THEORETICAL AND NUMERICAL STUDIES OF TWO POINT BOUNDARY VALUE PROBLEMS USING TRIGONOMETRIC AND EXPONENTIAL CUBIC B-SPLINES

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Abstract

A collocation method with different types of B-spline functions such as trigonometric and exponential cubic B-splines functions are proposed from many researchers for studying the numerical solutions of many physical nonlinear partial differential equations and ordinary differential equations. In this paper, a collocation finite difference scheme based on trigonometric and exponential cubic B-splines functions are investigated and applied for the numerical solution of two-point boundary value problems. The convergence of trigonometric and exponential cubic B-splines is proved using diagonally dominant matrices. The numerical examples show that our method is very effective and the maximum absolute error is acceptable.

Keywords: Two point boundary value problems, Trigonometric cubic B-spline, Exponential cubic B-spline.

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1 Introduction

We consider the two boundary point value problem, given by

$$\alpha(x) V''(x) + \beta(x) V'(x) + \gamma(x) V(x) = g(x, V) \quad \text{in } [a, b], \quad (1)$$

subject to boundary conditions

$$V(a) = V_0, \quad \text{and} \quad V(b) = V_n, \quad (2)$$

where $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are coefficients and $g(x, V)$ is forcing function. Cubic splines are used in [1-8] and B-splines as cubic, quintic and septic are presented in [9-15]. The trigonometric cubic B-spline is studied to solve numerical solutions of various partial differential equations [16-20]. The exponential cubic B-spline is piecewise polynomial functions containing a free parameter and its properties are presented in [21], and is used for finding numerical solutions of various differential equations [22-31]. This paper is organized as follows: In section 2, we describe the proposed method. In section 3, some numerical examples are discussed. Finally, the conclusion of this study is given in section 4.

2 Description of cubic B-spline methods

Firstly, we assume that the problem domain $[a, b]$ is equally divided into n subintervals $[x_j, x_{j+1}]$, $j = 0, 1, 2, \dots, n-1$ by the knots $x_j = a + jh$ where $a = x_0 < x_1 < \dots < x_n = b$ and the step size $h = \frac{b-a}{n}$. Then cubic B-spline collocation methods for solving a two point value problems (1) numerically are presented.

2.1 Trigonometric cubic B-spline method

On the above partition together with additional knots $x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}$ outside the problem domain, the trigonometric cubic B-spline $TCB_j(x)$ is defined as follows

$$TCB_j(x) = \frac{1}{\rho} \begin{cases} \varphi^3(x_{j-2}), & x \in [x_{j-2}, x_{j-1}] \\ \varphi(x_{j-2})(\varphi(x_{j-2})\theta(x_j) + \varphi(x_{j-1})\theta(x_{j+1})) + \varphi^2(x_{j-1})\theta(x_{j+2}), & x \in [x_{j-1}, x_j] \\ \varphi(x_{j-2})\theta^2(x_{j+1}) + \theta(x_{j+2})(\varphi(x_{j-1})\theta(x_{j+1}) + \varphi(x_j)\theta(x_{j+2})), & x \in [x_j, x_{j+1}] \\ \theta^3(x_{j+2}), & x \in [x_{j+1}, x_{j+2}] \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

$j = -1, 0, 1, \dots, n+1,$

where $\rho = \sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)$, $\varphi(x_j) = \sin\left(\frac{x-x_j}{2}\right)$ and $\theta(x_j) = \sin\left(\frac{x_j-x}{2}\right)$.

An approximate solution $v(x)$ to the unknown $V(x)$, of the problem (1) and (2), is written in terms of the expansion of the $TCB_j(x)$ as

$$v(x) = \sum_{j=-1}^{n+1} \tau_j TCB_j(x), \quad (4)$$

where constants τ_j 's are determined from the collocation points x_j , $j = 0, 1, 2, \dots, n$ and the boundary and initial conditions. The values of the trigonometric cubic B-spline functions $TCB_j(x)$ and its principle two derivatives $TCB'_j(x)$ and $TCB''_j(x)$ at the knots and their values are summarized in Table 1.

Table 1. Values of $TCB_j(x)$ and its principle two derivatives at the knot points

x	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
$TCB_j(x)$	0	δ_1	δ_2	δ_1	0
$TCB'_j(x)$	0	$-\delta_3$	0	δ_3	0
$TCB''_j(x)$	0	δ_4	δ_5	δ_4	0

where

$$\delta_1 = \frac{\sin^2\left(\frac{h}{2}\right)}{\sin(h)\sin\left(\frac{3h}{2}\right)}, \delta_2 = \frac{2}{1+2\cos(h)}, \delta_3 = \frac{3}{4\sin\left(\frac{3h}{2}\right)}, \delta_4 = \frac{3(1+3\cos(h))}{16\sin^2\left(\frac{h}{2}\right)(2\cos\left(\frac{h}{2}\right) + \cos\left(\frac{3h}{2}\right))},$$

$$\delta_5 = \frac{-3\cos^2\left(\frac{h}{2}\right)}{\sin^2\left(\frac{h}{2}\right)(2+4\cos(h))}.$$

Using equations (3) and (4), the values of v_j and their derivatives up to second order at the knots are

$$\begin{aligned} v_j &= \delta_1\tau_{j-1} + \delta_2\tau_j + \delta_1\tau_{j+1} \\ v'_j &= -\delta_3\tau_{j-1} + \delta_3\tau_{j+1} \\ v''_j &= \delta_4\tau_{j-1} + \delta_5\tau_j + \delta_4\tau_{j+1} \end{aligned}, \quad j = 0, 1, 2, \dots, n. \quad (5)$$

Substituting from (5) in Eq. (1) and Eq. (2) we find,

$$\begin{aligned} &(\delta_4\alpha(x_j) - \delta_3\beta(x_j) + \delta_1\gamma(x_j))\tau_{j-1} + (\delta_5\alpha(x_j) + \delta_2\gamma(x_j))\tau_j \\ &+ (\delta_4\alpha(x_j) + \delta_3\beta(x_j) + \delta_1\gamma(x_j))\tau_{j+1} = g_j, \quad j = 0, 1, \dots, n, \end{aligned} \quad (6)$$

and the boundary conditions (2) are given as

$$\begin{aligned} \delta_1\tau_{-1} + \delta_2\tau_0 + \delta_1\tau_1 &= V_0 \\ \delta_1\tau_{n-1} + \delta_2\tau_n + \delta_1\tau_{n+1} &= V_n. \end{aligned} \quad (7)$$

Solving the system of equations (7) in τ_{-1} and τ_{n+1} , the linear algebraic system of equations (6) can be converted to the following matrix form ;

$$A\tau = g(\tau), \quad (8)$$

where A is an $(n+1) \times (n+1)$ matrix, τ is an $(n+1)$ dimensional vector with components τ_j and the right hand side $g(\tau)$ is an $(n+1)$ dimensional vector.

For appropriately enough small h , the matrix A is diagonally dominant if

$$\begin{cases} |\delta_5\alpha + \delta_2\gamma| - \{|\delta_4\alpha - \delta_3\beta + \delta_1\gamma| + |\delta_4\alpha + \delta_3\beta + \delta_1\gamma|\} = \\ \quad \begin{cases} (-\delta_5\alpha - \delta_2\gamma) - \{(\delta_4\alpha - \delta_3\beta + \delta_1\gamma) + (\delta_4\alpha + \delta_3\beta + \delta_1\gamma)\} \\ \quad = \left(\frac{3 \sec(\frac{h}{2}) \tan(\frac{h}{4})^2}{-4+8 \cos(\frac{h}{2})} \right) \alpha - \left(\frac{2+\sec(\frac{h}{2})}{1+\cos(h)} \right) \gamma \\ \quad \approx -\gamma > 0, \quad (\alpha > 0), \\ (\delta_5\alpha + \delta_2\gamma) - \{(-\delta_4\alpha + \delta_3\beta - \delta_1\gamma) + (-\delta_4\alpha - \delta_3\beta - \delta_1\gamma)\} \\ \quad = \left(\frac{3 \sec(\frac{h}{2}) \tan(\frac{h}{4})^2}{4-8 \cos(\frac{h}{2})} \right) \alpha + \left(\frac{2+\sec(\frac{h}{2})}{1+\cos(h)} \right) \gamma \\ \quad \approx \gamma > 0, \quad (\alpha < 0), \end{cases} \end{cases} \quad (9)$$

and

$$\alpha(x_j)\gamma(x_j) < 0, \quad j = 0, 1, \dots, n. \quad (10)$$

Furthermore,

$$\|A^{-1}\|_\infty \leq \frac{1}{\min_{0 \leq j \leq n} |\gamma(x_j)|} \equiv M, \quad (11)$$

where $\|\cdot\|_\infty$ indicates the maximum norm in $[a, b]$. Since D is a linear second order differential operator at the knots, we write,

$$D(\bar{v}(x_j)) = g(x_j, \bar{v}(x_j)) + \epsilon(x_j), \quad j = 0, 1, \dots, n, \quad (12)$$

where $\bar{v}(x)$ be the trigonometric cubic B-spline of interpolation to the unique solution of the problems (1) and (2). $\epsilon(x_j)$ mean that error function.

Thus, the equation (8) becomes

$$A\bar{\tau} = g(\bar{\tau}) + \epsilon, \quad (13)$$

where A is an $(n+1) \times (n+1)$ matrix, $\bar{\tau}$ is an $(n+1)$ dimensional vector with components $\bar{\tau}_j$ and the right hand side $g(\bar{\tau})$ and ϵ are an $(n+1)$ dimensional vector.

Let the Lipschitz condition on the forcing function is of the form:

$$|g(x, v_1) - g(x, v_2)| \leq L |v_1 - v_2| \quad \text{for all } x \in [a, b], \quad (14)$$

where the constant L is independent of x .

From the equations (8) and (13), we obtain

$$Ae = \epsilon + g(\tau) - g(\bar{\tau}), \quad (15)$$

where $e_j = \tau_j - \bar{\tau}_j = (e_0, e_1, \dots, e_n)^T$ and applying the Lipschitz condition (14) on the forcing function, then

$$\begin{aligned} |g(x_j, v(x_j)) - g(x_j, \bar{v}(x_j))| &= L_j |v(x_j) - \bar{v}(x_j)| \\ &= \begin{cases} 0, & j = 0 \\ L_j \left(\frac{1}{6}e_{j-1} + \frac{2}{3}e_j + \frac{1}{6}e_{j+1} \right), & 1 \leq j \leq n-1 \\ 0, & j = n. \end{cases} \end{aligned} \quad (16)$$

Substituting the equation (16) in the equation (15), we obtain

$$Ae = \epsilon + \tilde{L}Ne, \quad (17)$$

$$\text{where } \tilde{L} = \text{Diag} \{L_0, L_1, \dots, L_n\}, N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So that $e = A^{-1}\epsilon + A^{-1}\tilde{L}Ne$, when A^{-1} exists and bounded as in (11), then

$$\|e\|_{\infty} = M\|\epsilon\|_{\infty} + ML\|e\|_{\infty}, \quad (18)$$

where M is abounded on $\|A^{-1}\|_{\infty}$ and $\|N\|_{\infty} = 1$, then

$$1 - ML > 0, \quad (19)$$

and we have

$$\|e\|_{\infty} \leq \frac{\|\epsilon\|_{\infty}}{\min_{0 \leq j \leq n} |\gamma(x_j)| - L}. \quad (20)$$

Then the following theorem shows the convergence of the trigonometric cubic B-spline.

Theorem 2.1 Let a two point boundary value problem have the form (1) and (2) where the coefficients $\alpha(x)$, $\beta(x)$, $\gamma(x)$ and the forcing function $g(x, V)$ are satisfied the conditions (10), (14) and (19), then the trigonometric cubic B-spline $\bar{v}(x)$ interpolating to $v(x)$ converges to $V(x)$ in the maximum error over $[a, b]$ with error norm as (20).

2.2 Exponential cubic B-spline method

The exponential cubic B-spline is

$$ECB_j(x) = \begin{cases} \omega_1 \left((x_{j-2} - x) - \frac{1}{\rho} (\sinh(\rho(x_{j-2} - x))) \right), & x \in [x_{j-2}, x_{j-1}] \\ \omega_2 + \omega_3(x_j - x) + \omega_4 \exp(\rho(x_j - x)) + \omega_5 \exp(-\rho(x_j - x)), & x \in [x_{j-1}, x_j] \\ \omega_2 + \omega_3(x - x_j) + \omega_4 \exp(\rho(x - x_j)) + \omega_5 \exp(-\rho(x - x_j)), & x \in [x_j, x_{j+1}] \\ \omega_1 \left((x - x_{j+2}) - \frac{1}{\rho} (\sinh(\rho(x - x_{j+2}))) \right), & x \in [x_{j+1}, x_{j+2}] \\ 0, & otherwise \end{cases} \quad (21)$$

$j = -1, 0, 1, \dots, n+1,$

where

$$\omega_1 = \frac{\rho}{2(\rho h C - S)}, \omega_2 = \frac{\rho h C}{\rho h C - S}, \omega_3 = \frac{\rho}{2} \left(\frac{C(C-1) + S^2}{(\rho h C - S)(1-C)} \right),$$

$$\omega_4 = \frac{1}{4} \left(\frac{e^{-\rho h} (1-C) + S(e^{-\rho h} - 1)}{(\rho h C - S)(1-C)} \right), \omega_5 = \frac{1}{4} \left(\frac{e^{\rho h} (C-1) + S(e^{\rho h} - 1)}{(\rho h C - S)(1-C)} \right), C = \cosh(\rho h),$$

$S = \sinh(\rho h)$ and ρ is a free parameter.

We let

$$v(x) = \sum_{j=-1}^{n+1} \varepsilon_j ECB_j(x), \quad (22)$$

be the solution of the problem (1) and (2) as a cubic B-spline of the interpolation to the true solution $V(x)$, where constants ε_j 's are determined. To solve a two point value boundary value problems, we find $ECB_j(x)$, $ECB'_j(x)$ and $ECB''_j(x)$ at the knots and their values are summarized in Table 2.

Table 2. Values of $ECB_j(x)$, $ECB'_j(x)$ and $ECB''_j(x)$ at the knot points.

x	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
$ECB_j(x)$	0	μ_1	μ_2	μ_1	0
$ECB'_j(x)$	0	$-\mu_3$	0	μ_3	0
$ECB''_j(x)$	0	μ_4	μ_5	μ_4	0

where

$$\mu_1 = \frac{S - \rho h}{2(\rho h C - S)}, \mu_2 = 1, \mu_3 = \frac{\rho(C-1)}{2(\rho h C - S)}, \mu_4 = \frac{\rho^2 S}{2(\rho h C - S)},$$

$$\mu_5 = -\frac{\rho^2 S}{\rho h C - S}.$$

Using equations (21) and (22), the values of v_j and their derivatives up to second order at the knots are

$$\begin{aligned} v_j &= \mu_1 \varepsilon_{j-1} + \mu_2 \varepsilon_j + \mu_1 \varepsilon_{j+1} \\ v'_j &= -\mu_3 \varepsilon_{j-1} + \mu_3 \varepsilon_{j+1} \\ v''_j &= \mu_4 \varepsilon_{j-1} + \mu_5 \varepsilon_j + \mu_4 \varepsilon_{j+1} \end{aligned} \quad , \quad j = 0, 1, 2, \dots, n. \quad (23)$$

Substituting from (23) in Eq. (1) and Eq. (2) we find,

$$\begin{aligned} &(\mu_4 \alpha(x_j) - \mu_3 \beta(x_j) + \mu_1 \gamma(x_j)) \varepsilon_{j-1} + (\mu_5 \alpha(x_j) + \mu_2 \gamma(x_j)) \varepsilon_j \\ &+ (\mu_4 \alpha(x_j) + \mu_3 \beta(x_j) + \mu_1 \gamma(x_j)) \varepsilon_{j+1} = g_j, \quad j = 0, 1, \dots, n, \end{aligned} \quad (24)$$

and the boundary conditions (2) are given as

$$\begin{aligned} \mu_1 \varepsilon_{-1} + \mu_2 \varepsilon_0 + \mu_1 \varepsilon_1 &= V_0 \\ \mu_1 \varepsilon_{n-1} + \mu_2 \varepsilon_n + \mu_1 \varepsilon_{n+1} &= V_n. \end{aligned} \quad (25)$$

From the equations (24) and (25), we obtain

$$A\varepsilon = g(\varepsilon), \quad (26)$$

where A is an $(n+1) \times (n+1)$ matrix, ε is an $(n+1)$ dimensional vector with components ε_j and the right hand side $g(\varepsilon)$ is an $(n+1)$ dimensional vector.

For appropriately enough small h the matrix A is diagonally dominant if

$$\left\{ \begin{aligned} &|\mu_5 \alpha + \mu_2 \gamma| - \{|\mu_4 \alpha - \mu_3 \beta + \mu_1 \gamma| + |\mu_4 \alpha + \mu_3 \beta + \mu_1 \gamma|\} = \\ &\quad (-\mu_5 \alpha - \mu_2 \gamma) - \{(\mu_4 \alpha - \mu_3 \beta + \mu_1 \gamma) + (\mu_4 \alpha + \mu_3 \beta + \mu_1 \gamma)\} \\ &\quad = -\left(1 + \frac{S-h}{hC-S}\right) \gamma \\ &\quad \approx -\frac{3}{2} \gamma > 0, \quad (\alpha > 0), \\ &(\mu_5 \alpha + \mu_2 \gamma) - \{(-\mu_4 \alpha + \mu_3 \beta - \mu_1 \gamma) + (-\mu_4 \alpha - \mu_3 \beta - \mu_1 \gamma)\} \\ &\quad = \left(1 + \frac{S-h}{hC-S}\right) \gamma \\ &\quad \approx \frac{3}{2} \gamma > 0, \quad (\alpha < 0), \end{aligned} \right. \quad (27)$$

and

$$\alpha(x_j) \gamma(x_j) < 0, \quad 0 \leq j \leq n. \quad (28)$$

Furthermore,

$$\|A^{-1}\|_{\infty} \leq \frac{2}{3 \min_{0 \leq j \leq n} |\gamma(x_j)|} \equiv M, \quad (29)$$

where $\|\cdot\|_{\infty}$ indicates the maximum norm in $[a, b]$, since D is a linear second order differential operator at the knots, we write,

$$D(\bar{v}(x_j)) = g(x_j, \bar{v}(x_j)) + \epsilon(x_j), \quad 0 \leq j \leq n, \quad (30)$$

where $\bar{v}(x)$ be the exponential cubic B-spline of interpolation to the unique solution of the problems (1) and (2). $\epsilon(x_j)$ mean that error function.

Thus, the equation (26) becomes

$$A\bar{\varepsilon} = g(\bar{\varepsilon}) + \epsilon, \quad (31)$$

where A is an $(n+1) \times (n+1)$ matrix, $\bar{\varepsilon}$ is an $(n+1)$ dimensional vector with components, $\bar{\varepsilon}_j$ and the right hand side $g(\bar{\varepsilon})$ and ϵ are an $(n+1)$ dimensional vector.

Let the Lipschitz condition on the forcing function:

$$|g(x, v_1) - g(x, v_2)| \leq L |v_1 - v_2| \quad \text{for all } x \in [a, b], \quad (32)$$

where the constant L is independent of x .

From the equations (26) and (31), we obtain

$$Ae = \epsilon + g(\varepsilon) - g(\bar{\varepsilon}), \quad (33)$$

where $e_j = \varepsilon_j - \bar{\varepsilon}_j = (e_0, e_1, \dots, e_n)^T$ and applying the Lipschitz condition (32) on the forcing function, then

$$|g(x_j, v(x_j)) - g(x_j, \bar{v}(x_j))| = L_j |v(x_j) - \bar{v}(x_j)| = \begin{cases} 0, & j = 0 \\ L_j (\frac{1}{4}e_{j-1} + e_j + \frac{1}{4}e_{j+1}), & 1 \leq j \leq n-1 \\ 0, & j = n. \end{cases} \quad (34)$$

Substituting the equation (34) in the equation (33), we obtain

$$Ae = \epsilon + \tilde{L}Ne, \quad (35)$$

$$\text{where } \tilde{L} = \text{Diag} \{L_0, L_1, \dots, L_n\}, N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 1 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So that $e = A^{-1}\epsilon + A^{-1}\tilde{L}Ne$, when A^{-1} exists and bounded as in (29), then

$$\|e\|_\infty = M\|\epsilon\|_\infty + \frac{3}{2}ML\|e\|_\infty, \quad (36)$$

where M is abounded on $\|A^{-1}\|_\infty$ and $\|N\|_\infty = \frac{3}{2}$, then

$$1 - \frac{3}{2}ML > 0, \quad (37)$$

and we have

$$\|e\|_\infty \leq \frac{2\|\epsilon\|_\infty}{3\min_{0 \leq j \leq n} |\gamma(x_j)| - L}. \quad (38)$$

Then the following theorem shows the convergence of the trigonometric cubic B-spline.

Theorem 2.2 Let a two point boundary value problems have the form (1) and (2) where the coefficients $\alpha(x)$, $\beta(x)$, $\gamma(x)$ and the forcing function $g(x, V)$ are satisfied the conditions (28), (32) and (37), then the exponential cubic B-spline $\bar{v}(x)$ interpolating to $v(x)$ converges to $V(x)$ in the maximum error over $[a, b]$ with error norm as (38).

3 Numerical examples

In this section, we present some examples of two point boundary value problems. We take different values of h on the interval $[0, 1]$ and the results are generated with Mathematica using FindRoot function to solve the emerging algebraic equations. At each h , we evaluated the difference between approximate solution and exact solution, and then take the absolute errors of this difference.

Example 1. We consider linear boundary value problem [9]

$$V''(x) - 100V(x) = 0, \quad V(0) = V(1) = 1, \quad (39)$$

and the exact solution is given by $V(x) = \text{Cosh}(10x - 5)/\text{Cosh}(5)$. Thus, we can compare our numerical estimates with this solution to obtain the exact maximum errors of the approximation which summarized in Table 3.

Table 3. Comparison of maximum absolute errors for Example 1.

h	Trigonometric Cubic B-spline errors	CPU times (s)	Exponential Cubic B-spline errors	CPU times (s)	Cubic B-spline errors [9]
1/5	6.58×10^{-2}	0.265	6.30×10^{-2}	0.25	1.00×10^{-1}
1/10	1.73×10^{-2}	0.28	1.67×10^{-2}	0.28	1.69×10^{-2}
1/15	6.96×10^{-3}	0.312	6.71×10^{-3}	0.281	7.30×10^{-3}
1/20	4.03×10^{-3}	0.343	3.89×10^{-3}	0.297	3.93×10^{-3}

Figure 1. Numerical solutions of Example 1 at $n = 20$.

Example 2. We present another linear boundary value problem [9]

$$V''(x) - 4V(x) - 4\text{Cosh}(1) = 0, \quad V(0) = V(1) = 0, \quad (40)$$

and the exact solution is given by $V(x) = \text{Cosh}(2x - 1) - \text{Cosh}(1)$. Thus, we can compare our numerical estimates with this solution to obtain the exact maximum errors of the approximation which summarized in Table 4.

Table 4. Comparison of maximum absolute errors for Example 2.

h	Trigonometric Cubic B-spline errors	Exponential Cubic B-spline errors	Cubic B-spline errors [9]
1/3	2.21×10^{-2}	9.99×10^{-3}	1.53×10^{-2}
1/5	8.10×10^{-3}	3.73×10^{-3}	5.23×10^{-3}
1/7	4.15×10^{-3}	1.92×10^{-3}	2.63×10^{-3}
1/9	2.51×10^{-3}	1.16×10^{-3}	1.58×10^{-3}

Figure 2. Numerical solutions of Example 2 at $n = 10$.

Example 3. We examine nonlinear boundary value problem [9]

$$V''(x) - e^{V(x)} = 0, \quad V(0) = V(1) = 0, \quad (41)$$

and the exact solution is given by $V(x) = \ln(0.5) + 2\ln\left(c \text{Sec}\left(\frac{c(x-0.5)}{2}\right)\right)$, $c = 1.3360556949$. Thus, we can compare our numerical estimates with this solution to obtain the exact maximum errors of the approximation which summarized in Table 5.

Table 5. Comparison of maximum absolute errors for Example 3.

h	Trigonometric Cubic B-spline errors	Exponential Cubic B-spline errors	Cubic B-spline errors [9]
1/3	3.04×10^{-3}	2.92×10^{-5}	9.59×10^{-4}
1/4	1.87×10^{-3}	2.53×10^{-5}	5.20×10^{-4}
1/6	8.23×10^{-4}	1.24×10^{-5}	2.29×10^{-4}
1/8	4.60×10^{-4}	7.19×10^{-6}	1.28×10^{-4}

Figure 3. Numerical solutions of Example 3 at $n = 8$.

Example 4. We investigate another nonlinear boundary value problem [9]

$$V''(x) - 0.5(V(x) + x + 1)^3 = 0, \quad V(0) = V(1) = 0, \quad (42)$$

and the exact solution is given by $V(x) = \frac{2}{2-x} - x - 1$. Thus, we can compare our numerical estimates with this solution to obtain the exact maximum errors of the approximation which summarized in Table 6.

Table 6. Comparison of maximum absolute errors for Example 4.

h	Trigonometric Cubic B-spline errors	Exponential Cubic B-spline errors	Cubic B-spline errors [9]
1/4	6.45×10^{-3}	4.13×10^{-3}	5.04×10^{-3}
1/6	2.89×10^{-3}	1.82×10^{-3}	2.13×10^{-3}
1/8	1.60×10^{-3}	1.00×10^{-3}	1.18×10^{-3}

Figure 4. Numerical solutions of Example 4 at $n = 8$.

4 Conclusion

Numerical treatment of two point boundary value problems is carried out in the paper using trigonometric and exponential cubic B-splines. From the computational results, the proposed methods give reliable solutions of the considered problem if compared with the existing results in literature. The numerical experiments are compared with the analytic solutions by finding the maximum absolute errors and are compared with numerical results in [9] as shown in tables 3, 4, 5 and 6 where we can see that the numerical accuracy is similar to results obtained in [9].

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