = 1, 2$. Utilizing Equations (1.2) and (1.1) in Equation (2.1) and ignoring the normalized constant, the likelihood function of β and θ is obtained as

$$L(\underline{x}; \beta, \theta) \propto \beta^D \theta^D e^{-(\beta+1) \sum_{i=1}^D \ln(1+x_i)} e^{(\theta-1) \sum_{i=1}^D \ln \left(1 - (1+x_i)^{-\beta}\right)} \left(1 - \left(1 - (1+T^*)^{-\beta}\right)^\theta\right)^{n-D}. \quad (2.3)$$

The log-likelihood function $\ell(\underline{x}; \beta, \theta)$ of β and θ is obtained from Equation (2.3) as

$$\begin{aligned} \ell(\underline{x}; \beta, \theta) &= D \ln \beta + D \ln \theta - (\beta+1) \sum_{i=1}^D \ln(1+x_i) + (\theta-1) \sum_{i=1}^D \ln \left(1 - (1+x_i)^{-\beta}\right) \\ &\quad + (n-D) \ln \left(1 - \left(1 - (1+T^*)^{-\beta}\right)^\theta\right). \end{aligned} \quad (2.4)$$

Differentiating Equation (2.4) with respect to β and θ and equating each result to zero, we obtaine likelihood equations as

$$\begin{aligned} \frac{\partial \ell(\underline{x}; \beta, \theta)}{\partial \beta} &= \frac{D}{\beta} - \sum_{i=1}^D \ln(1+x_i) + (\theta-1) \sum_{i=1}^D \frac{\ln(1+x_i)}{(1+x_i)^\beta - 1} \\ &\quad - \frac{\theta(n-D) \left(1 - (1+T^*)^{-\beta}\right)^{\theta-1} (1+T^*)^{-\beta} \ln(1+T^*)}{1 - \left(1 - (1+T^*)^{-\beta}\right)^\theta} = 0, \end{aligned} \quad (2.5)$$

and

$$\frac{\partial \ell(\underline{x}; \beta, \theta)}{\partial \theta} = \frac{D}{\theta} + \sum_{i=1}^D \ln \left(1 - (1+x_i)^{-\beta}\right) - \frac{(n-D) \ln \left(1 - (1+T^*)^{-\beta}\right)}{\left(1 - (1+T^*)^{-\beta}\right)^{-\theta} - 1} = 0. \quad (2.6)$$

Equations (2.5) and (2.6) cannot be solved analytically for β and θ , hence, we utilize numerical methods such as Newton's Raphson method to compute maximum likelihood estimates for β and θ , say $\hat{\beta}_{ML}$ and $\hat{\theta}_{ML}$, respectively. Consequently, utilizing the invariance property, the MLEs of $R(t)$, $H(t)$ and $CV(X)$ can be computed after substituting β and θ by $\hat{\beta}_{ML}$

and $\hat{\theta}_{\text{ML}}$, as

$$\begin{cases} \hat{R}_{\text{ML}}(t) = 1 - \left(1 - (1+t)^{-\hat{\beta}_{\text{ML}}}\right)^{\hat{\theta}_{\text{ML}}}, \\ \hat{H}_{\text{ML}}(t) = \frac{\hat{\beta}_{\text{ML}} \hat{\theta}_{\text{ML}} (1+t)^{-(\hat{\beta}_{\text{ML}}+1)} \left(1 - (1+t)^{-\hat{\beta}_{\text{ML}}}\right)^{\hat{\theta}_{\text{ML}}-1}}{1 - \left(1 - (1+t)^{-\hat{\beta}_{\text{ML}}}\right)^{\hat{\theta}_{\text{ML}}}}, \\ \widehat{CV}_{\text{ML}}(X) = \frac{\sqrt{\hat{\theta}_{\text{ML}} B\left(1 - \frac{2}{\hat{\beta}_{\text{ML}}}, \hat{\theta}_{\text{ML}}\right) - \hat{\theta}_{\text{ML}}^2 \left(B\left(1 - \frac{1}{\hat{\beta}_{\text{ML}}}, \hat{\theta}_{\text{ML}}\right)\right)^2}}{\hat{\theta}_{\text{ML}} B\left(1 - \frac{1}{\hat{\beta}_{\text{ML}}}, \hat{\theta}_{\text{ML}}\right) - 1}. \end{cases} \quad (2.7)$$

2.2 Approximate Interval Estimation

The Fisher information matrix $I(\beta, \theta)$ is then obtained by expecting $I_{ij} = -\frac{\partial^2 \ell(\underline{x}; \beta, \theta)}{\partial \phi_i \partial \phi_j}$ where $\phi_1 = \beta$, $\phi_2 = \theta$ for $i, j = 1, 2$. Evidently, obtaining the exact expressions of the above expectation represents a challenge. Accordingly, we extract the approximate asymptotic variance-covariance matrix of MLEs given as

$$I_0(\hat{\beta}, \hat{\theta}) = [\hat{V}] = \begin{pmatrix} -\frac{\partial^2 \ell(\underline{x}; \beta, \theta)}{\partial \beta^2} & -\frac{\partial^2 \ell(\underline{x}; \beta, \theta)}{\partial \beta \partial \theta} \\ -\frac{\partial^2 \ell(\underline{x}; \beta, \theta)}{\partial \theta \partial \beta} & -\frac{\partial^2 \ell(\underline{x}; \beta, \theta)}{\partial \theta^2} \end{pmatrix}_{(\hat{\beta}, \hat{\theta})}^{-1} = \begin{pmatrix} \widehat{\text{Var}}(\hat{\beta}) & \text{Cov}(\hat{\beta}, \hat{\theta}) \\ \text{Cov}(\hat{\beta}, \hat{\theta}) & \widehat{\text{Var}}(\hat{\theta}) \end{pmatrix}. \quad (2.8)$$

Thus, it can be utilized to calculate the $100(1-\tau)\%$ approximate CIs for the parameters β and θ which are given, respectively, by

$$\left(\hat{\beta}_{\text{ML}} \mp Z_{\frac{\tau}{2}} \sqrt{\widehat{\text{Var}}(\hat{\beta}_{\text{ML}})}\right), \quad \left(\hat{\theta}_{\text{ML}} \mp Z_{\frac{\tau}{2}} \sqrt{\widehat{\text{Var}}(\hat{\theta}_{\text{ML}})}\right), \quad (2.9)$$

where $Z_{\frac{\tau}{2}}$ is the percentile of $N(0, 1)$ with right-tail probability $\frac{\tau}{2}$.

Moreover, to construct the asymptotic CIs of the reliability, hazard functions and coefficient of variation, we need to compute the variances of them by utilizing the delta method, see Greene [17]. Assume that $A_1 = \begin{pmatrix} \frac{\partial R(t)}{\partial \beta} & \frac{\partial R(t)}{\partial \theta} \end{pmatrix}$, $A_2 = \begin{pmatrix} \frac{\partial H(t)}{\partial \beta} & \frac{\partial H(t)}{\partial \theta} \end{pmatrix}$ and $A_3 = \begin{pmatrix} \frac{\partial CV(X)}{\partial \beta} & \frac{\partial CV(X)}{\partial \theta} \end{pmatrix}$, where $\frac{\partial R(t)}{\partial \beta}$, $\frac{\partial R(t)}{\partial \theta}$, $\frac{\partial H(t)}{\partial \beta}$, $\frac{\partial H(t)}{\partial \theta}$, $\frac{\partial CV(X)}{\partial \beta}$ and $\frac{\partial CV(X)}{\partial \theta}$ are the first derivatives of the $R(t)$, $H(t)$ and $CV(X)$ with respect to the parameters β and θ , respectively. The approximate asymptotic variances of $\hat{R}_{\text{ML}}(t)$, $\hat{H}_{\text{ML}}(t)$ and $\widehat{CV}_{\text{ML}}(X)$ can be computed, respectively, by

$$\widehat{\text{Var}}(\hat{R}_{\text{ML}}(t)) = \left(A_1 \hat{V} A_1^T\right)_{(\hat{\beta}, \hat{\theta})}, \quad \widehat{\text{Var}}(\hat{H}_{\text{ML}}(t)) = \left(A_2 \hat{V} A_2^T\right)_{(\hat{\beta}, \hat{\theta})}, \quad \widehat{\text{Var}}(\widehat{CV}_{\text{ML}}(X)) = \left(A_3 \hat{V} A_3^T\right)_{(\hat{\beta}, \hat{\theta})}, \quad (2.10)$$

where, A_k^T is the transpose of A_k , $k = 1, 2, 3$. These findings yield the approximate CIs for $R(t)$, $H(t)$ and $CV(X)$ as follows:

$$\left(\hat{R}_{\text{ML}}(t) \mp Z_{\frac{\tau}{2}} \sqrt{\widehat{\text{Var}}(\hat{R}_{\text{ML}}(t))}\right), \left(\hat{H}_{\text{ML}}(t) \mp Z_{\frac{\tau}{2}} \sqrt{\widehat{\text{Var}}(\hat{H}_{\text{ML}}(t))}\right), \left(\widehat{CV}_{\text{ML}}(X) \mp Z_{\frac{\tau}{2}} \sqrt{\widehat{\text{Var}}(\widehat{CV}_{\text{ML}}(X))}\right). \quad (2.11)$$

3 Bootstrap confidence intervals

In the present section, two parametric bootstrap procedures introduced to construct the bootstrap CIs of β , θ , $R(t)$, $H(t)$ and $CV(X)$: The percentile bootstrap (Boot-p) confidence interval by Efron [18] and the bootstrap-t (Boot-t) confidence interval proposed by Hall [19]. The following algorithms show the estimation of CIs utilizing both methods.

3.1 Parametric Boot-p

1. Utilize the original data $\underline{x} = x_{1:n}, \dots, x_{n:n}$ compute $\hat{\beta}$ and $\hat{\theta}$ via maximizing Equations (2.5) and (2.6).
2. Obtain a bootstrap sample $\underline{x}^* = x_{1:n}^*, \dots, x_{n:n}^*$ through resampling with replacement.
3. Compute the MLEs depending on the bootstrap sample and indicate this bootstrap estimate by $\hat{\varphi}^*$ ($\varphi = \beta, \theta, R(t), H(t)$ or $CV(X)$).

4. Duplicate 2 and 3 NBoot times, then compute $\hat{\varphi}_1^*, \hat{\varphi}_2^*, \dots, \hat{\varphi}_{\text{NBoot}}^*$, $\hat{\varphi}_l^* = (\hat{\beta}_l^*, \hat{\theta}_l^*, \hat{R}_l^*(t), \hat{H}_l^*(t), \widehat{CV}_l^*(X))$, $l = 1, 2, \dots, \text{NBoot}$.
5. Arrange $\hat{\varphi}_l^*$, $l = 1, 2, \dots, \text{NBoot}$ in ascending orders, thence compute $\hat{\varphi}_{(1)}^*, \hat{\varphi}_{(2)}^*, \dots, \hat{\varphi}_{(\text{NBoot})}^*$.
6. Suppose that $\hat{G}_1(u) = P(\hat{\varphi}^* \leq u)$ be the CDF of $\hat{\varphi}^*$. Identify $\hat{\varphi}_{\text{NBootP}} = \hat{G}_1^{-1}(u)$ for given u . The approximate bootstrap-p 100(1 - τ)% CIs for $\hat{\varphi}$, can be obtained as

$$\left(\hat{\varphi}_{\text{NBootP}}\left(\frac{\tau}{2}\right), \hat{\varphi}_{\text{NBootP}}\left(1 - \frac{\tau}{2}\right) \right). \quad (3.1)$$

3.2 Parametric Boot-t

1-3. The same as above-mentioned steps.

4. Based on Equation (2.8) calculate $I^{-1*}(\hat{\beta}^*, \hat{\theta}^*)$.

5. Calculate $T^{*\varphi}$ statistic from $T^{*\varphi} = \frac{(\hat{\varphi}^* - \hat{\varphi})}{\sqrt{\widehat{\text{Var}}(\hat{\varphi}^*)}}$.

6. Duplicate the procedues 2-5, NBoot times, then calculate $T_1^{*\varphi}, T_2^{*\varphi}, \dots, T_{\text{NBoot}}^{*\varphi}$.

7. Sort $T_1^{*\varphi}, T_2^{*\varphi}, \dots, T_{\text{NBoot}}^{*\varphi}$ in ascending orders, then calculate the following ordered sequences

$$T_{(1)}^{*\varphi}, T_{(2)}^{*\varphi}, \dots, T_{(\text{NBoot})}^{*\varphi}.$$

8. Suppose that $\hat{G}_2(u) = P(T^* \leq u)$ be the CDF of T^* . For a given u , define

$$\hat{\varphi}_{\text{NBootT}} = \hat{\varphi} + \hat{G}_2^{-1}(u) \sqrt{\widehat{\text{Var}}(\hat{\varphi}^*)}.$$

Subsequently, the approximate bootstrap-t 100(1 - τ)% CIs of $\hat{\varphi}$ is given by

$$\left(\hat{\varphi}_{\text{NBootT}}\left(\frac{\tau}{2}\right), \hat{\varphi}_{\text{NBootT}}\left(1 - \frac{\tau}{2}\right) \right). \quad (3.2)$$

4 Bayes estimation utilizing MCMC technique

In the present section, we compute Bayesian estimates and the corresponding credible intervals of β and θ in addition to some lifetime parameters, including $R(t)$, $H(t)$ and $CV(X)$. We suppose that the parameters β and θ are independent and follow the gamma prior distributions

$$\begin{cases} \pi_1(\beta) \propto \beta^{a_1-1} e^{-b_1\beta}, & \beta > 0, \\ \pi_2(\theta) \propto \theta^{a_2-1} e^{-b_2\theta}, & \theta > 0. \end{cases} \quad (4.1)$$

Here, all the hyper parameters a_1, a_2, b_1 and b_2 are known and positive. As a consequence, the joint prior distribution $\pi(\beta, \theta)$ for β and θ is

$$\pi(\beta, \theta) \propto \beta^{a_1-1} \theta^{a_2-1} e^{-(b_1\beta + b_2\theta)}. \quad (4.2)$$

The posterior distribution of the parameters β and θ indicated by $\pi^*(\beta, \theta|\underline{x})$ up to proportionality is determined by combining the likelihood function Equation (2.3) with the prior Equation (4.1) via Bayes' theorem, then $\pi^*(\beta, \theta|\underline{x})$ is formulated as

$$\pi^*(\beta, \theta|\underline{x}) = \frac{L(\underline{x}|\beta, \theta) \times \pi_1(\beta) \times \pi_2(\theta)}{\int_0^\infty \int_0^\infty L(\underline{x}|\beta, \theta) \times \pi_1(\beta) \times \pi_2(\theta) d\beta d\theta}. \quad (4.3)$$

Consequently, we can determine the Bayes estimate of any parameters function, such as $h(\beta, \theta)$, based on squared error loss function, as

$$E_{\beta, \theta|\underline{x}}(h(\beta, \theta|\underline{x})) = \frac{\int_0^\infty \int_0^\infty h(\beta, \theta|\underline{x}) \times L(\underline{x}|\beta, \theta) \times \pi_1(\beta) \times \pi_2(\theta) d\beta d\theta}{\int_0^\infty \int_0^\infty L(\underline{x}|\beta, \theta) \times \pi_1(\beta) \times \pi_2(\theta) d\beta d\theta}. \quad (4.4)$$

It may be noted that, the computation of (4.4) cannot be solved analytically. It is due to the complicated form of the likelihood function given in Equation (2.3). Consequently, MCMC technique is utilized to generate samples from Equation (4.3). Then, we use samples to calculate the Bayes estimate of β and θ and any function of them such as $R(t)$, $H(t)$ and $CV(X)$ and to construct associated CIs. To apply the MCMC technique, we use the Gibbs within Metropolis sampler which requires the derivation of the complete set of conditional posterior distribution. From (4.3), $\pi^*(\beta, \theta | \underline{x})$ up to proportionality is formulated as

$$\begin{aligned} \pi^*(\beta, \theta | \underline{x}) &\propto \beta^{R+a_1-1} \theta^{R+a_2-1} e^{-\beta \left(b_1 + \sum_{i=1}^R \ln(1+x_i) \right)} e^{-b_2 \theta + (\theta-1) \sum_{i=1}^R \ln \left(1 - (1+x_i)^{-\beta} \right)} \\ &\times \left(1 - \left(1 - (1+C)^{-\beta} \right)^\theta \right)^{n-R}. \end{aligned} \quad (4.5)$$

From (4.5), the conditional posterior densities of β and θ are given, respectively, by

$$\pi^*(\beta | \theta, \underline{x}) \propto \beta^{R+a_1-1} e^{-\beta \left(b_1 + \sum_{i=1}^R \ln(1+x_i) \right)}, \quad (4.6)$$

$$\pi^*(\theta | \beta, \underline{x}) \propto \theta^{R+a_2-1} e^{-b_2 \theta + (\theta-1) \sum_{i=1}^R \ln \left(1 - (1+x_i)^{-\beta} \right)} \left(1 - \left(1 - (1+C)^{-\beta} \right)^\theta \right)^{n-R}. \quad (4.7)$$

The conditional posterior density of β in Equation (4.6) is gamma density with shape parameter $(R+a_1)$ and scale parameter $\left(b_1 + \sum_{i=1}^R \ln(1+x_i) \right)$. Subsequently, samples of β can be easily generated utilizing any gamma generating routine. The conditional posterior distribution of θ in Equation (4.7) can not be present in standard form, but its plot is similar to normal distribution. So, Gibbs sampling is not a straight-forward option. Using Metropolis-Hasting (M-H) sampler is necessary for applying MCMC methodology. The following procedures show the process of the M-H algorithm within Gibbs sampling:

1. For given t , start with initial guess (β^0, θ^0) .
2. Put $j = 1$.
3. Generate β^j from Gamma $\left(R + a_1, \left(b_1 + \sum_{i=1}^R \ln(1+x_i) \right) \right)$.
4. Use the following M-H algorithm, generate $\theta^{(j)}$ from $\pi_2^*(\theta^{j-1} | \beta^j, \underline{x})$ with $N(\theta^{(j-1)}, \sigma^2(\theta))$.
 - (i) Generate a proposal θ^* from $N(\theta^{(j-1)}, \sigma^2(\theta))$.
 - (ii) Compute the acceptance probability
$$\rho = \min \left[1, \frac{\pi_2^*(\theta^* | \beta^j, \underline{x})}{\pi_2^*(\theta^{j-1} | \beta^j, \underline{x})} \right].$$
 - (iii) Generate u from a Uniform $(0, 1)$ distribution.
 - (iv) If $u < \rho$. accept the proposal and put $\theta^{(j)} = \theta^*$. Otherwise, put $\theta^{(j)} = \theta^{(j-1)}$.
5. Calculate $R(t)$, $H(t)$ and $CV(X)$ as

$$\begin{cases} R^{(j)}(t) = 1 - \left(1 - (1+t)^{-\beta^{(j)}} \right)^{\theta^{(j)}}, \\ H^{(j)}(t) = \frac{\beta^{(j)} \theta^{(j)} (1+t)^{-(\beta^{(j)}+1)} \left(1 - (1+t)^{-\beta^{(j)}} \right)^{\theta^{(j)}-1}}{1 - \left(1 - (1+t)^{-\beta^{(j)}} \right)^{\theta^{(j)}}}, \\ CV^{(j)}(X) = \frac{\sqrt{\theta^{(j)} B\left(1 - \frac{2}{\beta^{(j)}}, \theta^{(j)}\right) - (\theta^{(j)})^2 \left(B\left(1 - \frac{1}{\beta^{(j)}}, \theta^{(j)}\right) \right)^2}}{\theta^{(j)} B\left(1 - \frac{1}{\beta^{(j)}}, \theta^{(j)}\right) - 1}. \end{cases} \quad (4.8)$$

6. Put $j = j + 1$.

7. Duplicate 3-6, N times.

For ensuring the convergence and removing the affection of the selection of initial value, we have to discard the first M simulated varieties. Thence the selected samples are $\beta^{(j)}$, $\theta^{(j)}$, $R^{(j)}(t)$, $H^{(j)}(t)$ and $CV^{(j)}(X)$, $j = M+1, \dots, N$, for sufficiently large N , compose an approximate posterior sample that is utilized to improve the Bayes estimates of $\Psi = \beta, \theta, R(t), H(t)$ or $CV(X)$ is given by

$$\hat{\Psi}_{MC} = \frac{1}{N-M} \sum_{j=M+1}^N \Psi^{(j)}. \quad (4.9)$$

To calculate the CIs of $\Psi = \beta, \theta, R(t), H(t)$ and $CV(X)$, order $\beta^{(k)}$, $\theta^{(k)}$, $R^{(k)}(t)$, $H^{(k)}(t)$ and $CV^{(k)}(X)$, $k = 1, \dots, N$ as $\{\beta^{(1)} < \dots < \beta^{(N)}\}$, $\{\theta^{(1)} < \dots < \theta^{(N)}\}$, $\{R^{(1)}(t) < \dots < R^{(N)}(t)\}$, $\{H^{(1)}(t) < \dots < H^{(N)}(t)\}$ and $\{CV^{(1)}(X) < \dots < CV^{(N)}(X)\}$. Thus, the $100(1 - \tau)\%$ approximate CIs for $\beta, \theta, R(t), H(t)$ or $CV(X)$ become

$$\left[\Psi_{\left(N \frac{\tau}{2}\right)}, \Psi_{\left(N \left(1 - \frac{\tau}{2}\right)\right)} \right]. \quad (4.10)$$

5 Bayes estimation utilizing balanced loss functions

Asymmetric loss function is often choosen, since it makes the statistical inferences more practical and applicable. Linear-exponential loss function (see Varian [20]) and General entropy loss function (see Calabria et al. [21]) are the most well-known asymmetric loss functions. However, a more generalized loss function named the balanced loss function introduced by Jozani et al. [22] is given by

$$L_{\rho, \omega, \delta_0}(\Theta, \delta) = \omega \rho(\delta, \delta_0) + (1 - \omega) \rho(\Theta, \delta), \quad (5.1)$$

where $\rho(\Theta, \delta)$ is an arbitrary loss function and δ_0 is chosen as a prior ‘target’ estimator of Θ , computed for the utilization of the standard of maximum likelihood, least-squares or unbiasedness. Depending on the observed value of $\delta_0(X)$, loss $L_{\rho, \omega, \delta_0}$ reflects a desire of closeness of δ to the target estimator δ_0 and the unknown parameter Θ . With the relative importance of these criteria governed by the choice of $0 \leq \omega < 1$. $L_{\rho, \omega, \delta_0}$ is specified to different choices of loss function for instance, for absolute value, balanced squared error loss (BSEL), balanced linear-exponential loss (BLINEXL) and balanced general entropy loss (BGEL) functions.

If we put $\rho(\Theta, \delta) = (\Theta - \delta)^2$ in Equation (5.1), it leads to BSEL function, see Ahmadi et al. [23], is

$$L_{\omega, \delta_0}(\Theta, \delta) = \omega (\delta - \delta_0)^2 + (1 - \omega) (\delta - \Theta)^2, \quad (5.2)$$

then the Bayes estimator of the unknown parameter Θ based on BSEL is

$$\delta_{\omega, \delta_0}(\underline{x}) = \omega \delta_0(\underline{x}) + (1 - \omega) E(\Theta | \underline{x}). \quad (5.3)$$

Also, if we put $\rho(\Theta, \delta) = e^{c(\delta - h(\Theta))} - c(\Theta - \delta) - 1$ with shape parameter c , $c \neq 0$ in Equation (5.1), it leads to BLINEXL function, see Zellner [24]. Thus, the Bayes estimation of Θ based on BLINEXL function is

$$\delta_{\omega}(\underline{x}) = \frac{-1}{c} \ln \left[\omega e^{-c \delta_0} + (1 - \omega) E(e^{-c \Theta} | \underline{x}) \right]. \quad (5.4)$$

In addition, the BGEL function with shape parameter q is obtained with the choice of $\rho(\Theta, \delta) = \left(\frac{\delta}{\Theta}\right)^q - q \ln\left(\frac{\delta}{\Theta}\right) - 1$. Thence, the Bayes estimation of Θ based on BGEL function is

$$\delta_{\omega}(\underline{x}) = \left[\omega (\delta_0)^{-q} + (1 - \omega) E(\Theta^{-q} | \underline{x}) \right]^{\frac{-1}{q}}. \quad (5.5)$$

The balanced loss functions are more comprehensive, since they include the MLE and both symmetric and asymmetric Bayes estimates as special cases. For instance, when $\omega = 1$ in Equation (5.3) the Bayes estimate based on BSEL function limited

to ML estimate and when $\omega = 0$ it limited to the Bayes estimate relative to SEL function (symmetric). As well, the Bayes estimator based on BLINEXL function in (5.4) limited to ML estimate when $\omega = 1$ and if $\omega = 0$ it limited to the case of LINEXL function (asymmetric). Furthermore, the Bayes estimator based on BGEL function reduces to MLE if $\omega = 1$ in Equation (5.5) and when $\omega = 0$, it reduces to the case of general entropy loss function. When $\Theta = (\beta, \theta, R(t), H(t), CV(X))$ and according to MCMC technique described above, the approximate posterior means under BSEL, BLINEXL and BGEL functions can be formulated, respectively, by

$$E[\Theta] = \omega \delta_0(\underline{x}) + \frac{1 - \omega}{N - M} \sum_{j=M+1}^N \Theta^{(j)}, \quad (5.6)$$

$$E[\Theta] = \frac{-1}{c} \ln \left[\omega e^{-c \delta_0(\underline{x})} + \frac{1 - \omega}{N - M} \sum_{j=M+1}^N e^{-c \Theta^{(j)}} \right], \quad (5.7)$$

$$E[\Theta] = \left[\omega (\delta_0(\underline{x}))^{-q} + \frac{1 - \omega}{N - M} \sum_{j=M+1}^N (\Theta^{(j)})^{-q} \right]^{-q}. \quad (5.8)$$

Subsequently, the approximate Bayes estimates for $\Theta = \beta, \theta, R(t), H(t)$ or $CV(X)$ based on BSEL, BLINEXL and BGEL functions can be computed, respectively, by

$$\hat{\Theta}_{BS} = \omega \hat{\Theta} + \frac{1 - \omega}{N - M} \sum_{j=M+1}^N \Theta^{(j)}, \quad (5.9)$$

$$\hat{\Theta}_{BL} = \frac{-1}{c} \ln \left[\omega e^{-c \hat{\Theta}} + \frac{1 - \omega}{N - M} \sum_{j=M+1}^N e^{-c \Theta^{(j)}} \right], \quad (5.10)$$

$$\hat{\Theta}_{BGE} = \left[\omega (\hat{\Theta})^{-q} + \frac{1 - \omega}{N - M} \sum_{j=M+1}^N (\Theta^{(j)})^{-q} \right]^{-q}. \quad (5.11)$$

6 Application a set of real data

We introduce an example utilizing a set of real data to explain the computations of the methods proposed in this article. The set of real data is extracted from Ghazal and Hasaballah [25]. The K-S test value is 0.0658573 of the EPD. From Figure 1, it is clearly seen that the EPD is a good model fitting this data. To facilitate the computation processes, we have divided each data point by 10. According to unified HCS, we consider the six cases as following:

Figure 1: Empirical and Fitted Survival Functions.

- Case I: $T_1 = 0.8, T_2 = 1.1, k = 60, r = 65$. ($D = 67, T^* = T_1 = 0.8$).
- Case II: $T_1 = 0.8, T_2 = 1.1, k = 60, r = 70$. ($D = 70, T^* = x_{r:n} = 0.82$).
- Case III: $T_1 = 0.8, T_2 = 1.1, k = 60, r = 92$. ($D = 90, T^* = T_2 = 1.1$).
- Case IV: $T_1 = 0.8, T_2 = 1.1, k = 80, r = 85$. ($D = 85, T^* = x_{r:n} = 0.99$).
- Case V: $T_1 = 0.8, T_2 = 1.05, k = 83, r = 91$. ($D = 88, T^* = T_2 = 1.05$).
- Case VI: $T_1 = 0.6, T_2 = 1.05, k = 93, r = 94$. ($D = 93, T^* = x_{k:n} = 1.2$).

The MLEs for β and θ of complete sample ($n = 100$) are $\hat{\beta}_{ML} = 7.45561$ and $\hat{\theta}_{ML} = 28.1245$. Also, using Equation (2.7) with $t = 0.6$ the MLEs for $R(t)$ and $H(t)$ are $\hat{R}_{ML}(t) = 0.576303$ and $\hat{H}_{ML}(t) = 2.98727$, as well as the MLEs for $CV(X)$ is $\widehat{CV}_{ML}(X) = 0.459203$. Based on different methods (MLEs, Boot-p, Boot-t and MCMC) discussed in the above sections, we compute the point and interval estimations for $\beta, \theta, R(t), H(t)$ and $CV(X)$ under unified HCS. To calculate the Bayes estimates for $\beta, \theta, R(t), H(t)$ and $CV(X)$ against BSEL, BLINEXL and BGEL functions, we suppose the informative priors on both β and θ as follows:

Case I ($a_1 = 133$), Case II ($a_1 = 133$), Case III ($a_1 = 225$), Case IV ($a_1 = 205$), Case V ($a_1 = 215$) and Case VI ($a_1 = 240$) and $b_1 = a_2 = b_2 = 0$ in the six cases. Using algorithm in Section 4, we generate $N = 16000$ MCMC samples $\{(\beta^{(j)}, \theta^{(j)}), j = 1, 2, 3, \dots, 16000\}$ with discarding the first $M = 1000$ values as ‘burn-in’. The MCMC findings of the posterior mean, median, mode, standard deviation (SD), standard errors (SE) and skewness for $\beta, \theta, R(t), H(t)$ and $CV(X)$ are reported in Table 1. The estimates and the corresponding 95% CIs of $\beta, \theta, R(t), H(t)$ and $CV(X)$ displayed in Tables 2 and 3. With $c = -7, 0.3.7$ and $q = -7, 0.3.7$, the findings of Bayes estimates relative to BSEL, BLINEXL and BGEL functions with various values of $w = 0, 0.2, 0.5, 0.9$ for $\beta, \theta, R(t), H(t)$ and $CV(X)$ are presented in Tables 4, 5, 6, 7, 8 and 9.

7 Concluding remarks

The aim of this article is to utilize various methods (MLEs, Boot-p, Boot-t and MCMC) to estimate and construct CIs of the parameters β and θ also reliability, hazard functions and coefficient of variation of the exponentiated Pareto distribution based on unified hybrid censoring data. We assume that the gamma priors of β and θ and the Bayes estimators based on the assumptions for balanced squared error loss, balanced linear-exponential loss and general entropy loss functions. In this article, the Bayes estimates cannot be computed in closed form. Thus, we utilize the MCMC method to calculate the approximate Bayes estimates and the corresponding CIs. Further, the details have been explained utilizing a real life data example.

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Table 1: MCMC findings for some posterior characteristics of six cases.

	Parameters	Mean	Median	Mode	SD	SE	Skewness
Case I	β	6.861	6.8517	6.8331	0.4854	6.8782	0.1465
	θ	22.7281	22.3002	21.4444	4.6592	23.2007	0.5576
	$R(t = 0.6)$	0.5959	0.5951	0.5933	0.0938	0.6033	-0.0086
	$H(t = 0.6)$	2.6809	2.6864	2.6972	0.5019	2.7275	-0.0352
	$CV(X)$	0.4964	0.4953	0.493	0.0263	0.4971	0.3936
Case II	β	6.5621	6.5534	6.536	0.4581	6.5781	0.1035
	θ	20.113	19.7736	19.0948	4.0566	20.518	0.4241
	$R(t = 0.6)$	0.6036	0.6023	0.5995	0.0906	0.6104	-0.0064
	$H(t = 0.6)$	2.5437	2.5523	2.5695	0.4649	2.5858	-0.0476
	$CV(X)$	0.5168	0.5152	0.5121	0.028	0.5176	0.5016
Case III	β	7.115	7.1068	7.0906	0.3999	7.1262	0.1322
	θ	24.9147	24.6865	24.23	4.4561	25.31	0.3945
	$R(t = 0.6)$	0.5867	0.5859	0.5843	0.0799	0.5921	0.0154
	$H(t = 0.6)$	2.8116	2.8177	2.8301	0.4264	2.8437	-0.0515
	$CV(X)$	0.4803	0.4791	0.4768	0.0212	0.4808	0.4648
Case IV	β	7.1156	7.1114	7.1032	0.4191	7.1279	0.0658
	θ	24.9453	24.541	23.7324	4.4981	25.3476	0.4303
	$R(t = 0.6)$	0.587	0.5859	0.5836	0.0822	0.5928	0.0053
	$H(t = 0.6)$	2.8106	2.8192	2.8363	0.4435	2.8454	-0.0463
	$CV(X)$	0.4803	0.479	0.4765	0.0215	0.4807	0.3962
Case V	β	7.0745	7.067	7.0518	0.4071	7.0862	0.111
	θ	24.7039	24.4841	24.0445	4.3868	25.0903	0.3756
	$R(t = 0.6)$	0.5907	0.5895	0.5871	0.0804	0.5961	0.0228
	$H(t = 0.6)$	2.7812	2.7883	2.8026	0.4314	2.8144	-0.0534
	$CV(X)$	0.4821	0.4807	0.478	0.0213	0.4826	0.4176
Case VI	β	7.1504	7.1427	7.1274	0.3923	7.1611	0.137
	θ	25.1272	24.7544	24.0089	4.401	25.5097	0.4166
	$R(t = 0.6)$	0.5839	0.583	0.5813	0.0788	0.5892	0.014
	$H(t = 0.6)$	2.8358	2.8392	2.8459	0.4216	2.867	-0.0485
	$CV(X)$	0.4785	0.4776	0.4758	0.0203	0.4789	0.3511

Table 2: Estimates for β , θ , $R(t)$, $H(t)$ and $CV(X)$ under MLE, parametric bootstrap and MCMC technique of six cases.

	Method	β	θ	$R(t = 0.6)$	$H(t = 0.6)$	$CV(X)$
Case I	ML	6.97144	23.1841	0.59029	2.03667	0.48754
	Boot-p	7.0952	24.8524	0.58239	2.82435	0.48319
	Boot-t	7.46417	24.5654	0.57049	3.07851	0.49106
	MCMC	6.85352	22.774	0.5985	2.66933	0.49623
Case II	ML	7.05924	23.9767	0.5872	2.79556	0.48226
	Boot-p	7.10847	25.9136	0.58903	2.81192	0.48336
	Boot-t	7.00733	25.4001	0.59383	2.72784	0.46934
	MCMC	6.56213	20.113	0.60364	2.54371	0.51681
Case III	ML	7.20674	25.448	0.58318	2.86623	0.47339
	Boot-p	7.25657	26.8257	0.58456	2.87815	0.47185
	Boot-t	7.36938	21.2201	0.58647	3.02731	0.53394
	MCMC	7.11497	24.9147	0.58669	2.81155	0.48028
Case IV	ML	7.16195	25.0024	0.58455	2.84412	0.47601
	Boot-p	7.28059	27.2329	0.58389	2.89147	0.47107
	Boot-t	7.64466	26.9221	0.5905	3.19965	0.40925
	MCMC	7.11556	24.9453	0.58705	2.81059	0.48025
Case V	ML	7.189	25.2702	0.58372	2.85748	0.47442
	Boot-p	7.26187	26.8987	0.58511	2.87826	0.47133
	Boot-t	7.30431	22.1696	0.58736	2.98266	0.51331
	MCMC	7.07455	24.7039	0.59067	2.78119	0.48208
Case VI	ML	7.22595	25.6438	0.58263	2.8756	0.47226
	Boot-p	7.3335	27.7874	0.58172	2.92049	0.46819
	Boot-t	7.14026	23.969	0.5781	2.7977	0.52805
	MCMC	7.15036	25.1272	0.58392	2.83584	0.4785

Table 3: 95% confidence lengths for β , θ , $R(t)$, $H(t)$ and $CV(X)$ under MLEs, parametric bootstrap and MCMC technique of six cases.

Cases	Parameters	MLE		Boot-p		Boot-t		MCMC	
		Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
Case I	β	5.57444	8.36845	6.28984	8.13133	7.08997	7.88393	5.94551	7.84689
	θ	9.25782	37.1103	15.6495	41.3592	21.8757	26.8198	14.6674	32.9259
	$R(t = 0.6)$	0.50942	0.67115	0.53016	0.63142	0.49261	0.63629	0.3995	0.78791
	$H(t = 0.6)$	2.03667	3.46563	2.52776	3.17531	2.96291	3.26006	1.65458	3.70527
	$CV(X)$	0.3979	0.57719	0.41706	0.54645	0.32684	0.70127	0.44953	0.55235
Case II	β	5.68382	8.43466	5.67865	8.7106	6.52859	7.45695	5.69059	7.47855
	θ	9.77644	38.1771	13.8901	48.61	22.2252	27.8813	12.7653	29.1825
	$R(t = 0.6)$	0.50678	0.66763	0.49853	0.67581	0.50402	0.6903	0.41843	0.78841
	$H(t = 0.6)$	2.08686	3.50425	2.10467	3.62305	2.53351	2.91348	1.58963	3.47845
	$CV(X)$	0.39594	0.56857	0.39821	0.58255	0.23724	0.72332	0.46647	0.57734
Case III	β	5.97634	8.43714	6.37307	8.2871	7.10278	7.65482	6.3573	7.91398
	θ	11.7262	39.1698	17.2515	40.9707	19.0619	23.1343	16.7529	34.4034
	$R(t = 0.6)$	0.5051	0.66126	0.51404	0.65463	0.51373	0.66256	0.42672	0.7514
	$H(t = 0.6)$	2.21839	3.51407	2.42478	3.39854	2.89376	3.15228	1.93355	3.65404
	$CV(X)$	0.39852	0.54826	0.41568	0.52986	0.3998	0.68004	0.44239	0.52669
Case IV	β	5.90362	8.42027	6.25235	8.46247	7.30592	7.99233	6.3058	7.93951
	θ	11.2562	38.7486	16.6775	44.2927	24.1808	29.4337	16.636	34.5881
	$R(t = 0.6)$	0.50609	0.66301	0.50384	0.65875	0.51659	0.67767	0.41951	0.75514
	$H(t = 0.6)$	2.18582	3.50241	2.36907	3.55776	3.02233	3.35233	1.90981	3.70613
	$CV(X)$	0.39865	0.55337	0.40656	0.53931	0.25995	0.57195	0.44056	0.52678
Case V	β	5.94792	8.43008	6.41227	8.22615	7.04456	7.5644	6.30024	7.89359
	θ	11.5412	38.9991	17.5747	41.9445	20.0557	24.0727	16.6075	34.1971
	$R(t = 0.6)$	0.5055	0.66194	0.51543	0.65841	0.51421	0.66318	0.42847	0.75636
	$H(t = 0.6)$	2.20572	3.50925	2.40971	3.42833	2.85774	3.094	1.89419	3.64812
	$CV(X)$	0.39859	0.55025	0.41601	0.52875	0.3904	0.65372	0.44349	0.5276
Case VI	β	6.01103	8.44088	6.32234	8.56298	6.8357	7.471	6.40075	7.93886
	θ	11.9676	39.32	17.1728	46.1368	21.3695	26.308	17.1935	34.6209
	$R(t = 0.6)$	0.50473	0.66052	0.50331	0.65522	0.50349	0.657	0.42343	0.74704
	$H(t = 0.6)$	2.23369	3.51751	2.38698	3.58632	2.65494	2.94333	1.96199	3.68566
	$CV(X)$	0.39865	0.54588	0.40205	0.53428	0.36355	0.71378	0.44135	0.52132

Table 4: Bayes MCMC estimates based on BSEL, BLINEXL and BGEL functions for Case I.

Parameters	ω	BSEL	BLINEXL			BGEL		
			$c = -7$	$c = 0.3$	$c = 7$	$q = -7$	$q = 0.3$	$q = 7$
β	0.0	6.86103	7.90801	6.82593	6.16709	6.96378	6.83872	6.72357
	0.2	6.88311	7.87618	6.85453	6.19884	6.96532	6.865	6.76771
	0.5	6.91623	7.80919	6.89789	6.2656	6.96762	6.90466	6.83856
	0.9	6.9604	7.58088	6.9566	6.49149	6.97068	6.95802	6.94318
θ	0.0	22.7281	53.3899	20.0002	10.9066	25.7905	22.1154	18.6907
	0.2	22.8193	53.3581	20.4379	10.9385	25.3845	22.3239	19.1482
	0.5	22.9561	53.2909	21.2256	11.0056	24.6911	22.6415	20.0551
	0.9	23.1385	53.061	22.6896	11.2355	23.5347	23.0742	22.2069
$R(t = 0.6)$	0.0	0.59595	0.62711	0.59463	0.56464	0.63702	0.58587	0.51361
	0.2	0.59482	0.62047	0.59376	0.56942	0.62921	0.58675	0.52346
	0.5	0.59312	0.60988	0.59246	0.57689	0.61629	0.58807	0.54171
	0.9	0.59085	0.59443	0.59072	0.5875	0.59606	0.58984	0.57756
$H(t = 0.6)$	0.0	2.68093	3.63633	2.64304	1.63225	2.93443	2.61466	1.49219
	0.2	2.69498	3.60453	2.66439	1.66412	2.90298	2.64124	1.53976
	0.5	2.71604	3.5376	2.69666	1.73122	2.85159	2.68178	1.64429
	0.9	2.74413	3.30999	2.74018	1.96068	2.77304	2.73709	2.03898
$CV(X)$	0.0	0.49641	0.49889	0.49631	0.49405	0.50069	0.49552	0.49105
	0.2	0.49464	0.49669	0.49455	0.49273	0.49822	0.49391	0.49033
	0.5	0.49198	0.49333	0.49192	0.49076	0.49438	0.49151	0.48927
	0.9	0.48843	0.48872	0.48842	0.48818	0.48896	0.48833	0.48789

Table 5: Bayes MCMC estimates based on BSEL, BLINEXL and BGEL functions for Case II.

Parameters	ω	BSEL	BLINEXL			BGEL		
			$c = -7$	$c = 0.3$	$c = 7$	$q = -7$	$q = 0.3$	$q = 7$
β	0.0	6.56213	7.32888	6.5308	5.94403	6.65731	6.54131	6.43332
	0.2	6.66155	7.30231	6.63	5.97589	6.74986	6.64084	6.52632
	0.5	6.81069	7.25001	6.78456	6.04299	6.87583	6.79387	6.68924
	0.9	7.00953	7.12279	7.00247	6.27245	7.0248	7.0051	6.97146
θ	0.0	20.113	35.4447	17.9547	8.44189	22.5324	19.581	16.4863
	0.2	20.8857	35.4128	18.5644	8.47377	22.8678	20.3706	16.9765
	0.5	22.0449	35.3457	19.7583	8.54091	23.3214	21.6344	18.0208
	0.9	23.5904	35.1158	22.6053	8.77083	23.8541	23.4828	21.3187
$R(t = 0.6)$	0.0	0.60364	0.63254	0.60241	0.57462	0.64152	0.59443	0.53311
	0.2	0.60035	0.62455	0.59936	0.57705	0.6327	0.59298	0.54105
	0.5	0.59542	0.61166	0.5948	0.58077	0.61791	0.5908	0.55503
	0.9	0.58885	0.59244	0.58872	0.58589	0.59415	0.58792	0.57951
$H(t = 0.6)$	0.0	2.54371	3.29342	2.5112	1.72334	2.77178	2.48433	1.83342
	0.2	2.59408	3.26263	2.56616	1.7552	2.77664	2.54283	1.88931
	0.5	2.66963	3.19871	2.65035	1.82229	2.78382	2.63398	2.0096
	0.9	2.77037	2.99928	2.766	2.05158	2.79323	2.76223	2.41119
$CV(X)$	0.0	0.51681	0.51965	0.51669	0.51415	0.52152	0.51584	0.51105
	0.2	0.5099	0.51291	0.50978	0.50718	0.51499	0.50888	0.50413
	0.5	0.49953	0.50217	0.49943	0.49732	0.50417	0.49868	0.49499
	0.9	0.48571	0.48647	0.48568	0.48514	0.48713	0.48549	0.4846

Table 6: Bayes MCMC estimates based on BSEL, BLINEXL and BGEL functions for Case III.

Parameters	ω	BSEL	BLINEXL			BGEL		
			$c = -7$	$c = 0.3$	$c = 7$	$q = -7$	$q = 0.3$	$q = 7$
β	0.0	7.11497	7.73853	7.09111	6.61629	7.1824	7.10037	7.02519
	0.2	7.13333	7.70751	7.11391	6.64759	7.18731	7.12148	7.05864
	0.5	7.16086	7.64292	7.14842	6.71304	7.19463	7.1533	7.11134
	0.9	7.19756	7.43771	7.19499	6.92597	7.20433	7.19601	7.18683
θ	0.0	24.9147	45.9819	22.2421	11.6296	27.3333	24.395	21.2545
	0.2	25.0213	45.95	22.6817	11.6615	27.015	24.601	21.7293
	0.5	25.1813	45.8829	23.4736	11.7286	26.4907	24.9142	22.6448
	0.9	25.3947	45.653	24.9486	11.9586	25.6777	25.3401	24.6419
$R(t = 0.6)$	0.0	0.58669	0.60928	0.58573	0.56416	0.61758	0.57937	0.53056
	0.2	0.58598	0.60443	0.58522	0.56777	0.61158	0.58013	0.53834
	0.5	0.58493	0.59683	0.58445	0.57336	0.60185	0.58127	0.55199
	0.9	0.58353	0.58602	0.58344	0.58116	0.58721	0.5828	0.57576
$H(t = 0.6)$	0.0	2.81155	3.50086	2.78421	2.119	2.9907	2.76737	2.45064
	0.2	2.82249	3.4694	2.80045	2.15068	2.9682	2.78678	2.5012
	0.5	2.83889	3.40351	2.82497	2.21725	2.93242	2.81623	2.59659
	0.9	2.86076	3.1863	2.85793	2.44122	2.88021	2.85614	2.79275
$CV(X)$	0.0	0.48028	0.48189	0.48022	0.47875	0.48317	0.47968	0.4767
	0.2	0.4789	0.48023	0.47885	0.47766	0.48131	0.47842	0.47602
	0.5	0.47684	0.4777	0.4768	0.47604	0.47843	0.47652	0.47502
	0.9	0.47408	0.47426	0.47407	0.47391	0.47442	0.47401	0.47371

Table 7: Bayes MCMC estimates based on BSEL, BLINEXL and BGEL functions for Case IV.

Parameters	ω	BSEL	BLINEXL			BGEL		
			$c = -7$	$c = 0.3$	$c = 7$	$q = -7$	$q = 0.3$	$q = 7$
β	0.0	7.11556	7.72509	7.08929	6.56409	7.18904	7.09948	7.01587
	0.2	7.12484	7.6939	7.1037	6.59542	7.18367	7.11192	7.04322
	0.5	7.13875	7.62881	7.12542	6.66095	7.17557	7.13063	7.0859
	0.9	7.15731	7.41915	7.15461	6.87469	7.16469	7.15567	7.14621
θ	0.0	24.9453	44.2832	22.3306	11.377	27.3889	24.4219	21.557
	0.2	24.9567	44.2513	22.72	11.4088	27.0039	24.5366	21.9871
	0.5	24.9738	44.1842	23.4056	11.476	26.3563	24.7099	22.7919
	0.9	24.9966	43.9542	24.616	11.7059	25.3097	24.9435	24.4112
$R(t = 0.6)$	0.0	0.58705	0.61092	0.58604	0.56318	0.61955	0.5793	0.52762
	0.2	0.58655	0.60602	0.58574	0.56721	0.61345	0.58034	0.53583
	0.5	0.5858	0.59835	0.58529	0.57347	0.60357	0.58192	0.55038
	0.9	0.5848	0.58742	0.5847	0.58226	0.58866	0.58402	0.57628
$H(t = 0.6)$	0.0	2.81059	3.55008	2.781	2.03854	3.00386	2.76255	2.38411
	0.2	2.81729	3.51845	2.79353	2.07029	2.97581	2.77862	2.43678
	0.5	2.82735	3.45207	2.81241	2.13705	2.9305	2.80295	2.538
	0.9	2.84076	3.23003	2.83775	2.36298	2.86268	2.83582	2.75683
$CV(X)$	0.0	0.48025	0.4819	0.48018	0.47867	0.4832	0.47963	0.47653
	0.2	0.4794	0.48074	0.47935	0.47813	0.48181	0.47891	0.47643
	0.5	0.47813	0.47899	0.47809	0.47733	0.47969	0.47782	0.47627
	0.9	0.47643	0.47661	0.47643	0.47627	0.47676	0.47637	0.47606

Table 8: Bayes MCMC estimates based on BSEL, BLINEXL and BGEL functions for Case V.

Parameters	ω	BSEL	BLINEXL			BGEL		
			$c = -7$	$c = 0.3$	$c = 7$	$q = -7$	$q = 0.3$	$q = 7$
β	0.0	7.07455	7.75339	7.04981	6.54455	7.14464	7.05932	6.98056
	0.2	7.09744	7.7222	7.07718	6.57603	7.15364	7.08501	7.0185
	0.5	7.13177	7.6571	7.11868	6.64201	7.16702	7.12378	7.07866
	0.9	7.17755	7.44727	7.17481	6.86002	7.18463	7.17589	7.16581
θ	0.0	24.7039	41.9896	22.1605	12.318	27.0212	24.1977	21.361
	0.2	24.8171	41.9578	22.5916	12.3499	26.7224	24.4074	21.8201
	0.5	24.987	41.8906	23.3652	12.417	26.2329	24.7264	22.696
	0.9	25.2135	41.6607	24.7922	12.647	25.4808	25.1602	24.5501
$R(t = 0.6)$	0.0	0.59067	0.61361	0.5897	0.56786	0.62182	0.58331	0.53368
	0.2	0.58928	0.60811	0.5885	0.5709	0.61526	0.5834	0.54119
	0.5	0.58719	0.59944	0.58671	0.57557	0.60456	0.58352	0.55429
	0.9	0.58441	0.587	0.58432	0.58205	0.58825	0.58368	0.57678
$H(t = 0.6)$	0.0	2.78119	3.5998	2.75319	2.06942	2.9665	2.73528	2.39518
	0.2	2.79645	3.56812	2.77379	2.10115	2.94656	2.75917	2.4481
	0.5	2.81934	3.50157	2.80493	2.16787	2.91505	2.79551	2.54982
	0.9	2.84985	3.27781	2.84691	2.39328	2.86956	2.84495	2.76974
$CV(X)$	0.0	0.48208	0.48371	0.48201	0.48052	0.48498	0.48147	0.47844
	0.2	0.48055	0.4819	0.48049	0.47928	0.48297	0.48005	0.47762
	0.5	0.47825	0.47914	0.47822	0.47744	0.47987	0.47793	0.4764
	0.9	0.47519	0.47538	0.47518	0.47502	0.47554	0.47512	0.47481

Table 9: Bayes MCMC estimates based on BSEL, BLINEXL and BGEL functions for Case VI.

Parameters	ω	BSEL	BLINEXL			BGEL		
			$c = -7$	$c = 0.3$	$c = 7$	$q = -7$	$q = 0.3$	$q = 7$
β	0.0	7.15036	7.73126	7.12739	6.66774	7.21497	7.13638	7.06448
	0.2	7.16548	7.70042	7.14687	6.6989	7.21718	7.15418	7.09451
	0.5	7.18816	7.63634	7.17631	6.76392	7.22048	7.18099	7.14157
	0.9	7.21839	7.43555	7.21597	6.97294	7.22486	7.21693	7.20843
θ	0.0	25.1272	43.1342	22.5886	11.0114	27.4518	24.629	21.8384
	0.2	25.2305	43.1023	23.0148	11.0433	27.1442	24.8277	22.2957
	0.5	25.3855	43.0352	23.7778	11.1105	26.6394	25.1297	23.1619
	0.9	25.5921	42.8053	25.1777	11.3404	25.862	25.5399	24.9613
$R(t = 0.6)$	0.0	0.58392	0.60594	0.58299	0.56198	0.6142	0.57677	0.52961
	0.2	0.58366	0.60158	0.58292	0.56588	0.60863	0.57794	0.53742
	0.5	0.58327	0.59476	0.58281	0.57193	0.59966	0.57969	0.55116
	0.9	0.58276	0.58514	0.58266	0.58042	0.58628	0.58204	0.57512
$H(t = 0.6)$	0.0	2.83584	3.50885	2.80911	2.09672	3.01039	2.79303	2.45969
	0.2	2.84379	3.47739	2.82231	2.12845	2.98622	2.8093	2.51035
	0.5	2.85572	3.41151	2.84219	2.19513	2.94761	2.83393	2.6059
	0.9	2.87163	3.19442	2.86889	2.42025	2.89089	2.8672	2.80217
$CV(X)$	0.0	0.4785	0.47997	0.47844	0.47708	0.48113	0.47794	0.47514
	0.2	0.47725	0.47846	0.4772	0.4761	0.47944	0.4768	0.47455
	0.5	0.47538	0.47617	0.47535	0.47465	0.47682	0.47509	0.47368
	0.9	0.47289	0.47305	0.47288	0.47274	0.4732	0.47283	0.47255