VARIABLE ORDER FRACTIONAL PERMANENT MAGNET SYNCHRONOUS MOTOR: DYNAMICAL ANALYSIS AND NUMERICAL SIMULATION

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Abstract— In this paper, the variable order fractional permanent magnet synchronous motor (VOFPMSM) is investigated. Conditions for existence and uniqueness of the solution of the VOFPMSSM are proposed. The stability behavior of the system's equilibrium points along with the variation of the motor parameters and the order of differentiation is discussed. Sufficient conditions that guarantee the asymptotic stability of each of the equilibrium points of the system are established. Also, the required conditions that give the effect of Hopf bifurcation of the system are established in terms of the system parameters and the order of differentiation and consequently the appearance of the chaotic behavior of the VOFPMSSM. New numerical techniques based on the modified backward Euler's schemes for continuous and discontinuous variable order fractional model are presented. The obtained numerical results demonstrate the merits of the proposed method and the variable order fractional permanent magnet synchronous motor over the fractional permanent magnet synchronous motor.

Keywords: Variable order fractional dynamical system, Permanent magnet synchronous motor, Stability, Hopf Bifurcation, Chaos.

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1. Introduction

Fractional calculus plays a significant role in describing natural phenomena, and it is considered as a generalization of integral calculus [1]-[5]. For example, the fractional derivative concerns with the history of the considered variable hence it is more suitable for representing memory and hereditary behavior of physical models as in [6]-[12]. Electromagnetic fields with their applications are reformulated using a fractional derivative, and many important enhancements are satisfied in this area. The permanent magnet synchronous motor (PMSM) is one of the most critical applications that widely used in industrial applications due to its high desired performance [13]-[15]. Many researchers presented the fractional order PMSM as fractional order Lorenz system [16], [17] (FPMSM) which has high chaotic behavior if the parameters of the motor lie in a particular area. Chaos weakens the efficiency of the FPMSM by operating it in undesired conditions, and it occurs in a wide range of the operating parameters. Many studies were presented to overcome this chaotic behavior [15], [18], [19]. Chun et al. [20] showed an adaptive feedback controller includes one single state variable to eliminate chaos. Also, Zhou et al. [16] applied the adaptive sliding mode proposed to control the chaotic behavior in the FPMSM, while Rajagopal et al. [15] used recursive extended back stepping controller.

Due to the existence of the long memory effect, some physical phenomena can be modelled by variable order fractional differential equations, [10], [11]. This is due to the fractional order derivative needs infinitely many terms, namely memory effect, on the other hand, the integer order derivative requires only one term. The aim of this paper is to demonstrate and discuss the

behavior of the memory property characterized by the variable order fractional permanent magnet synchronous motor (VOFPMSM). Based on the FPMSM considered by [17], [19], this paper presents a theoretical model of VOFPMSSM. We study the stability conditions of our model along with the variation of the motor parameters and the order of differentiation. Also, the system behavior is investigated when the differential order is fractional, continuous and discontinuous function. The required conditions for the system to have Hopf bifurcation are established.

This paper is organized as follows: Section 2 contains some relevant definitions, resulting in fractional order derivatives and the modeling of VOFPMSSM. In section 3, we discussed the stability analysis of the model. Two new numerical techniques to get approximate solutions of VOFPMSSM are presented in section 4. Numerical examples with their approximate solutions are given in section 5 to illustrate the idea of our work. Finally, we concluded our results in section 6.

2. Variable Order Fractional Permanent Magnet Synchronous Motor (VOFPMSM)

Definition 1 [21] The fractional order derivative of a function $f(t)$, $f : R^+ \rightarrow R$ in Caputo sense is

$${}_a D_t^\alpha f(t) = (1 / \Gamma(n - \alpha)) \int_a^t (t - x)^{n-\alpha-1} f^{(n)}(x) dx, \\ n - 1 < \alpha < n. \quad (1)$$

Definition 2 [21] The variable order fractional derivative of the function $f(t)$, $f : R^+ \rightarrow R$ Caputo sense is:

$${}_a D_t^{\alpha(t)} f(t) = (1 / \Gamma(n - \alpha(t))) \int_a^t (t-x)^{n-\alpha(t)-1} f^{(n)}(x) dx, \quad n-1 < \alpha(t) < n. \quad (2)$$

Let us, for simplicity write ${}_a D_t^{\alpha} f(t) = D^{\alpha} f(t)$ when $a = 0$.

Now consider the following fractional system:

$$D^{\alpha} X(t) = F(X), \quad X(0) = X_0, \quad (3)$$

where $X(t) = (x_1, x_2, x_3)^T \in R^3$ and

$$F(X) = (F_1, F_2, F_3), \quad F_i : R^3 \rightarrow R, \quad i = 1, 2, 3. \quad (4)$$

The stability behavior of the system (3) is assessed by studying the stability of the system equilibria, where the corresponding linearized system at certain equilibrium point X^* is given in the form:

$$D^{\alpha} X(t) = J(X^*) X. \quad (5)$$

We need the following lemma.

Lemma 1 [20] If all the eigenvalues $\lambda_i, i = 1, 2, 3$ of $J(X^*)$ for the system (5) satisfy that

$$|\arg(\lambda_i)| > \alpha \pi / 2, \quad i = 1, 2, 3, \quad (6)$$

then X^* of the system (3) is locally asymptotically stable.

Using [17], we can write the mathematical model of a PMSM as:

$$\begin{aligned} L I_q' &= -R I_q - L I_d \omega - K_T \omega + V_q, \\ L I_d' &= -R I_d + L I_q \omega + V_d, \\ J \omega' &= K_T I_q - b \omega - T_L. \end{aligned} \quad (7)$$

Here, we used the q - d rotor coordinate system with the parameters: I_q, I_d stator currents on rotating frame; V_q, V_d stator voltages on rotating frame; L : Stator self-inductances; ω : Rotor angular speed (rotor angular frequency) at electrical angle; R : Stator per-phase residence; T_L : External load torque; J : Moment of inertia and $K_T = p\psi$, p : rotor pole pairs and ψ is the permanent magnet flux; b : viscous damping.

The above system is simplified to the famous Lorenz system using time scaling with the subsequent transformation, [17]:

$$\begin{aligned} Y(t) &= C X(t / \tau) + D, \quad \tau = L / R, \\ Y &= \begin{bmatrix} I_q \\ I_d \\ \omega \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad C = \begin{bmatrix} 0 & b R / L K_T & 0 \\ 0 & 0 & b R / L K_T \\ R / L & 0 & 0 \end{bmatrix}, \\ D &= \begin{bmatrix} 0 \\ -d_2 \\ 0 \end{bmatrix}, \quad d_2 = -\gamma \frac{b R}{L K_T} - \frac{K_T}{L}, \end{aligned} \quad (8)$$

where γ is a free positive constant.

The Lorenz system obtained has the form:

$$\begin{aligned} x' &= \beta(y - x) + T_L^*, \\ y' &= \gamma x - y - x z + V_q^*, \\ z' &= x y - z + V_d^*, \end{aligned} \quad (10)$$

where

$$\beta = b \tau / J, \quad T_L^* = -L^2 T_L / J R^2, \quad V_q^* = K_T L V_q / b R^2, \quad (11)$$

$$V_d^* = \gamma + K_T^2 / b R + L^2 K_T V_d / R^2 b,$$

where x, y and z indicate directly to ω, I_q and I_d respectively through the following relations:

$$x = (L / R) \omega, \quad y = (L K_T / R b) I_q \quad \text{and}$$

$$z = (L K_T / R b) I_d - \gamma - (K_T^2 / R b).$$

In this work, we consider that the operation of the PMSM when a power interrupt occurs suddenly, this means that $T_L^* = V_q^* = V_d^* = 0$.

For the PMSM, the variable order fractional permanent magnet synchronous motor (VOFPMSM) can be considered as a more realistic model. This is due to the fractional order derivative needs infinitely many terms, namely memory effect, on the other hand, the integer order derivative requires only one term. So, it is noteworthy to discuss the dynamical behavior and the numerical simulation of the VOFPMMSM which is represented by:

$$\begin{aligned} D^{\alpha(t)} X &= F(X), \quad X(0) = X_0, \\ 0 < \alpha_{\min} < \alpha_i(t) < \alpha_{\max} &\leq 1, \quad i = 1, 2, 3, \end{aligned} \quad (12)$$

$$D^{\alpha(t)} X = \begin{pmatrix} D^{\alpha_1(t)} x \\ D^{\alpha_2(t)} y \\ D^{\alpha_3(t)} z \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad X_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \quad F(X) = \begin{pmatrix} \beta(y-x) \\ \gamma x - y - x z \\ x y - z \end{pmatrix},$$

where $F(X)$ satisfies the following Lipschitz condition

$$\|F(X) - F(Y)\| \leq L \|X - Y\|, \quad L > 0. \quad (13)$$

For the system (12), we obtain the solution as:

$$X = X_0 + \int_0^t \frac{(t-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))} F(X(s)) ds = \Omega(X). \quad (14)$$

Then, one gets

$$\begin{aligned} \|\Omega(X) - \Omega(Y)\| &\leq \int_0^t \frac{(t-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))} \|F(X(s)) - F(Y(s))\| ds \\ &\leq L \int_0^t \frac{(t-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))} \sup_{s \in [0, T]} \|X(s) - Y(s)\| ds \\ &\leq \frac{L}{\Gamma(\alpha_{\min})} \|X - Y\| \int_0^t (t-s)^{\alpha_{\min}-1} ds \\ &\leq \frac{L T^{\alpha_{\min}}}{\Gamma(\alpha_{\min} + 1)} \|X - Y\|. \end{aligned}$$

If $L T^{\alpha_{\max}} \leq \Gamma(\alpha_{\min} + 1)$, we have $\|\Omega(X) - \Omega(Y)\| \leq \|X - Y\|$ which implies that our VOFPMMSM model has a unique solution. The above results are summarized in the following theorem.

Theorem 1 If the Lipschitz condition (13) is satisfied and $L T^{\alpha_{\max}} \leq \Gamma(\alpha_{\min} + 1)$, then the system given by (12) has a unique solution.

3. Stability analysis

The equilibrium points of the system (12) are presented in the following theorem:

Theorem 2 System (12) has the following equilibrium points:

- (i) Only the static solution $S_0(0, 0, 0)$ when $\gamma \leq 1$,

(ii) S_0 and $S_{1,2}(\pm\sqrt{\gamma-1}, \pm\sqrt{\gamma-1}, \gamma-1)$ when $\gamma > 1$.

Proof:

The equilibrium points are obtained directly by solving the nonlinear system that satisfies the right-hand side of the system (12) to be zero.

Now we study the stability behavior of the system (12) for different values of the system parameters and the value of the order of differentiation. Firstly, we discuss the stability conditions of the static equilibrium point.

Let $0 < \alpha_{\min} < \alpha_i(t) < \alpha_{\max} \leq 1$, $i = 1, 2, 3$ in system (12). Then the equilibrium point S_0 is asymptotically stable when $\gamma \leq 1$ and unstable when $\gamma > 1$.

Proof:

The characteristic equation of the Jacobin matrix of the system (12) at S_0 has the following form:

$$(1 + \lambda)(\lambda^2 + (1 + \beta)\lambda + \beta(1 - \gamma)) = 0. \quad (15)$$

The eigenvalues of the characteristic equation (15) are $\lambda_1 = -1$, and the two roots of the quadratic equation are $\lambda_{2,3} = 0.5[-(1 + \beta) \pm \sqrt{(1 + \beta)^2 - 4\beta(1 - \gamma)}]$. Since β and γ having positive values, then the Eigenvalues λ_i , $i = 1, 2, 3$ satisfy condition (6) when $\gamma \leq 1$, hence S_0 is asymptotically stable while if $\gamma > 1$, Eq. (15) has a positive real root, consequently S_0 is unstable.

When $\gamma > 1$ and the static equilibrium point S_0 is unstable, we devoted our effort to study the stability of the operating points. The characteristic equation of the Jacobin matrix of the system (12) at S_1 and S_2 are both given by the equation:

$$\lambda^3 + (2 + \beta)\lambda^2 + (\beta + \gamma)\lambda + 2\beta(\gamma - 1) = 0. \quad (16)$$

The position of the eigenvalues of Eq. (16) classifies the stability behavior of S_1 and S_2 and consequently the system (12). An important tool in defining the type of the roots of a cubic equation $p(x) = x^3 + a_1x^2 + a_2x + a_3$ is its discriminant $D(p)$ which is determined by the following relation [22]:

$$D(p) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3(a_1)^3 - 4(a_2)^3 - 27(a_3)^2. \quad (17)$$

Considering $p(\lambda) = \lambda^3 + (2 + \beta)\lambda^2 + (\beta + \gamma)\lambda + 2\beta(\gamma - 1)$, then its discriminant has the form as in Eq. (17) with the following parameters

$$a_1 = 2 + \beta, \quad a_2 = \beta + \gamma \quad \text{and} \quad a_3 = 2\beta(\gamma - 1). \quad (18)$$

Now, we give the stability conditions of the operating points of the system (12) in the following theorem.

Theorem 4 The operating points of the system (12) S_1 and S_2 are asymptotically stable in the following cases:

- (i) $D(p) > 0$, $0 < \alpha_{\min} < \alpha_i(t) < \alpha_{\max} \leq 1$,
 $i = 1, 2, 3$ if $\gamma_1 < \gamma < \gamma^*$ or $\gamma > \gamma^*$ and $\beta > \beta^*$
- (ii) $D(p) = 0$ and $0 < \alpha_{\min} < \alpha_i(t) < \alpha_{\max} \leq 1$, $i = 1, 2, 3$
- (iii) $D(p) < 0$ and $2/3 < \alpha_{\min} < \alpha_i(t) < \alpha_{\max} \leq 1$, $i = 1, 2, 3$

(iv) $D(p) < 0$, $0 < \alpha_{\min} < \alpha_i(t) < \alpha_{\max} < 1$,

$i = 1, 2, 3$, $\gamma > \gamma^*$ and $\beta = \beta^*$ or $\beta = \beta_1$,

where $D(p)$ is defined by Eq. (17) with the parameters displayed in (18),

$$\begin{aligned} \gamma_1 &= 1.0717, \quad \gamma^* = 14.93, \quad \beta^* = 0.5[(\gamma - 4) + \sqrt{\gamma^2 - 16(\gamma - 1)}] \text{ and} \\ \beta_1 &= 0.5[(\gamma - 4) - \sqrt{\gamma^2 - 16(\gamma - 1)}]. \end{aligned} \quad (19)$$

Proof:

For $D(p) > 0$, the eigenvalues of the characteristic equation (16) are three distinct real roots, and by using Routh-Hurwitz conditions, we have that the operating points S_1 and S_2 are asymptotic stables and the conditions in case (i) are fulfilled. Also, for (ii) when $D(p) = 0$ then $p(\lambda) = 0$ has either a triple root or one real double root and one single real root all these roots are negative where $a_i > 0$, $i = 1, 2, 3$.

For $D(p) < 0$, $p(\lambda) = 0$ has one real root b and two complex conjugate roots $b_1 \pm i b_2$.

Hence $b_1^2 / (b_1^2 + b_2^2) < 0.25$ is sufficient and necessary to satisfy condition (6) which also, applies to the other two cases (iii) and (iv).

We are familiar with the concept of Hopf bifurcation for ordinary differential equations that can be seen in phase portraits of the solutions as periodic solutions or limit cycles. However, in the fractional order differential equation there is no solution exactly said to be periodic or limit cycle, but we may find solutions converge to have a periodic form or a limit cycle. We may call this behavior as Hopf bifurcation for fractional order differential equations. Hence in the following theorem, we present the required conditions for the system (12) to conform Hopf bifurcation at the certain order of differentiation when $\alpha_i(t)$, $i = 1, 2, 3$ are constants.

Theorem 5 In system (12) with constant fractional order $\alpha_i(t) = 1$, $i = 1, 2, 3$, S_1 and S_2 have a Hopf bifurcation in the following cases:

- (i) $\alpha_i(t) = 1$, $i = 1, 2, 3$, $\gamma > \gamma^*$ and $\beta = \beta^*$,
- (ii) If
 $D(p) < 0$, $\alpha_i(t) = \alpha$, $2/3 < \alpha < 1$, $\beta > 2/(1 - 8\cos^2(\alpha\pi/2))$
and
 $\gamma > (\beta^2 + 4\beta(1 - 2\cos^2(\alpha\pi/2)) / (\beta(1 - 8\cos^2(\alpha\pi/2)) - 2)$,
(20)

where γ^* and β^* are defined in (19).

Proof:

The required conditions for Hopf bifurcation with constant fractional derivative $\alpha_i(t) = 1$, $i = 1, 2, 3$ are $(2 + \beta)(\beta + \gamma) - 2\beta(\gamma - 1) = 0$, which is satisfied if $\gamma > \gamma^*$ and $\beta = \beta^*$. If $D(p) < 0$ and $\alpha_i(t) = \alpha$, $2/3 < \alpha < 1$, then the characteristic equation has three eigenvalues namely $\lambda = \lambda_1 < 0$ and $\lambda = b_1 \pm i b_2$, $b_2 / b_1 = \tan(\alpha\pi/2)$, these roots must obey the following relations:

$$-\lambda_1 = 2 + \beta + 2b_1, \quad (22)$$

$$b_1^2 \sec^2(\alpha\pi/2) = 2\beta(\gamma-1)/(2+\beta+2b_1), \quad (23)$$

$$(2b_1)^3 + 2(2+\beta)(2b_1)^2 + ((2+\beta)^2 + \gamma + \beta)(2b_1) + (2\gamma + 4\beta - \gamma\beta + \beta^2) = 0. \quad (24)$$

Conditions (20-21) satisfy equations (22-24) and the value of b_1 is suggested to be:

$$b_1 = (1/4) [-\eta_1 + \sqrt{\eta_1^2 - 4\eta_2}], \text{ where}$$

$$\eta_1 = [(2+\beta)^2 + \gamma + \beta]/(2+\beta) \text{ and}$$

$$\eta_2 = [2\gamma + 4\beta - \gamma\beta + \beta^2 + 8\beta(\gamma-1)\cos^2(\alpha\pi/2)]/(2+\beta).$$

Also, we note that the chaotic behavior of the VOFPSM occurs near the value of α that the Hopf bifurcation occurs and disappear by decreasing α .

4. Discretization of Caputo derivative

In this section, we discrete the first order derivative $u'(t)$ by the modified difference method that is given by Mickens [23] as follows:

$$u'(t_k) = \frac{u_{k+1} - u_k}{\eta(\tau)} + O(\eta), \quad \eta(\tau) = \tau + O(\tau^2), \quad (25)$$

where τ is the step size, examples of $\eta(\tau)$ are $1 - e^{-\tau}$, $\sin \tau$ and $\sinh \tau$, see [6], [24].

4.1. Modified Backward Euler's Scheme I (MBEI)

To get an approximation for the time variable order Caputo fractional derivative given by Eq. (13), we use the discretization given by Eq.(25) at the point $t = t_{k+1}$, then the discretization of the Caputo fractional derivative is:

$$\begin{aligned} D^{\alpha(t_{k+1})} u(t_{k+1}) &= \int_0^{t_{k+1}} \frac{u'(s)}{\Gamma(1-\alpha(t_{k+1}))} (t_{k+1} - s)^{-\alpha(t_{k+1})} ds \\ &= \int_0^{t_{k+1}} \frac{1}{\Gamma(1-\alpha(t_{k+1}))} \left[\frac{u_{k+1} - u_k}{\eta(\tau)} + O(\eta) \right] (t_{k+1} - s)^{-\alpha(t_{k+1})} ds \\ &= \sum_{j=0}^k \left[\frac{u_{k+1} - u_k}{\eta} + O(\eta) \right] \int_{t_{k-j}}^{t_{k+1}} \frac{1}{\Gamma(1-\alpha(t_{k+1}))} (t_{k+1} - s)^{-\alpha(t_{k+1})} ds \\ &= \sum_{j=0}^k \left[\frac{u_{k+1} - u_k}{\eta} + O(\eta) \right] \int_{t_j}^{t_{j+1}} \frac{1}{\Gamma(1-\alpha(t_{j+1}))} (t_{k+1} - s)^{-\alpha(t_{j+1})} ds \\ &= \sum_{j=0}^k \frac{\eta^{-\alpha(t_{j+1})}}{\Gamma(2-\alpha(t_{j+1}))} [u_{k-j+1} - u_{k-j}] [(j+1)^{1-\alpha(t_{j+1})} - j^{1-\alpha(t_{j+1})}] + T \\ &= \sum_{j=0}^k \omega_j(\eta) J_j [u_{k-j+1} - u_{k-j}] + T, \end{aligned} \quad (26)$$

$$\text{where } \omega_j(\eta) = \frac{\eta^{-\alpha(t_{j+1})}}{\Gamma(2-\alpha(t_{j+1}))}, \quad J_j = [(j+1)^{1-\alpha(t_{j+1})} - j^{1-\alpha(t_{j+1})}]$$

and T is the truncation error.

Then, we have

$$D^{\alpha(t_{k+1})} u(t_{k+1}) \approx \sum_{j=0}^k \omega_j(\eta) J_j [u_{k-j+1} - u_{k-j}]. \quad (27)$$

Lemma 2 [13] for $j = 0, 1, 2, \dots, k$ and

$$0 < \alpha_{\min} < \alpha(t) < \alpha_{\max} \leq 1, \text{ then}$$

$$(i) \quad 1 = J_0 > J_1 > J_2 > J_3 > \dots > J_k > 0,$$

$$(ii) \quad 1 - J_1 + \sum_{j=1}^{k-1} (J_j - J_{j+1}) + J_k = 1,$$

Lemma 3 [13] The bound of the truncation error T is:

$$|T| \leq c\eta^{2-\alpha_{\min}}, \text{ where}$$

$$c = \frac{1}{\Gamma(2-\alpha_{\min})} \left[\frac{1-\alpha_{\min}}{12} + \frac{2^{2-\alpha_{\min}}}{2-\alpha_{\max}} - 1 + 2^{-\alpha_{\min}} \right] \max_{t_0 \leq t \leq t_k} u''(t).$$

4.2. Modified Backward Euler's Scheme II (MBE II)

For better accuracy, we write Eq. (26) as

$$D^{\alpha(t_{k+1})} u(t_{k+1}) = \mathcal{G}(\eta) + c\eta^{2-\alpha_{\min}}, \quad (28)$$

$$\text{where } \mathcal{G}(\eta) = \sum_{j=0}^k \omega_j(\eta) J_j [u_{k-j+1} - u_{k-j}],$$

then

$$D^{\alpha(t_{k+1})} u(t_{k+1}) = \mathcal{G}\left(\frac{\eta}{2}\right) + c\left(\frac{\eta}{2}\right)^{2-\alpha_{\min}}, \quad (29)$$

Multiply Eq. (29) by $2^{2-\alpha_{\min}}$, then subtracting the results from Eq. (28), we get

$$(1 - 2^{2-\alpha_{\min}}) D^{\alpha(t_{k+1})} u(t_{k+1}) = \mathcal{G}(\eta) - 2^{2-\alpha_{\min}} \mathcal{G}\left(\frac{\eta}{2}\right) + C\eta^{3-\alpha_{\min}},$$

$$D^{\alpha(t_{k+1})} u(t_{k+1}) \approx \frac{1}{1 - 2^{2-\alpha_{\min}}} (\mathcal{G}(\eta) - 2^{2-\alpha_{\min}} \mathcal{G}\left(\frac{\eta}{2}\right)). \quad (30)$$

Now, applying the modified backward Euler's scheme I (MBE I) given by (27) to the system (12), we get the following scheme,

$$\begin{aligned} x(t_{k+1}) &= \frac{1}{\omega_{0\alpha_1}} [(\omega_{0\alpha_1} - \beta) x(t_k) \\ &\quad - \sum_{j=1}^k \omega_{j\alpha_1} J_{j\alpha_1} (x(t_{k-j+1}) - x(t_{k-j})) + \beta y(t_k)], \\ y(t_{k+1}) &= \frac{1}{\omega_{0\alpha_2}} [(\omega_{0\alpha_2} - 1) y(t_k) \\ &\quad - \sum_{j=1}^k \omega_{j\alpha_2} J_{j\alpha_2} (y(t_{k-j+1}) - y(t_{k-j})) + \gamma x(t_k) - x(t_k) z(t_k)], \\ z(t_{k+1}) &= \frac{1}{\omega_{0\alpha_3}} [(\omega_{0\alpha_3} - 1) z(t_k) \\ &\quad - \sum_{j=1}^k \omega_{j\alpha_3} J_{j\alpha_3} (z(t_{k-j+1}) - z(t_{k-j})) + x(t_k) y(t_k)], \end{aligned} \quad (31)$$

where

$$\omega_{0\alpha_1} = \frac{\eta^{-\alpha_1(t_1)}}{\Gamma[2-\alpha_1(t_1)]}, \quad \omega_{0\alpha_2} = \frac{\eta^{-\alpha_2(t_1)}}{\Gamma[2-\alpha_2(t_1)]}, \quad (32)$$

$$\omega_{0\alpha_3} = \frac{\eta^{-\alpha_3(t_1)}}{\Gamma[2-\alpha_3(t_1)]}, \quad \omega_{j\alpha_i} = \frac{\eta^{-\alpha_i(t_{j+1})}}{\Gamma[2-\alpha_i(t_{j+1})]},$$

and

$$J_{j\alpha_i} = [(j+1)^{1-\alpha_i(t_{j+1})} - j^{1-\alpha_i(t_{j+1})}], \quad i = 1, 2, 3. \quad (33)$$

Also, applying the modified backward Euler's scheme II (MBE II) given by (30) to the system (12), we get,

$$\begin{aligned}
x(t_{k+1}) &= \frac{1}{(\omega_{1x0} - 2^{-\alpha_{1\min}})} [(1 - 2^{-\alpha_{1\min}}) \beta(y(t_k) - x(t_k)) \\
&+ \omega_{1x0} x(t_k) - \sum_{j=1}^k \omega_{1xj} J_{1xj} (x(t_{k-j+1}) - x(t_{k-j})) \\
&+ 2^{-\alpha_{1\min}} (\sum_{j=1}^{2k} \omega_{2xj} J_{2xj} (x_2(t_{2k-j+1}) - x_2(t_{2k-j}) - \omega_{2x0} x_2(2k))],
\end{aligned} \tag{34}$$

$$\begin{aligned}
y(t_{k+1}) &= \frac{1}{(\omega_{1y0} - 2^{-\alpha_{1\min}})} [(1 - 2^{-\alpha_{1\min}}) (\gamma x(t_k) \\
&- y(t_k) - x(t_k) z(t_k)) + \omega_{1y0} y(t_k) \\
&- \sum_{j=1}^k \omega_{1yj} J_{1yj} (y(t_{k-j+1}) - y(t_{k-j})) \\
&+ 2^{-\alpha_{1\min}} (\sum_{j=1}^{2k} \omega_{2yj} J_{2yj} (y_2(t_{2k-j+1}) - y_2(t_{2k-j}) - \omega_{2y0} x_2(2k))],
\end{aligned} \tag{35}$$

$$\begin{aligned}
z(t_{k+1}) &= \frac{1}{(\omega_{1z0} - 2^{-\alpha_{1\min}})} [(1 - 2^{-\alpha_{1\min}}) (x(t_k) y(t_k) - z(t_k)) \\
&+ \omega_{1z0} z(t_k) - \sum_{j=1}^k \omega_{1zj} J_{1zj} (z(t_{k-j+1}) - z(t_{k-j})) \\
&+ 2^{-\alpha_{1\min}} (\sum_{j=1}^{2k} \omega_{2zj} J_{2zj} (z_2(t_{2k-j+1}) - z_2(t_{2k-j}) - \omega_{2z0} z_2(2k))],
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
\omega_{1xj} &= \frac{\eta^{-\alpha_1(t_{j+1})}}{\Gamma[2 - \alpha_1(t_{j+1})]}, \quad \omega_{1yj} = \frac{\eta^{-\alpha_2(t_{j+1})}}{\Gamma[2 - \alpha_2(t_{j+1})]}, \\
\omega_{1zj} &= \frac{\eta^{-\alpha_3(t_{j+1})}}{\Gamma[2 - \alpha_3(t_{j+1})]}, \quad j = 0, 1, \dots, n-1, \\
\omega_{2xj} &= \frac{(\eta/2)^{-\alpha_1(t_{j+1})}}{\Gamma[2 - \alpha_1(t_{j+1})]}, \quad \omega_{2yj} = \frac{(\eta/2)^{-\alpha_2(t_{j+1})}}{\Gamma[2 - \alpha_2(t_{j+1})]}, \\
\omega_{2zj} &= \frac{(\eta/2)^{-\alpha_3(t_{j+1})}}{\Gamma[2 - \alpha_3(t_{j+1})]}, \quad j = 0, 1, \dots, 2n-1, \\
\text{and } J_{j\alpha_i} &= [(j+1)^{1-\alpha_i(t_{j+1})} - j^{1-\alpha_i(t_{j+1})}], \quad i = 1, 2, 3, \quad j = 0, 1, 2, \dots
\end{aligned} \tag{37}$$

5. Numerical examples

We present the numerical results that carried out by modified Backward Euler's schemes (MBE I and MBE II) to discuss the dynamical behavior of VOFPSM. Throughout this section, we use the initial condition of the system (12) as $x(0) = 0.01$, $y(0) = 0.02$ and $z(0) = 35$.

Example 5.1 Consider the system (12) with the parameters given as $\gamma = 0.2$, $\beta = 2$ and $\alpha_i(t) = 1, i = 1, 2, 3$. As stated in Theorem 2, the system (12) has the static solution $S_0(0, 0, 0)$, as the only critical point for $\gamma \leq 1$ and it is asymptotically stable through Theorem 3. Fig. 1 shows the numerical solutions of system (12) for the time interval $T = 4$ and $\eta = \tau = 0.01$ for the integer order derivative ($\alpha(t) = 1$). The fourth-order Runge-Kutta method

(RK4) is used to compare the numerical solutions giving by the modified backward Euler schemes MBE I and MBE II. As depicted in Figure 1, we conclude that the numerical results given by MBE II are better than the numerical results given by the other two methods MBE I for $\eta = \tau$, $\eta = \tau/2$ and RK4.

(a)

(b)

Figure 1: Numerical simulation of example 5.1, (a, b) the time response of $x(t)$ and $y(t)$

Example 5.2 Assume that $\gamma > 1$, hence applying Theorem 2, the system (12) has the static solution $S_0(0, 0, 0)$, in addition to the two operating equilibrium points S_1 and S_2 . Following Theorem 3, then S_0 is unstable while Theorem 4 indicates that S_1 and S_2 are asymptotically stable for $\alpha_i(t) < \alpha$, $(2/3) < \alpha \leq 1$, $i = 1, 2, 3$. Fig. 2 (a, b, c, d, e) gives the numerical solution of system (12) using the approximation provided by MBE I with the parameters $\gamma = 25$, $\beta = 17$, and $T = 100$ for different values of the order of differentiation. The system is asymptotically stable for $\alpha_i(t) = 0.9$ and $\alpha_i(t) = 0.7, i = 1, 2, 3$ as displayed in Fig. 2 (a, b, c, d), while the system goes to be unstable for $\alpha_i(t) = 0.6, i = 1, 2, 3$ as shown in Fig. 2 (e, f), which agree with Theorem 5.

(a)

(b)

(c)

(d)

(e)

(f)

Figure 2 Phase portrait and time response of example 5.2

Example 5.3 Again, in the system (12) set $\alpha_i(t) = 0.9$, $i = 1, 2, 3$ and $\beta = 17$, then there is a Hopf bifurcation at $\gamma > 30.3$ as discussed in Theorem 5. Figure 3 (a, b, c, d, e, f, g, h, I, j) give the numerical solution of the system (12) using the discretization (31) with $T = 100$, for $\gamma = 40, 30.5, 27.7, 27.5$ and 27 respectively. The behavior of the numerical solutions of example 5.3 for different values of γ satisfies Theorem 5 and the system is asymptotically stable for $\alpha_i(t) = 0.9$, $i = 1, 2, 3$ and $\gamma < 30.3$ as shown in figure 3 (e, f, g, h, i, j).

(a)

(b)

(c)

(d)

(i)

(j)

Figure 3 Phase portrait and time response of example 5.3

Example 5.4 Let the order of differentiations $\alpha_i(t)$, $i = 1, 2, 3$ in the system (12) have the following form $\alpha_1(t) = 1 - 0.006t$, $\alpha_2(t) = 1 - 0.002t$, $\alpha_3(t) = 1 - 0.008t$, $\gamma = 35 > 30.3$ and $\beta = 17$. Then by Theorem 4, the system will be asymptotically stable. Figure 4 (a, b) gives the numerical solution of the system (12) for $T = 20$. A chaotic behavior appears in the solution of system (12), for the values $\gamma > 30.3$, $\beta = 17$ and $\alpha_i(t) = 0.9$, $i = 1, 2, 3$ which are together the thresholds for Hopf bifurcation.

(a)

(e)

(f)

(g)

(h)

(e)

(f)

Figure 4 Phase portrait and time response of example 5.4 for $\gamma = 40, 30.5, 27.7, 27.5$ and 27

Example 5.5 Now we consider the order of differentiation of the system (12) are discontinuous function as given below:

$$\alpha_i(t) = \begin{cases} 0.9 & \text{for } 0 < t < 10 \\ 0.7 & \text{for } 10 < t < 20 \end{cases}, i = 1, 2, 3. \quad (39)$$

Assume that, $\gamma = 50$ and $\beta = 9$. Back to Theorem 4, we get that the system (12) is asymptotically stable. Figure 5 gives the numerical solution for $T = 20$, (a, b) and (c, d) give the numerical solutions for $\alpha_i(t) = 0.9, 0.7$, $i = 1, 2, 3$ respectively in the interval $0 < t < 20$. While in Fig. 5(e, f), the numerical results are considered for the discontinuous function of the order of derivatives as defined in (39).

(a)

(b)

(c)

(f)

Figure 5 Phase portrait and time response of example 5.5

6. Conclusion

In this paper, we proposed a variable order fractional permanent magnet synchronous motor (VOFPMSM) in the sense of Caputo derivative. The existence and the uniqueness of the solution of the VOFPSM are investigated. The dynamical behavior of the system is discussed, sufficient conditions for the equilibrium point to be asymptotically stable are given, and the effect of the variable order of differentiation on the stability of the system is displayed. The existence of Hopf bifurcation of the VOFPSM is discussed and the conditions that guarantee that behavior are established. So we can avoid the occurrence of chaotic phenomena either by changing the value of the parameters or the order of differentiation. A modified backward Euler's scheme I (MBE I) and a modified backward Euler's scheme II (MBE II) are presented to get the numerical solutions of the system of variable order fractional differential equations. Moreover, applying these techniques on VOFPSM, we can get numerical solutions for this system. As a particular case, a comparison between the numerical solutions using MBE I, MBE II and the well-known fourth order Runge-Kutta method for $\alpha_i(t) = 1, i = 1, 2, 3$ are given which clears that our modified schemes MBE I and MBE II are very helpful for obtaining an accurate numerical simulation of VOFPSM. The effect of the variation of the order of differentiation $\alpha_i(t)$ on the stability of the equilibrium points is presented in Theorem 4 and example 5.2. Also, threshold values for the existence of Hopf bifurcation are obtainable in Theorem 5. Furthermore, in example 5.3, we note that the chaotic behavior of the VOFPSM occurs near the value of α that the Hopf bifurcation occurs and disappears by

(d)

(e)

decreasing α . The discontinuous form of the order of differentiation is documented in example 5.5.

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