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NUMERICAL TREATMENT OF FIRST ORDER MATRIX DIFFERENTIAL EQUATIONS USING DIFFERENT CUBIC B-SPLINE FUNCTIONS

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Abstract

The aim of the present paper is to present numerical treatments for solving Sylvester and Riccati matrix differential equations of first order with polynomial, exponential and trigonometric cubic B-spline methods. Exactness and accuracy of the proposed methods are illustrated by calculating the maximum errors. The results of numerical experiments shown by these methods are convenient to be implemented and effective numerical technique for solving matrix differential equations.

Keywords: Sylvester matrix differential equations, Riccati matrix differential equations, Polynomial cubic B-spline, Exponential cubic B-spline, Trigonometric cubic B-spline, Kronecker product, and Frobenius norm.

Mathematics Subject Classification. 41A15, 65D07, 65M70, 65M12.

1 Introduction

Given a boundary matrix differential equation of first order

$$\left. \begin{aligned} U'(\tau) &= f(\tau, U(\tau)) \\ U(a) &= U_a, U(b) = U_b \end{aligned} \right\}, a \leq \tau \leq b, \quad (1)$$

where matrices $U_a, U_b, U(\tau) \in C^{m \times n}$ and matrix function $f: [a, b] \times C^{m \times n} \rightarrow C^{m \times n}$, different examples of problem (??) can be found in [1]. Sylvester matrix differential equations arise in many fields of science and engineering [2, 3], and Riccati matrix differential equations emerge a lot throughout science, applied mathematics and engineering. In particular, they play major roles in optimal control, filtering and estimation [4-6] and in solving linear, two-point boundary value problems of ordinary differential equations [7-10]. If $Y \in C^{m \times n}$ and $X \in C^{p \times q}$, we define the Kronecker product by $Y \otimes X$ [11]

$$Y \otimes X = \begin{pmatrix} y_{11}X & \cdots & y_{1n}X \\ \vdots & \ddots & \vdots \\ y_{m1}X & \cdots & y_{mn}X \end{pmatrix}. \quad (2)$$

The column vector operator on a matrix $Y \in C^{n \times m}$ is given by [11]:

$$Vec(Y) = \begin{bmatrix} Y_{\bullet 1} \\ \vdots \\ Y_{\bullet m} \end{bmatrix}, \quad \text{where} \quad Y_{\bullet k} = \begin{bmatrix} Y_{1k} \\ \vdots \\ Y_{mk} \end{bmatrix}. \quad (3)$$

Also, the derivative of a matrix $U \in C^{m \times n}$ with respect to a matrix $V \in C^{p \times q}$ is defined by [11]:

$$\frac{\partial U}{\partial V} = \begin{pmatrix} \frac{\partial U}{\partial v_{11}} & \cdots & \frac{\partial U}{\partial v_{1q}} \\ \vdots & \ddots & \vdots \\ \frac{\partial U}{\partial v_{p1}} & \cdots & \frac{\partial U}{\partial v_{pq}} \end{pmatrix}, \quad \text{where} \quad \frac{\partial U}{\partial v_n} = \begin{pmatrix} \frac{\partial U_{11}}{\partial v_n} & \cdots & \frac{\partial U_{1n}}{\partial v_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial U_{m1}}{\partial v_n} & \cdots & \frac{\partial U_{mn}}{\partial v_n} \end{pmatrix}. \quad (4)$$

The derivative of a matrix product $V \in C^{p \times q}$ and $U \in C^{q \times v}$ with respect to another matrix $W \in C^{m \times n}$ is given by [11]:

$$\frac{\partial VU}{\partial W} = \frac{\partial V}{\partial W} [I_n \otimes U] + [I_m \otimes V] \frac{\partial U}{\partial W}, \quad (5)$$

where the identity matrices of dimensions m and n denoted by I_m and I_n respectively.

The chain rule and derivative of a Kronecker product of matrices $V \otimes U$ with respect to a matrix W are given by [11]:

$$\frac{\partial W}{\partial V} = \left[\frac{\partial [\text{Vec}(U)]^T}{\partial V} \otimes I_m \right] + \left[I_q \otimes \frac{\partial W}{\partial [\text{Vec}(U)]} \right], \quad (6)$$

$$\frac{\partial (V \otimes U)}{\partial W} = \frac{\partial V}{\partial W} \otimes U + [I_m \otimes U_1] \left[\frac{\partial U}{\partial W} \otimes V \right] [I_n \otimes U_2], \quad (7)$$

where $V \in C^{p \times q}$, $U \in C^{u \times v}$, $W \in C^{m \times n}$ and U_1 , U_2 are permutation matrices.

The frobenius norm of $U \in C^{m \times n}$ is given by [12]:

$$\|U\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |u_{ij}|^2}. \quad (8)$$

The Frobenius norm and 2-norm holds [12]:

$$\|U\|_2 \leq \|U\|_F \leq \sqrt{n} \|U\|_2, \quad (9)$$

Cubic splines are discussed in [13-15], matrix differential equations are studied in [16-19] and exponential cubic B-splines are piecewise polynomial functions containing a free parameter and its properties are presented in [20]. The exponential and trigonometric cubic B-spline methods are studied to solve numerical solutions of various ordinary and partial differential equations [21-27] and the sextic and septic B-spline methods are introduced to solve Rosenau-KdV equation [28, 29]. This paper is organized as follows: In section 2, we present the polynomial, exponential and trigonometric cubic B-spline methods. In section 3, some numerical examples are discussed. Finally, the conclusion of this study is given in section 4.

2 Description of cubic B-spline methods

Firstly, we assume that the problem domain $[a, b]$ is equally divided into N subintervals $[\tau_i, \tau_{i+1}]$, $i = 0, 1, \dots, N-1$ by the knots $\tau_i = a + ih$ where $a = \tau_0 < \tau_1 < \dots < \tau_{N-1} < \tau_N = b$ and the step size $h = \frac{b-a}{N}$. Then cubic B-spline collocation methods for solving matrix boundary value problems (1) numerically are presented.

2.1 Polynomial cubic B-spline method (PCBSM)

The polynomial cubic B-spline can be defined as follows:

$$B_i(\tau) = \frac{1}{h^3} \begin{cases} (\tau - \tau_{i-2})^3 & \tau \in [\tau_{i-2}, \tau_{i-1}], \\ h^3 + 3h^2(\tau - \tau_{i-1}) + 3h(\tau - \tau_{i-1})^2 - 3(\tau - \tau_{i-1})^3 & \tau \in [\tau_{i-1}, \tau_i], \\ h^3 + 3h^2(\tau_{i+1} - \tau) + 3h(\tau_{i+1} - \tau)^2 - 3(\tau_{i+1} - \tau)^3 & \tau \in [\tau_i, \tau_{i+1}], \\ (\tau_{i+2} - \tau)^3 & \tau \in [\tau_{i+1}, \tau_{i+2}], \\ 0 & \text{elsewhere.} \end{cases} \quad (10)$$

$$(i = -1, 0, \dots, N+1).$$

We consider the spline function is interpolation to the solutions $u^{kl}(\tau)$ of the problem (1):

$$u^{kl}(\tau) = \sum_{i=-1}^{N+1} \omega_i^{kl}(\tau) PCB_i(\tau); 1 \leq k \leq n, 1 \leq l \leq m \quad (11)$$

where constants $\omega_i^{kl}(\tau)$'s are be determined. To solve boundary matrix differential equation of first order, we find PCB_i and PCB'_i at the nodal points are needed. Their coefficients are summarized in Table 1.

Table 1. values of PCB_i and PCB'_i .

τ	τ_{i-2}	τ_{i-1}	τ_i	τ_{i+1}	τ_{i+2}
PCB_i	0	1	4	1	0
PCB'_i	0	$-\frac{3}{h}$	0	$\frac{3}{h}$	0

Using Eqs. (??) and (??), the values of u_i^{kl} and their first derivatives at the knots are

$$\left. \begin{aligned} u_i^{kl} &= \omega_{i-1}^{kl} + 4\omega_i^{kl} + \omega_{i+1}^{kl} \\ u_i'^{kl} &= -\frac{3}{h}\omega_{i-1}^{kl} + \frac{3}{h}\omega_{i+1}^{kl} \end{aligned} \right\}, i = 0, 1, \dots, N. \quad (12)$$

Substituting from Eq. (??) in Eq. (??) we find,

$$\left[-\frac{3}{h}\omega_{i-1}^{kl} + \frac{3}{h}\omega_{i+1}^{kl} = f\left(ih, \omega_{i-1}^{kl} + 4\omega_i^{kl} + \omega_{i+1}^{kl}\right),\right] \quad (13)$$

where $i = 0, 1, \dots, N, k = 1, 2, \dots, n$ and $l = 1, 2, \dots, m$.

and the boundary conditions are given as

$$\begin{aligned} \omega_{-1}^{kl} + 4\omega_0^{kl} + \omega_1^{kl} &= u_a, \\ \omega_{N-1}^{kl} + 4\omega_N^{kl} + \omega_{N+1}^{kl} &= u_b. \end{aligned} \quad (14)$$

Solving the system of Eqs. (??) in ω_{-1}^{kl} and ω_{N+1}^{kl} , the linear algebraic system of Eqs. (??) can be converted to the following matrix form ;

$$A^{kl} \omega = F^{kl}, 1 \leq k \leq n, 1 \leq l \leq m. \quad (15)$$

where A^{kl} is an $(N+1) \times (N+1)$ matrix, ω is an $N+1$ dimensional vector with components ω_i^{kl} and the right hand side F^{kl} is an $N+1$ dimensional vector;

$$\omega = [\omega_0^{kl}, \omega_1^{kl}, \dots, \omega_N^{kl}]^T, F = [f_0^{kl}, f_1^{kl}, \dots, f_{N-1}^{kl}, f_N^{kl}]^T. \quad (16)$$

2.2 Exponential cubic B-spline method (ECBSM)

The exponential cubic B-spline can be defined as follows:

$$ECB_i(\tau) = \begin{cases} w_1 \left[(\tau_{i-2} - \tau) - \frac{1}{\eta} (\sinh(\eta(\tau_{i-2} - \tau))) \right] & \tau \in [\tau_{i-2}, \tau_{i-1}], \\ w_2 + w_3 (\tau_i - \tau) + w_4 e^{\eta(\tau_i - \tau)} + w_5 e^{-\eta(\tau_i - \tau)} & \tau \in [\tau_{i-1}, \tau_i], \\ w_2 + w_3 (\tau - \tau_i) + w_4 e^{\eta(\tau - \tau_i)} + w_5 e^{-\eta(\tau - \tau_i)} & \tau \in [\tau_i, \tau_{i+1}], \\ w_1 \left[(\tau - \tau_{i+2}) - \frac{1}{\eta} (\sinh(\eta(\tau - \tau_{i+2}))) \right] & \tau \in [\tau_{i+1}, \tau_{i+2}], \\ 0 & elsewhere. \end{cases} \quad (17)$$

$$(i = -1, 0, \dots, N+1).$$

where,

$$\begin{aligned} w_1 &= \frac{\eta}{2(\eta h C - S)}, w_2 = \frac{\eta h C}{\eta h C - S}, w_3 = \frac{\eta}{2} \left[\frac{C(C-1) + S^2}{(\eta h C - S)(1-C)} \right], \\ w_4 &= \frac{1}{4} \left[\frac{e^{-\eta h}(1-C) + S(e^{-\eta h} - 1)}{(\eta h C - S)(1-C)} \right], w_5 = \frac{1}{4} \left[\frac{e^{\eta h}(C-1) + S(e^{\eta h} - 1)}{(\eta h C - S)(1-C)} \right], \\ C &= \cosh(\eta h), S = \sinh(\eta h), \end{aligned}$$

and η is a free parameter.

We consider the spline function is interpolation to the solutions $u^{kl}(\tau)$ of the problem (??):

$$u^{kl}(\tau) = \sum_{i=-1}^{N+1} \zeta_i^{kl}(\tau) ECB_i(\tau); 1 \leq k \leq n, 1 \leq l \leq m, \quad (18)$$

where constants $\zeta_i^{kl}(\tau)$'s are to be determined. To solve boundary matrix differential equation of first order, we find ECB_i and ECB'_i at the nodal points are needed. Their coefficients are summarized in Table 2.

Table 2. Values of ECB_i and ECB'_i

τ	τ_{i-2}	τ_{i-1}	τ_i	τ_{i+1}	τ_{i+2}
ECB_i	0	β_1	1	β_1	0
ECB'_i	0	$-\beta_2$	0	β_2	0

where,

$$\beta_1 = \frac{S - \eta h}{2(\eta h C - S)}, \quad \beta_2 = \frac{\eta(C - 1)}{2(\eta h C - S)}, \quad \beta_3 = \frac{\eta^2 S}{2(\eta h C - S)}, \quad \beta_4 = \frac{-\eta^2 S}{\eta h C - S}.$$

Using Eqs. (17) and (18), the values of u_i^{kl} and their first derivatives at the knots are

$$\left. \begin{aligned} u_i^{kl} &= \beta_1 \zeta_{i-1}^{kl} + \zeta_i^{kl} + \beta_1 \zeta_{i+1}^{kl} \\ u'_i &= -\beta_2 \zeta_{i-1}^{kl} + \beta_2 \zeta_{i+1}^{kl} \end{aligned} \right\}, i = 0, 1, \dots, N. \quad (19)$$

Substituting from Eq. (??) in Eq. (??) we find,

$$[-\beta_2 \zeta_{i-1}^{kl} + \beta_2 \zeta_{i+1}^{kl} = f(ih, \beta_1 \zeta_{i-1}^{kl} + \zeta_i^{kl} + \beta_1 \zeta_{i+1}^{kl})], \quad (20)$$

where $i = 0, 1, \dots, N$, $k = 1, 2, \dots, n$ and $l = 1, 2, \dots, m$.

and the boundary conditions are given as

$$\left. \begin{aligned} \beta_1 \zeta_{-1}^{kl} + \zeta_0^{kl} + \beta_1 \zeta_1^{kl} &= u_a, \\ \beta_1 \zeta_{N-1}^{kl} + \zeta_N^{kl} + \beta_1 \zeta_{N+1}^{kl} &= u_b. \end{aligned} \right\} \quad (21)$$

Solving the system of Eqs. (21) in ζ_{-1}^{kl} and ζ_{N+1}^{kl} , the linear algebraic system of Eqs. (??) can be converted to the following matrix form ;

$$A^{kl} \zeta = F^{kl}, 1 \leq k \leq n, 1 \leq l \leq m. \quad (22)$$

where A^{kl} is an $(N+1) \times (N+1)$ matrix, ζ^{kl} is an $(N+1)$ dimensional vector with components ζ_i^{kl} and the right hand side F^{kl} is an $(N+1)$ dimensional vector;

$$\zeta^{kl} = [\zeta_0^{kl}, \zeta_1^{kl}, \dots, \zeta_N^{kl}]^T, F^{kl} = [f_0^{kl}, f_1^{kl}, \dots, f_{N-1}^{kl}, f_N^{kl}]^T. \quad (23)$$

2.3 Trigonometric cubic B-spline method (TCBSM)

The trigonometric cubic B-spline can be defined as follows:

$$TCB_i(\tau) = \frac{1}{\rho} \begin{cases} \varphi^3(\tau_{i-2}) & \tau \in [\tau_{i-2}, \tau_{i-1}], \\ \varphi(\tau_{i-2})[\varphi(\tau_{i-2})\vartheta(\tau_i) + \varphi(\tau_{i-1})\vartheta(\tau_{i+1})] + \varphi^2(\tau_{i-1})\vartheta(\tau_{i+2}) & \tau \in [\tau_{i-1}, \tau_i], \\ \vartheta(\tau_{i+2})[\vartheta(\tau_{i+2})\varphi(\tau_i) + \vartheta(\tau_{i+1})\varphi(\tau_{i+1})] + \vartheta^2(\tau_{i+1})\varphi(\tau_{i-2}) & \tau \in [\tau_i, \tau_{i+1}], \\ \vartheta^3(\tau_{i+2}) & \tau \in [\tau_{i+1}, \tau_{i+2}], \\ 0 & \text{elsewhere.} \end{cases} \quad (24)$$

$$(i = -1, 0, 1, \dots, N+1).$$

where,

$$\rho = \sin\left(\frac{h}{2}\right) \sin(h) \sin\left(\frac{3h}{2}\right), \quad \varphi(\tau_i) = \sin\left(\frac{\tau - \tau_i}{2}\right), \quad \vartheta(\tau_i) = \sin\left(\frac{\tau_i - \tau}{2}\right).$$

we consider the spline function is interpolation to the solutions $u^{kl}(\tau)$ of the problem (1):

$$u^{kl}(\tau) = \sum_{i=-1}^{N+1} T_i^{kl}(\tau) B_i(\tau) \quad ; 1 \leq k \leq n, 1 \leq l \leq m, \quad (25)$$

where constants $T_i^{kl}(\tau)$'s are to be determined. To solve boundary matrix differential equation of first order, we find TCB_i and TCB'_i at the nodal points are needed. Their coefficients are summarized in Table 3.

Table 3. values of TCB_i and TCB'_i .

τ	τ_{i-2}	τ_{i-1}	τ_i	τ_{i+1}	τ_{i+2}
TCB_i	0	Ω_1	Ω_2	Ω_1	0
TCB'_i	0	$-\Omega_3$	0	Ω_3	0

where,

$$\Omega_1 = \frac{\sin^2\left(\frac{h}{2}\right)}{\sin(h)\sin\left(\frac{3h}{2}\right)}, \quad \Omega_2 = \frac{2}{1+\cos(h)}, \quad \Omega_3 = \frac{3}{4\sin\left(\frac{3h}{2}\right)}, \quad \Omega_4 = \frac{3(1+3\cos(h))}{16\sin^2\left(\frac{h}{2}\right)(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right))},$$

$$\Omega_5 = \frac{-3\cos^2\left(\frac{h}{2}\right)}{\sin^2\left(\frac{h}{2}\right)(2+4\cos(h))}.$$

Using Eqs. (??) and (??), the values of u_i^{kl} and their first derivatives at the knots are

$$\left. \begin{aligned} u_i^{kl} &= \Omega_1 T_{i-1}^{kl} + \Omega_2 T_i^{kl} + \Omega_1 T_{i+1}^{kl} \\ u'_i &= -\Omega_3 T_{i-1}^{kl} + \Omega_3 T_{i+1}^{kl} \end{aligned} \right\}, i = 0, 1, \dots, N. \quad (26)$$

Substituting from Eq. (??) in Eq. (??) we find,

$$[-\Omega_3 T_{i-1}^{kl} + \Omega_3 T_{i+1}^{kl} = f(ih, \Omega_1 T_{i-1}^{kl} + \Omega_2 T_i^{kl} + \Omega_1 T_{i+1}^{kl})], \quad (27)$$

where $i = 0, 1, \dots, N$, $k = 1, 2, \dots, n$ and $l = 1, 2, \dots, m$.
and the boundary conditions are given as

$$\begin{aligned} \Omega_1 T_{-1}^{kl} + \Omega_2 T_0^{kl} + \Omega_1 T_1^{kl} &= u_a, \\ \Omega_1 T_{N-1}^{kl} + \Omega_2 T_N^{kl} + \Omega_1 T_{N+1}^{kl} &= u_b. \end{aligned} \quad (28)$$

Solving the system of Eqs. (??) in T_{-1}^{kl} and T_{N+1}^{kl} , the linear algebraic system of Eqs. (??) can be converted to the following matrix form;

$$A T = F, 1 \leq k \leq n, 1 \leq l \leq m. \quad (29)$$

where A is an $(N+1) \times (N+1)$ matrix, T is an $(N+1)$ dimensional vector with components T_i^{kl} and the right hand side F is an $(N+1)$ dimensional vector;

$$T = \begin{bmatrix} T_0^{kl} & T_1^{kl} & \dots & T_N^{kl} \end{bmatrix}^T, \quad F = \begin{bmatrix} f_0^{kl} & f_1^{kl} & \dots & f_{N-1}^{kl} & f_N^{kl} \end{bmatrix}^T. \quad (30)$$

3 Numerical examples

In this section, we present examples of matrix differential equations of first order to explore the efficiency and accuracy of the proposed methods using Frobenius norm of the difference between approximate solution and exact solution at each point in the interval $[a, b]$ taking a suitable step size h and the results are generated with Mathematica using Find Root function to solve the emerging algebraic equations.

Example 1. We examine the linear Sylvester matrix differential equation of the type [18]

$$U'(\tau) = A(\tau)U(\tau) + U(\tau)B(\tau) + C(\tau); a \leq \tau \leq b, U(\tau), A(\tau), B(\tau), C(\tau) \in C^{n \times n}, \quad (31)$$

where,

$$A(\tau) = \begin{pmatrix} 0 & \tau e^{-\tau} \\ \tau & 0 \end{pmatrix}, B(\tau) = \begin{pmatrix} 0 & \tau \\ 0 & 0 \end{pmatrix}, C(\tau) = \begin{pmatrix} -(1 + \tau^2)e^{-\tau} & -2\tau e^{-\tau} \\ 1 - \tau e^{-\tau} & -\tau^2 \end{pmatrix}, 0 \leq \tau \leq 1.$$

This example has an exact solution $U(\tau) = \begin{pmatrix} e^{-\tau} & 0 \\ \tau & 1 \end{pmatrix}$. Thus, we can compare our numerical estimates with this solution to obtain the exact errors of the approximation which summarized in Table 4 and figure 1.

For a free equilibrium points $u_j \equiv 0, (j = 1, 2, \dots, 4)$, we find that the Jacobian matrix of the Eq. (??) is

$$\begin{pmatrix} 0 & 0 & \tau e^{-\tau} & 0 \\ \tau & 0 & 0 & \tau e^{-\tau} \\ \tau & 0 & 0 & 0 \\ 0 & \tau & \tau & 0 \end{pmatrix}, \quad (32)$$

and its eigenvalues are evaluated from the equation

$$\lambda^4 - 2\tau^2 e^{-\tau} \lambda^2 + \tau^4 e^{-\tau} = 0$$

where $\lambda_{1,2} = -\tau e^{-\tau/2}$ and $\lambda_{3,4} = \tau e^{-\tau/2}$ have distinct signs for any value of τ in the interval $(0, 1]$, then the equilibrium points $u_j = 0$ of the Eq. (??) are unstable.; $(j = 1, 2, \dots, 4)$.

Table 4. Comparison of maximum absolute errors for Example 1.

τ	Ploynomial Cubic B-spline errors (PCBSM)	Exponential Cubic B-spline errors (ECBSM)	Trigonometric Cubic B-spline errors (TCBSM)	Cubic spline errors [18] (CSM)
0	6.28559×10^{-16}	1.11022×10^{-16}	1.11022×10^{-16}	1.33472×10^{-6}
0.1	2.79999×10^{-6}	8.83553×10^{-14}	1.14486×10^{-5}	1.33472×10^{-6}
0.2	7.66359×10^{-8}	2.38400×10^{-15}	3.17897×10^{-7}	1.24450×10^{-6}
0.3	2.79728×10^{-6}	8.94896×10^{-14}	1.14921×10^{-5}	1.24450×10^{-6}
0.4	1.00154×10^{-7}	3.57633×10^{-15}	4.19200×10^{-7}	1.17402×10^{-6}
0.5	2.79649×10^{-6}	9.04182×10^{-14}	1.15252×10^{-5}	1.17402×10^{-6}
0.6	8.73882×10^{-8}	3.31691×10^{-15}	3.68200×10^{-7}	1.12331×10^{-6}
0.7	2.79628×10^{-6}	9.10099×10^{-14}	1.15484×10^{-5}	1.12331×10^{-6}
0.8	5.09836×10^{-8}	2.26313×10^{-15}	2.15412×10^{-7}	1.09412×10^{-6}
0.9	2.79623×10^{-6}	9.13946×10^{-14}	1.15641×10^{-5}	1.09412×10^{-6}
1.0	3.18888×10^{-16}	1.11022×10^{-16}	5.55112×10^{-17}	1.09412×10^{-6}

Figure 1. Comparison of maximum absolute errors for Sylvester matrix differential equation in the interval $[0, 1]$ with step size $h = 0.1$.

Example 2. We investigate the rectangular non-symmetric Riccati matrix differential equation of the type [18]

$$U'(\tau) = C(\tau) - D(\tau)U(\tau) - U(\tau)A(\tau) - U(\tau)B(\tau)U(\tau); U(\tau) \in C^{n \times m}, U(\tau) \in C^{n \times m}, A(\tau) \in C^{m \times m}, B(\tau) \in C^{m \times n}, C(\tau) \in C^{n \times m}, D(\tau) \in C^{n \times n}, 0 \leq \tau \leq a \quad (33)$$

where,

$$A(\tau) = \begin{pmatrix} -\tau & 0 \\ -\tau & \tau \end{pmatrix}, B(\tau) = \begin{pmatrix} -\tau^2 & -2 \\ 0 & 1 \end{pmatrix}, D(\tau) = \begin{pmatrix} -1 & -\tau^{-2} \\ \tau & \tau \end{pmatrix}, C(\tau) = \begin{pmatrix} \tau(-e^\tau + \tau e^\tau - \tau^3) & \tau(2e^\tau - \tau^2) \\ \tau(1-\tau)(2+\tau+2\tau^2) & 1+\tau^2(3-2\tau)+e^\tau(\tau-\tau^4) \end{pmatrix}, 0 \leq \tau \leq 0.1.$$

This example has an exact solution $U(\tau) = \begin{pmatrix} 0 & e^\tau \\ \tau^2 & \tau \end{pmatrix}$. Thus, we can compare our numerical estimates with this solution to obtain the exact errors of the approximation which summarized in Table 5 and figure 2.

For a free equilibrium points $u_j \equiv 0, (j = 1, 2, \dots, 4)$, we find that the Jacobian matrix of the Eq. (33) is

$$\begin{pmatrix} 1+\tau & \tau & \tau^2 & 0 \\ 0 & 1-\tau & 0 & \tau^2 \\ -\tau & 0 & 0 & \tau \\ 0 & -\tau & 0 & -2\tau \end{pmatrix}, \quad (34)$$

and its eigenvalues are evaluated from the equation

$$\lambda^4 + 2(-1+\tau)\lambda^3 + (1-4\tau-\tau^2+2\tau^3)\lambda^2 + 2(\tau-2\tau^3+\tau^4)\lambda + (-2+2\tau+\tau^2)\tau^4 = 0$$

where $\lambda_{1,2} = \frac{1}{2}(1-3\tau \pm \sqrt{1+2\tau+\tau^2-4\tau^3})$ and $\lambda_{3,4} = \frac{1}{2}(1+\tau \pm \sqrt{1+2\tau+\tau^2-4\tau^3})$ have distinct sign for any value of τ in the interval $(0, 0.1]$, then the equilibrium points $u_j = 0$ of the Eq. (33) are unstable.; $(j = 1, 2, \dots, 4)$.

Table 5. Comparison of maximum absolute errors for Example 2.

τ	Polynomial Cubic B-spline errors (PCBSM)	Exponential Cubic B-spline errors (ECBSM)	Trigonometric Cubic B-spline errors (TCBSM)	Cubic spline errors [18] (CSM)
0	2.1684×10^{-19}	4.33681×10^{-19}	1.11023×10^{-16}	0
0.01	7.79457×10^{-11}	2.01112×10^{-15}	5.22446×10^{-10}	1.39903×10^{-10}
0.02	1.42204×10^{-13}	7.42212×10^{-19}	5.83501×10^{-13}	2.9602×10^{-12}
0.03	7.74756×10^{-11}	1.56922×10^{-15}	5.22841×10^{-10}	1.41976×10^{-10}
0.04	2.14239×10^{-13}	1.14405×10^{-18}	8.73255×10^{-13}	5.91358×10^{-12}
0.05	7.69626×10^{-11}	1.78941×10^{-15}	5.23167×10^{-10}	1.44084×10^{-10}
0.06	2.14852×10^{-13}	1.0766×10^{-18}	8.71469×10^{-13}	8.85918×10^{-12}
0.07	7.64061×10^{-11}	1.79108×10^{-15}	5.23433×10^{-10}	1.46223×10^{-10}
0.08	1.43865×10^{-13}	1.39261×10^{-17}	5.79685×10^{-13}	1.17952×10^{-11}
0.09	7.5805×10^{-11}	1.78943×10^{-15}	5.23647×10^{-10}	1.4839×10^{-10}
0.1	1.73472×10^{-18}	1.73472×10^{-18}	2.22485×10^{-16}	1.4721×10^{-11}

4 Conclusion

In this article, we have examined scheme treat numerically with the first-order matrix differential equations by cubic B-spline method and exponential and trigonometric cubic B-spline methods. From the computational results, we can view that the cubic B-spline and exponential and trigonometric cubic B-splines are summarized and easy to apply and the errors are acceptable. The numerical experiments are compared with the analytic solutions by finding Frobenius norm and are compared with Ref. [18] as shown in Tables (3-5) and Figures (1, 2).

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Figure 2. Comparison of maximum absolute errors for Riccati matrix differential equation in the interval $[0, 1]$ with step size $h = 0.01$.

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