SUBCLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH q -MITTAG LEFFLER FUNCTION

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Abstract

In the literature on geometric function theory, we can find many interesting applications of a variety of convolution operators which are defined by means of a number of special functions and analytic number theory. The main object of this paper is to examine two subclasses of multivalent functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,$$

which are analytic in the open unit disc $\mathbb{U}$. We investigate convolution properties and coefficient estimates for these subclasses.

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1 Introduction

Quantum calculus or q -calculus is an ordinary calculus without limit. In recent years, the study of q -theory attracted the researches due to its applications in various branches of mathematics and physics, for example, in the areas of special functions, ordinary fractional calculus, optimal control problems, q -difference, q -integral equations and in q -transform analysis (see, for instance, [1] – [7]). Our main objective in this paper is to introduce and study some subclasses of p -valently analytic functions in the open unit disk $\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ by applying the q -derivative operator in conjunction with the principle of subordination between analytic functions (see, for details, [8, 9]).

For a natural number p , let $\mathcal{A}(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and multivalent in $\mathbb{U}$. In particular, we write $\mathcal{A}(1) = \mathcal{A}$.

Let $\mathcal{S}_p^*(\alpha)$ and $\mathcal{C}_p(\alpha)$ denote the subclasses of multivalent starlike and convex functions of order α ($0 \leq \alpha < p$) (see, for example, Owa [10]). Also, for $f(z) \in \mathcal{A}(p)$ given by (1.1) and $0 < q < 1$, the q -derivative of $f(z)$ is defined by (see Gasper and Rahman [11])

$$D_{q,p}f(z) := \begin{cases} f'(0) & \text{if } z = 0, \\ \frac{f(qz) - f(z)}{(q-1)z} & \text{if } z \neq 0, \end{cases} \quad (1.2)$$

provided that $f'(0)$ exists. From (1.2), we deduce that

$$D_{q,p}f(z) = [p]_q z^{p-1} + \sum_{k=p+1}^{\infty} [k]_q a_k z^{k-1} \quad (z \neq 0), \quad (1.3)$$

where

$$[i]_q := \frac{1-q^i}{1-q} = 1 + q + q^2 + \dots + q^{i-1}, \quad (1.4)$$

and

$$\lim_{q \rightarrow 1^-} D_{q,p}f(z) = \lim_{q \rightarrow 1^-} \frac{f(qz) - f(z)}{(q-1)z} = f'(z),$$

for a function f which is differentiable in a given subset of $\mathbb{C}$. Further, for $p = 1$, we have $D_{q,1}f(z) = D_q f(z)$ (see Seoudy and Aouf [12]).

Making use of the q -derivative operator $D_{q,p}$ ($0 < q < 1$, $p \in \mathbb{N}$) given by (1.2), we introduce the subclass $\mathcal{S}_q^*(p, \alpha)$ of p -valently q -starlike functions of order α in $\mathbb{U}$ and the subclass $\mathcal{C}_q(p, \alpha)$ of p -valently q -convex functions of order α in $\mathbb{U}$, $0 \leq \alpha < 1$, as follows:

$$f(z) \in \mathcal{S}_q^*(p, \alpha) \iff \Re \left(\frac{1}{[p]_q} \frac{z D_{q,p}f(z)}{f(z)} \right) > \alpha,$$

and

$$f(z) \in \mathcal{C}_q^*(p, \alpha) \iff \Re \left(\frac{1}{[p]_q} \frac{D_{q,p}(z D_{q,p}f(z))}{D_{q,p}f(z)} \right) > \alpha,$$

respectively. It is easy to check that

$$f(z) \in \mathcal{C}_q(p, \alpha) \iff \frac{z D_{q,p}f(z)}{[p]_q} \in \mathcal{S}_q^*(p, \alpha).$$

We note also that $\lim_{q \rightarrow 1^-} \mathcal{S}_q^*(p, \alpha) = \mathcal{S}_p^*(\alpha)$ and $\lim_{q \rightarrow 1^-} \mathcal{C}_q(p, \alpha) = \mathcal{C}_p(\alpha)$.

We next introduce the subclasses $\mathcal{S}_{q,p,b}^*[A, B]$ and $\mathcal{C}_{q,p,b}[A, B]$ as follows.

Definition 1.1. For $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $0 < q < 1$, $-1 \leq B < A \leq 1$, $z \in \mathbb{U}$ and $p \in \mathbb{N}$, let $\mathcal{S}_{q,p,b}^*[A, B]$ and $\mathcal{C}_{q,p,b}[A, B]$ be the subclasses of $\mathcal{A}(p)$ consisting of functions $f(z)$ of the form (1.1) and satisfy the analytic criterion:

$$1 + \frac{1}{b} \left(\frac{1}{[p]_q} \frac{z D_{q,p}f(z)}{f(z)} - 1 \right) \prec \frac{1 + Az}{1 + Bz}, \quad (1.5)$$

and

$$1 + \frac{1}{b} \left(\frac{1}{[p]_q} \frac{D_{q,p}(z D_{q,p}f(z))}{D_{q,p}f(z)} - 1 \right) \prec \frac{1 + Az}{1 + Bz}, \quad (1.6)$$

respectively, where “ $\prec$ ” stands for subordination (see [8, 9]). From (1.5) and (1.6), it follows that

$$f(z) \in \mathcal{C}_{q,p,b}[A, B] \iff \frac{z D_{q,p}f(z)}{[p]_q} \in \mathcal{S}_{q,p,b}^*[A, B]. \quad (1.7)$$

We remark the following special cases:

- (i) $\mathcal{S}_{q,p,1}^*[1-2\alpha, -1] =: \mathcal{S}_q^*(p, \alpha)$ and $\mathcal{C}_{q,p,1}[1-2\alpha, -1] =: \mathcal{C}_q(p, \alpha)$ ($0 \leq \alpha < 1$);
- (ii) $\mathcal{S}_{q,1,1}^*[A, B] =: \mathcal{S}_q^*[A, B]$ and $\mathcal{C}_{q,1,1}[A, B] =: \mathcal{C}_q[A, B]$ (see Seoudy and Aouf [12]);
- (iii) $\lim_{q \rightarrow 1^-} \mathcal{S}_{q,p,1}^*[1-2\alpha, -1] =: \mathcal{S}_p^*(\alpha)$ and $\lim_{q \rightarrow 1^-} \mathcal{C}_{q,p,1}[1-2\alpha, -1] =: \mathcal{C}_p(\alpha)$ ($0 \leq \alpha < 1$) (see Owa [10]);
- (iv) $\lim_{q \rightarrow 1^-} \mathcal{S}_{q,p,1}^*[A, B] =: \mathcal{S}_p^0\left(\frac{1+Az}{1+Bz}\right) = \mathcal{S}_p[A, B]$ and $\lim_{q \rightarrow 1^-} \mathcal{C}_{q,p,1}[A, B] =: \mathcal{C}_p^0\left(\frac{1+Az}{1+Bz}\right) = \mathcal{C}_p[A, B]$ (see Sarkar et al. [13, with $\lambda = 0$ and $\phi(z) = \frac{1+Az}{1+Bz}$], see also Aouf [14]);
- (v) $\lim_{q \rightarrow 1^-} \mathcal{S}_{q,p,\left(1-\frac{\lambda}{[p]_q}\right)e^{-i\delta \cos \delta}}^*[1, -1] =: \mathcal{S}_p^\delta(\lambda)$ ($|\delta| < \frac{\pi}{2}$, $0 \leq \lambda < p$) (see Patil and Thakare [15]);
- (vi) $\lim_{q \rightarrow 1^-} \mathcal{C}_{q,p,\left(1-\frac{\lambda}{[p]_q}\right)e^{-i\delta \cos \delta}}[1, -1] =: \mathcal{C}_p^\delta(\lambda)$ ($|\delta| < \frac{\pi}{2}$, $0 \leq \lambda < p$) (see Aouf [16], see also Srivastava et al. [17]).

The q -shifted factorials, for any complex number α , are defined by

$$(\alpha; q)_0 := 1; \quad (\alpha; q)_n := \prod_{k=0}^{n-1} (1 - \alpha q^k), \quad n \in \mathbb{N}. \quad (1.8)$$

The definition (1.8) remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha; q)_\infty = \prod_{j=0}^{\infty} (1 - \alpha q^j) \quad \text{for } |q| < 1.$$

Furthermore, in terms of the basic (or q -) Gamma function $\Gamma_q(z)$ defined by

$$\Gamma_q(z) := \frac{(q; q)_\infty (1-q)^{1-z}}{(q^z; q)_\infty} \quad (0 < q < 1; z \in \mathbb{C}), \quad (1.9)$$

so that

$$\lim_{q \rightarrow 1^-} \{\Gamma_q(z)\} = \Gamma(z)$$

for the familiar Gamma function $\Gamma(z)$, we find from (1.8) that

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha + n)}{\Gamma_q(\alpha)} (1-q)^n \quad (n \in \mathbb{N}; \alpha \in \mathbb{C}).$$

We note that

$$\lim_{q \rightarrow 1^-} \frac{(q^\alpha; q)_n}{(1-q)^n} = (\alpha)_n,$$

where

$$(\alpha)_n = \begin{cases} 1, & \text{if } n = 0, \\ \alpha(\alpha+1)(\alpha+2) \dots (\alpha+n-1), & \text{if } n \in \mathbb{N}. \end{cases}$$

For $0 < q < 1$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, consider the q -analogue of Mittag Leffler defined by (see Sharma and Jain [18])

$$E_{\alpha,\beta}^\gamma(z; q) = \sum_{k=0}^{\infty} \frac{(q^\gamma; q)_k}{(q; q)_k} \frac{z^k}{\Gamma_q(\alpha k + \beta)}.$$

As $q \rightarrow 1^-$, the linear operator $E_{\alpha,\beta}^\gamma(z; q)$ reduces to $E_{\alpha,\beta}^\gamma(z)$ introduced by Prabhakar [19]. Now, let us define

$$\mathbb{E}_{\alpha,\beta}^{\gamma,p}(z; q) := z^p \Gamma_q(\beta) E_{\alpha,\beta}^\gamma(z; q) = z^p + \sum_{k=p+1}^{\infty} \frac{(q^\gamma; q)_{k-p}}{(q; q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p) + \beta)} z^k.$$

We remark that:

- (i) $\mathbb{E}_{1,1}^{1,p}(z; q) = z^p e_q(z)$;
- (ii) $\mathbb{E}_{1,2}^{1,p}(z; q) = z^{p-1} (e_q(z) - 1)$;

where $e_q(z)$ is one of the q -analogues of the exponential function e^z given by

$$e_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(k+1)} = \sum_{k=0}^{\infty} \frac{(1-q)^k z^k}{(q; q)_k} = \frac{1}{(z; q)_{\infty}}.$$

Using the *Hadamard product (or convolution)*, we define the linear operator $\mathbb{E}_{\alpha, \beta}^{\gamma, p} : \mathcal{A}(p) \rightarrow \mathcal{A}(p)$ by

$$(\mathbb{E}_{\alpha, \beta}^{\gamma, p} f)(z) := \mathbb{E}_{\alpha, \beta}^{\gamma, p}(z; q) * f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(q^{\gamma}; q)_{k-p}}{(q; q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p) + \beta)} a_k z^k, \quad z \in \mathbb{U}. \quad (1.10)$$

Definition 1.2. For $0 < q < 1$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $-1 \leq B < A \leq 1$ and $b \in \mathbb{C}^*$, let

$$\mathcal{S}_{q, p, b}^*[\alpha, \beta, \gamma; A, B] := \left\{ f \in \mathcal{A}(p) : \mathbb{E}_{\alpha, \beta}^{\gamma, p} f \in \mathcal{S}_{q, p, b}^*[A, B] \right\},$$

and

$$\mathcal{C}_{q, p, b}[\alpha, \beta, \gamma; A, B] := \left\{ f \in \mathcal{A}(p) : \mathbb{E}_{\alpha, \beta}^{\gamma, p} f \in \mathcal{C}_{q, p, b}[A, B] \right\}.$$

It is easy to check that

$$f \in \mathcal{C}_{q, p, b}[\alpha, \beta, \gamma; A, B] \Leftrightarrow \frac{z D_{q, p} f(z)}{[p]_q} \in \mathcal{S}_{q, p, b}^*[\alpha, \beta, \gamma; A, B].$$

Seoudy and Aouf in [12] and Mostafa et al. in [20] introduced new subclasses of q -starlike (meromorphic) and q -convex (meromorphic) functions involving q -derivative operator, they obtained convolution properties and coefficient estimates for functions belonging to these classes. In this paper, we introduce two subclasses of multivalent functions and investigate convolution properties and coefficient estimates for these subclasses.

2 Main Results

Unless otherwise mentioned, we assume throughout this paper that $0 < q < 1$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $-1 \leq B < A \leq 1$, $\mathbb{U}^* = \mathbb{U} \setminus \{0\}$ and $b \in \mathbb{C}^*$.

Theorem 2.1. If $f \in \mathcal{A}(p)$, then $f \in \mathcal{S}_{q, p, b}^*[A, B]$ if and only if

$$\frac{1}{z^p} \left[f(z) * \frac{z^p - (q + C(\theta)) z^{p+1}}{(1-z)(1-qz)} \right] \neq 0, \text{ for all } z \in \mathbb{U}^* \text{ and } \theta \in [0, 2\pi), \quad (2.1)$$

where

$$C(\theta) = C_{q, p}^{b, A, B}(\theta) := \frac{(1 + q[p]_q - [p]_q)(e^{-i\theta} + B)}{(A - B)b[p]_q}. \quad (2.2)$$

Proof. For any function $f \in \mathcal{A}(p)$, we can verify that

$$f(z) * \frac{z^p}{1-z} = f(z) \quad (2.3)$$

and

$$f(z) * \left[\frac{z^{p+1}}{(1-z)(1-qz)} + \frac{[p]_q z^p}{1-qz} \right] = z D_{q, p} f(z). \quad (2.4)$$

First, in order to prove that (2.1) holds, we will write (1.5) by using the principle of subordination between analytic functions, that is,

$$\frac{1}{[p]_q} \frac{z D_{q, p} f(z)}{f(z)} = \frac{1 + [B + (A - B)b]w(z)}{1 + Bw(z)},$$

where w is a Schwarz function, hence

$$\frac{1}{z^p} \left[z D_{q,p} f(z) (1 + B e^{i\theta}) - [p]_q \left[1 + [B + (A - B)b] e^{i\theta} \right] f(z) \right] \neq 0, \quad (2.5)$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$. Since the convolution operator satisfy the distributivity $f * (g + h) = f * g + f * h$ for any functions $f, g, h \in \mathcal{A}(p)$, and from (2.3) and (2.4), the relation (2.5) may be written as

$$\begin{aligned} & \frac{1}{z^p} \left[(1 + B e^{i\theta}) \left(f(z) * \left[\frac{z^{p+1}}{(1-z)(1-qz)} + \frac{[p]_q z^p}{1-qz} \right] \right) \right. \\ & \left. - [p]_q \left[1 + [B + (A - B)qb] e^{i\theta} \right] \left(f(z) * \frac{z^p}{1-z} \right) \right] \neq 0, \end{aligned}$$

which is equivalent to

$$\frac{1}{z^p} \left[f(z) * \frac{-[p]_q b(A - B) e^{i\theta} z^p + \left[(1 + B e^{i\theta}) \left(1 + q[p]_q - [p]_q \right) + [p]_q qb(A - B) e^{i\theta} \right] z^{p+1}}{(1-z)(1-qz)} \right] \neq 0,$$

or

$$\frac{1}{z^p} \left[f(z) * \frac{z^p - \left[q + \frac{(1 + q[p]_q - [p]_q)(e^{-i\theta} + B)}{(A - B)b[p]_q} \right] z^{p+1}}{(1-z)(1-qz)} \right] \neq 0, \quad z \in \mathbb{U}^*, \quad \theta \in [0, 2\pi),$$

that is (2.1).

Reversely, suppose that $f \in \mathcal{A}(p)$ satisfy the condition (2.1). Like it was previously shown, the assumption (2.1) is equivalent to (2.5), that is,

$$\frac{1}{[p]_q} \frac{z D_{q,p} f(z)}{f(z)} \neq \frac{1 + [B + (A - B)b] e^{i\theta}}{1 + B e^{i\theta}}, \quad \text{for all } z \in \mathbb{U}^* \text{ and } \theta \in [0, 2\pi). \quad (2.6)$$

Denoting

$$\varphi(z) = \frac{1}{[p]_q} \frac{z D_{q,p} f(z)}{f(z)} \quad \text{and} \quad \psi(z) = \frac{1 + [B + (A - B)b]z}{1 + Bz},$$

the relation (2.6) could be written as $\varphi(\mathbb{U}) \cap \psi(\partial\mathbb{U}) = \emptyset$. Therefore, the simply connected domain $\varphi(\mathbb{U})$ is included in a connected component of $\mathbb{C} \setminus \psi(\partial\mathbb{U})$. From this fact, using that $\varphi(0) = \psi(0)$ together with the univalence of the function ψ , it follows that $\varphi(z) \prec \psi(z)$, that is $f \in \mathcal{S}_{q,p,b}^*[A, B]$. $\square$

Remark 2.1. (i) Putting $q \rightarrow 1^-$, $b = 1$ and $e^{i\theta} = x$ in Theorem 2.1, we obtain the result of Sarkar et al. [13, Theorem 2.1] with $\lambda = 0$ and $\phi(z) = \frac{1+Az}{1+Bz}$;

(ii) Putting $b = p = 1$ in Theorem 2.1, we obtain the result of Seoudy and Aouf [12, Theorem 1].

Theorem 2.2. *If $f \in \mathcal{A}(p)$, then $f \in \mathcal{C}_{q,p,b}[A, B]$ if and only if*

$$\frac{1}{z^p} \left[f(z) * \frac{[p]_q z^p - \left[[p]_q - (q+1) + (q+C(\theta))(1+q[p]_q) \right] z^{p+1} + q([p]_q - 1)(q+C(\theta)) z^{p+2}}{(1-z)(1-qz)(1-q^2z)} \right] \neq 0, \quad (2.7)$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$, where $C(\theta)$ is given by (2.2).

Proof. From (1.7) it follows that $f \in \mathcal{C}_{q,p,b}[A, B]$ if and only if $\Phi_q(z) := \frac{zD_{q,p}f(z)}{[p]_q} \in \mathcal{S}_{q,p,b}^*[A, B]$. Then, according to Theorem 2.1, the function Φ_q belongs to $\mathcal{S}_{q,p,b}^*[A, B]$ if and only if

$$\frac{1}{z^p} [\Phi_q(z) * g(z)] \neq 0, \text{ for all } z \in \mathbb{U}^* \text{ and } \theta \in [0, 2\pi), \quad (2.8)$$

where

$$g(z) = \frac{z^p - (q + C(\theta))z^{p+1}}{(1-z)(1-qz)}.$$

But (2.8) is equivalent to

$$\frac{1}{z^p} \left[\frac{1}{[p]_q} (f(z) * zD_{q,p}g(z)) \right] \neq 0,$$

that is,

$$\frac{1}{[p]_q z^p} [f(z) * zD_{q,p}g(z)] \neq 0 \Leftrightarrow \frac{1}{z^p} [f(z) * zD_{q,p}g(z)] \neq 0,$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$. Using the fact that

$$zD_{q,p}g(z) = \frac{[p]_q z^p - \left[[p]_q - (q+1) + (q+C(\theta))(1+q[p]_q) \right] z^{p+1} + q([p]_q - 1)(q+C(\theta))z^{p+2}}{(1-z)(1-qz)(1-q^2z)},$$

it is easy to check that (2.8) is equivalent to (2.7). □

Remark 2.2. (i) For $q \rightarrow 1^-$, $b = 1$ and $e^{i\theta} = x$ in Theorem 2.2, we obtain the result of Sarkar et al. [13, Theorem 2.3 with $\lambda = 0$ and $\phi(z) = \frac{1+Az}{1+Bz}$];
(ii) For $b = p = 1$ in Theorem 2.2, we obtain the result of Seoudy and Aouf [12, Theorem 5].

Theorem 2.3. *If $f \in \mathcal{A}(p)$, then $f \in \mathcal{S}_{q,p,b}^*[\alpha, \beta, \gamma; A, B]$ if and only if*

$$\begin{aligned} & 1 + \sum_{k=p+1}^{\infty} \frac{(q^\gamma; q)_{k-p}}{(q; q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p) + \beta)} \\ & \times \frac{b(A-B)[p]_q[k-p+1]_q - [k-p]_q \left[qb(A-B)[p]_q + (1+q[p]_q - [p]_q)(e^{-i\theta} + B) \right]}{b(A-B)[p]_q} a_k z^{k-p} \neq 0, \end{aligned} \quad (2.9)$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$.

Proof. If $f \in \mathcal{A}(p)$, then from Definition 1.2 and according to Theorem 2.1, we have $f \in \mathcal{S}_{q,p,b}^*[\alpha, \beta, \gamma; A, B]$ if and only if

$$\frac{1}{z^p} \left[\left(\mathbb{E}_{\alpha, \beta}^{\gamma, p} f \right) (z) * \frac{z^p - (q + C(\theta))z^{p+1}}{(1-z)(1-qz)} \right] \neq 0, \text{ for all } z \in \mathbb{U}^* \text{ and all } \theta \in [0, 2\pi), \quad (2.10)$$

where $C(\theta)$ is given by (2.2). Since

$$\frac{z^p}{(1-z)(1-qz)} = z^p + \sum_{k=p+1}^{\infty} [k-p+1]_q z^k, \quad \frac{z^{p+1}}{(1-z)(1-qz)} = \sum_{k=p+1}^{\infty} [k-p]_q z^k.$$

After some computations, we get

$$\frac{z^p - (q + C(\theta))z^{p+1}}{(1-z)(1-qz)} = z^p + \sum_{k=p+1}^{\infty} \left([k-p+1]_q - [k-p]_q (q + C(\theta)) \right) z^k,$$

then we may deduce that (2.10) is equivalent to (2.9), and the proof is completed. □

Theorem 2.4. If $f \in \mathcal{A}(p)$, then $f \in \mathcal{C}_{q,p,b}[\alpha, \beta, \gamma; A, B]$ if and only if

$$1 + \sum_{k=p+1}^{\infty} \frac{[k]_q (q^\gamma; q)_{k-p}}{[p]_q (q; q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p) + \beta)} \times \frac{b(A-B)[p]_q [k-p+1]_q - [k-p]_q \left[qb(A-B)[p]_q + (1+q[p]_q - [p]_q)(e^{-i\theta} + B) \right]}{b(A-B)[p]_q} a_k z^{k-p} \neq 0, \quad (2.11)$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$.

Proof. If $f \in \mathcal{A}(p)$, then from Definition 1.2 and Theorem 2.2, we have that $f \in \mathcal{C}_{q,p,b}[\alpha, \beta, \gamma; A, B]$ if and only if

$$\frac{1}{z^p} \left[\left(\mathbb{E}_{\alpha, \beta}^{\gamma, p} f \right) (z) * \frac{[p]_q z^p - \left[[p]_q - (q+1) + (q+C(\theta))(1+q[p]_q) \right] z^{p+1} + q([p]_q - 1)(q+C(\theta))z^{p+2}}{(1-z)(1-qz)(1-q^2z)} \right] \neq 0, \quad (2.12)$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$, where $C(\theta)$ is given by (2.2). Since

$$\begin{aligned} & \frac{[p]_q z^p - \left[[p]_q - (q+1) + (q+C(\theta))(1+q[p]_q) \right] z^{p+1} + q([p]_q - 1)(q+C(\theta))z^{p+2}}{(1-z)(1-qz)(1-q^2z)} \\ &= [p]_q z^p + \sum_{k=p+1}^{\infty} [k]_q \left([k-p+1]_q - [k-p]_q (q+C(\theta)) \right) z^k, \quad z \in \mathbb{U}. \end{aligned}$$

Now, we may check that (2.12) is equivalent to (2.11) which proves our result. $\square$

Unless otherwise mentioned, we assume throughout the remainder part of this section that α, β and γ are real numbers.

Theorem 2.5. If $f \in \mathcal{A}(p)$ and satisfies the inequality

$$\begin{aligned} & \sum_{k=p+1}^{\infty} \frac{(q^\gamma; q)_{k-p}}{(q; q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p) + \beta)} \\ & \times \frac{|b|(A-B)[p]_q [k-p+1]_q + [k-p]_q \left[q|b|(A-B)[p]_q + (1+q[p]_q - [p]_q)(1+|B|) \right]}{[p]_q} |a_k| < |b|(A-B), \end{aligned} \quad (2.13)$$

then $f \in \mathcal{S}_{q,p,b}^*[\alpha, \beta, \gamma; A, B]$.

Proof. If $f \in \mathcal{A}(p)$ has the form (1.1) and assuming that (2.9) holds, we obtain

$$\begin{aligned} & \left| 1 + \sum_{k=p+1}^{\infty} \frac{(q^\gamma; q)_{k-p}}{(q; q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p) + \beta)} \times \frac{b(A-B)[p]_q [k-p+1]_q - [k-p]_q \left[qb(A-B)[p]_q + (1+q[p]_q - [p]_q)(e^{-i\theta} + B) \right]}{b(A-B)[p]_q} a_k z^{k-p} \right| \\ & \geq 1 - \left| \sum_{k=p+1}^{\infty} \frac{(q^\gamma; q)_{k-p}}{(q; q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p) + \beta)} \times \frac{b(A-B)[p]_q [k-p+1]_q - [k-p]_q \left[qb(A-B)[p]_q + (1+q[p]_q - [p]_q)(e^{-i\theta} + B) \right]}{b(A-B)[p]_q} a_k z^{k-p} \right| \\ & \geq 1 - \sum_{k=p+1}^{\infty} \frac{(q^\gamma; q)_{k-p}}{(q; q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p) + \beta)} \times \frac{|b|(A-B)[p]_q [k-p+1]_q + [k-p]_q \left[q|b|(A-B)[p]_q + (1+q[p]_q - [p]_q)(1+|B|) \right]}{|b|(A-B)[p]_q} |a_k| > 0, \end{aligned}$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$. It follows that (2.9) holds, and from Theorem 2.3 we obtain our conclusion. $\square$

Using similar arguments to those in the proof of Theorem 2.5, we obtain the following theorem:

Theorem 2.6. *If $f \in \mathcal{A}(p)$ and satisfies the inequality*

$$\sum_{k=p+1}^{\infty} \frac{[k]_q (q^\gamma; q)_{k-p}}{[p]_q (q; q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p) + \beta)} \\ \times \frac{|b|(A-B)[p]_q [k-p+1]_q + [k-p]_q \left[q|b|(A-B)[p]_q + (1+q[p]_q - [p]_q)(1+|B|) \right]}{[p]_q} |a_k| < |b|(A-B),$$

then $f \in \mathcal{C}_{q,p,b}[\alpha, \beta, \gamma; A, B]$.

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