Journal of the Egyptian Mathematical Society
Volume (26) - Issue (3) - 2018

ISSN:1110-256X
DOI:10.21608/joems.2018.3335.1075

PREDICTION UNDER BURR-XII DISTRIBUTION BASED ON GENERALIZED TYPE-II PROGRESSIVE HYBRID CENSORING SCHEME

Saieed F. Ateya[a,b] *and Heba S. Mohammed* [c,d]

[a]Department of Mathematics, Faculty of Science
Assiut University, Egypt
[b]Mathematics and statistics Department, Faculty of Science
Taif University, Saudi Arabia
[c]Department of Mathematics, Faculty of Science, New Valley Branch
Assiut University, Egypt
[d] Mathematical Science, Faculty of Science, Princess Nourah bint Abdularhman University
Riyadh, Saudi Arabia

Received 29/3/2018 Revised 30/3/2018 Accepted 18/7/2018

Abstract

In this paper, the prediction problem of future failure times from Burr-XII distribution based on generalized type-II progressive hybrid censoring scheme ($GTIIPHCS$) is considered by using maximum likelihood (ML) and ($Bayes$) methods for one and two-sample prediction schemes. Point and interval predictors, denoted by $PP's$ and $IP's$, of future failure times are computed based on simulated and real data. A comparative study is carried out between the two methods using the mean squared errors ($MSE's$) and the relative errors ($RE's$) criteria.

Key words and phrases: Burr-XII distribution; Statistical prediction; One and two-sample prediction schemes; Generalized type-II progressive hybrid censoring scheme; $MCMC$ algorithm.
Mathematics Subject Classification (2010) 62F10; 62F15; 62N01; 62N02.

1 INTRODUCTION

Suppose that n identical units from a certain distribution with probability density function (PDF), $f(x;\theta)$, where θ is a vector of parameters and reliability function (RF), $R(x;\theta)$, are placed on a lifetime test. At the time of the i^{th} failure, R_i surviving units are randomly withdrawn from the experiment, $1 \leq i \leq r$. Thus, if r failures are observed then $R_1+R_2+...+R_r$ units are progressively censored, hence $n = r+R_1+R_2+...+R_r$ and $X^M_{1:r:n} < X^M_{2:r:n} < ... < X^M_{r:r:n}$ describe the progressively type-II censored failure times, where $M = (R_1, R_2, ..., R_r)$ and $\sum_{i=1}^{r} R_i = n - r$.

The previous progressively type-II censored data can be written in the following form:
$x = (x^M_{1:r:n}, x^M_{2:r:n}, ..., x^M_{r:r:n})$ which can be written for simplicity as $x = (x_1, x_2, ..., x_r)$. For more details, see Balakrishnan and Aggrawala[1].

Two new censoring schemes related to the previous censoring scheme are introduced.The first is called Type-I progressive hybrid censoring scheme ($Type-I\ PHCS$) and studied by Kundu and Joarder[2] and Childs et al.[3] and the other is called Type-II progressive hybrid censoring scheme ($Type-II\ PHCS$) and introduced by Childs et al.[3].

Lee et al.[4] combined $Type-I\ PHCS$ and $Type-II\ PHCS$ to give a new censoring scheme called generalized Type-II progressive hybrid censoring scheme which can be described as follows. For fixed $r \in (1, 2, ..., n)$ and time points $T_1, T_2 \in (0, \infty)$ with $T1 < T2$. If the r^{th} failure occurs before the time point T_1, terminate the experiment at T_1. If the r^{th}

failure occurs between T_1 and T_2, terminate the experiment at $X_{r:n}$. Finally, if the r^{th} failure occurs after T_2, terminate the experiment at T_2. Under this censoring scheme, one of the following three forms can be observed:
(1) $0 < X_{r:n} < T_1 < T_2$ in which case we terminate at T_1,
(2) $0 < T_1 < T_2 < X_{r:n}$ in which case we terminate at T_2,
(3) $0 < T_1 < X_{r:n} < T_2$ in which case we terminate at $X_{r:n}$.

Let d_i denote the number of failures until time T_i, $i = 1, 2$. Then, the likelihood function of this $GTIIPHCS$ is as follows:

$$L(\theta|data) \propto \begin{cases} \left[\prod_{i=1}^{r-1} f(x_i;\theta)(R(x_i;\theta))^{R_i}\right]\left[\prod_{i=r}^{d_1} f(x_i;\theta)\right]\left[f(T_1;\theta)(R(T_1;\theta))^{R^*_{d_1}}\right], \\ R^*_{d_1} = n - d_1 - \sum_{i=1}^{r-1} R_i, R_r = 0 \text{ for } d_1 \geq r, d_1 = r, r+1, ..., n, \\ \left[\prod_{i=1}^{d_2} f(x_i;\theta)(R(x_i;\theta))^{R_i}\right]\left[f(T_2;\theta)(R(T_2;\theta))^{R^*_{d_2}}\right], \\ R^*_{d_2} = \sum_{i=d_2+1}^{r} R_i, d_2 = 1, 2, ..., r-1, \\ \prod_{i=1}^{r} f(x_i;\theta)(R(x_i;\theta))^{R_i}, d_1 = 0, 1, ..., r-1, d_2 = r, r+1, ..., n. \end{cases} \tag{1.1}$$

Special Cases:
From the proposed model, other well-known models can be obtained as special cases such as:

1. Type-I PHCS when $T_1 \to 0$.
2. Type-II PHCS when $T_2 \to \infty$.
3. Hybrid Type-I censoring scheme when $T_1 \to 0$, $R_i = 0, i = 1, 2, ..., r-1, R_r = n - r$.
4. Hybrid Type-II censoring scheme when $T_2 \to \infty$, $R_i = 0, i = 1, 2, ..., r-1, R_r = n - r$.

A random variable X is said to have a Burr-XII with vector of parameters $\theta = (\alpha, \beta)$ if its PDF is given by

$$f(x;\alpha,\beta) = \alpha\,\beta\,x^{\alpha-1}(1+x^{\alpha})^{-\beta-1}, \; x > 0, (\alpha > 0, \beta > 0). \tag{1.2}$$

The corresponding cumulative distribution function (CDF) and RF are given respectively as:

$$F(x;\alpha,\beta) = 1 - (1+x^{\alpha})^{-\beta}, \; x > 0, (\alpha > 0, \beta > 0), \tag{1.3}$$

$$R(x;\alpha,\beta) = (1+x^{\alpha})^{-\beta}, \; x > 0, (\alpha > 0, \beta > 0). \tag{1.4}$$

In this paper, the prediction problem under Burr-XII distribution based on generalized type-II progressive hybrid censoring scheme is studied extensively. For more details about Burr-XII, progressive censoring schemes with applications and prediction problem, see Amein[5], Balakrishnan and Cramer[6] and Geisser[7], respectively.

2 ONE-SAMPLE PREDICTION

Assume that n items are put in progressive type-II life-time experiment and this experiment will be terminated at fixed time T^* and the number of failures until this time is D. The observed failures are denoted by $x = (x_{1:n}, x_{2:n}, ..., x_{D:n})$,with progressive censoring scheme $(R_1, R_2, ..., R_D)$, which can be written for simplicity as $x = (x_1, x_2, ..., x_D)$ is called (***Informative sample***). In $GTIIPHCS$, T^* will equal to T_1 in the first case, T_2 in the second case and x_r in the third case. D will equal to d_1 in the first case, d_2 in the second case and r in the third case. Also, R_D will equal to $R^*_{d_1}$ in the first case, $R^*_{d_2}$ in the second case and R_r in the third case.

Let $y_i = (y_{i1}, y_{i2}, ..., y_{iR_i})$ denote the ordered failures of units removed at the i^{th} failure x_i. In one-sample prediction scheme, we wish to predict the future failures $y_D = (y_{Ds} \equiv y_s, s = 1, 2, ..., R_D)$ given the informative sample x.
In this section, the point and interval predictors of the future unknown failure time y_D are computed using maximum likelihood and Bayesian methods.
First, the conditional PDF of the future failure time y_s given the vector of parameters θ should be derived as follows:

Based on the informative sample $x = (x_1, x_2, ..., x_D)$, the PDF of y_s given θ will be the PDF of the s^{th} ordered value from R_D ordered values which can be written as:

$$g_1(y_s;\theta) \propto [R(T^*;\theta) - R(y_s;\theta)]^{s-1} \; [R(y_s;\theta)]^{R_D - s} \; [R(T^*;\theta)]^{-R_D} \; f(y_s;\theta), \; y_s > T^*. \tag{2.1}$$

Using this PDF, the conditional PDF of the future failure time y_s given θ based on all cases of $GTIIPHCS$ will be of the following form:

$$g_1(y_s;\theta) \propto \begin{cases} \Big[R(T_1;\theta) - R(y_s;\theta)\Big]^{s-1} \Big[R(y_s;\theta)\Big]^{R^*_{d_1}-s} \Big[R(T_1;\theta)\Big]^{-R^*_{d_1}} f(y_s;\theta), \; y_s > T_1, \\ R^*_{d_1} = n - d_1 - \sum_{i=1}^{r-1} R_i, R_r = 0 \text{ for } d_1 \geq r, d_1 = r, r+1, ..., n, \\ \Big[R(T_2;\theta) - R(y_s;\theta)\Big]^{s-1} \Big[R(y_s;\theta)\Big]^{R^*_{d_2}-s} \Big[R(T_2;\theta)\Big]^{-R^*_{d_2}} f(y_s;\theta), \; y_s > T_2, \\ R^*_{d_2} = \sum_{i=d_2+1}^{r} R_i, d_2 = 1, 2, ..., r-1, \\ \Big[R(x_r;\theta) - R(y_s;\theta)\Big]^{s-1} \Big[R(y_s;\theta)\Big]^{R_r-s} \Big[R(x_r;\theta)\Big]^{-R_r} f(y_s;\theta), \; y_s > x_r, \\ d_1 = 0, 1, ..., r-1, d_2 = r, r+1, ..., n. \end{cases} \tag{2.2}$$

2.1 Maximum Likelihood Prediction

In this subsection, the point and interval predictors of y_s are obtained using the maximum likelihood method, see Kaminsky[8], which depend on the the likelihood predictive function (LPF) which of the form:

$$g_1^*(y_s;\theta, x) \propto \; L(\theta; \; x) \; g_1(y_s;\theta), \; y_s > T^*. \tag{2.3}$$

Substituting from (1.1) and (2.2) in (2.3), we get

$$g_1^*(y_s;\theta,x) \propto \begin{cases} \Big[\prod_{i=1}^{r-1} f(x_i;\theta)(R(x_i;\theta))^{R_i}\Big]\Big[\prod_{i=r}^{d_1} f(x_i;\theta)\Big]\Big[R(T_1;\theta) - R(y_s;\theta)\Big]^{s-1} \times \\ \Big[R(y_s;\theta)\Big]^{R^*_{d_1}-s} f(T_1;\theta) \; f(y_s;\theta), \; y_s > T_1, \; R^*_{d_1} = n - d_1 - \sum_{i=1}^{r-1} R_i, \\ R_r = 0 \text{ for } d_1 \geq r, d_1 = r, r+1, ..., n, \\ \Big[\prod_{i=1}^{d_2} f(x_i;\theta)(R(x_i;\theta))^{R_i}\Big] \Big[R(T_2;\theta) - R(y_s;\theta)\Big]^{s-1} \times \\ \Big[R(y_s;\theta)\Big]^{R^*_{d_2}-s} f(T_2;\theta) \; f(y_s;\theta), \; y_s > T_2, \; R^*_{d_2} = \sum_{i=d_2+1}^{r} R_i, \\ d_2 = 1, 2, ..., r-1, \\ \Big[\prod_{i=1}^{r-1} f(x_i;\theta)(R(x_i;\theta))^{R_i}\Big]\Big[R(x_r;\theta) - R(y_s;\theta)\Big]^{s-1} \Big[R(y_s;\theta)\Big]^{R_r-s} \times \\ f(x_r;\theta) \; f(y_s;\theta), \; y_s > x_r, \; d_1 = 0, 1, ..., r-1, d_2 = r, r+1, ..., n. \end{cases} \tag{2.4}$$

Substituting from (1.2) - (1.4) in (2.4), we get

$$g_1^*(y_s;\theta,x) \propto \begin{cases} \alpha^{d_1+2}\,\beta^{d_1+2}\left(\prod_{i=1}^{d_1} x_i\right)^{\alpha-1}\left(\prod_{i=1}^{d_1}(1+x_i^\alpha)\right)^{-\beta-1}\left(\prod_{i=1}^{r-1}(1+x_i^\alpha)^{R_i}\right)^{-\beta}\times \\ \left((1+T_1^\alpha)^{-\beta}-(1+y_s^\alpha)^{-\beta}\right)^{s-1}(1+y_s^\alpha)^{-\beta(R_{d_1}^*-s)}\,(T_1\,y_s)^{\alpha-1}\times \\ \left((1+T_1^\alpha)\,(1+y_s^\alpha)\right)^{-\beta-1},\ y_s>T_1,\ R_{d_1}^*=n-d_1-\sum_{i=1}^{r-1}R_i, \\ R_r=0 \text{ for } d_1\geq r, d_1=r,r+1,...,n, \\ \alpha^{d_2+2}\,\beta^{d_2+2}\left(\prod_{i=1}^{d_2} x_i\right)^{\alpha-1}\left(\prod_{i=1}^{d_2}(1+x_i^\alpha)\right)^{-\beta-1}\left(\prod_{i=1}^{d_2}(1+x_i^\alpha)^{R_i}\right)^{-\beta}\times \\ \left((1+T_2^\alpha)^{-\beta}-(1+y_s^\alpha)^{-\beta}\right)^{s-1}(1+y_s^\alpha)^{-\beta(R_{d_2}^*-s)}\,(T_2\,y_s)^{\alpha-1}\times \\ \left((1+T_2^\alpha)\,(1+y_s^\alpha)\right)^{-\beta-1},\ y_s>T_2,\ R_{d_2}^*=\sum_{i=d_2+1}^{r}R_i, \\ d_2=1,2,...,r-1, \\ \alpha^{r+2}\,\beta^{r+2}\left(\prod_{i=1}^{r-1} x_i\right)^{\alpha-1}\left(\prod_{i=1}^{r-1}(1+x_i^\alpha)\right)^{-\beta-1}\left(\prod_{i=1}^{r-1}(1+x_i^\alpha)^{R_i}\right)^{-\beta}\times \\ \left((1+x_r^\alpha)^{-\beta}-(1+y_s^\alpha)^{-\beta}\right)^{s-1}(1+y_s^\alpha)^{-\beta(R_r-s)}\,(x_r\,y_s)^{\alpha-1}\times \\ \left((1+x_r^\alpha)\,(1+y_s^\alpha)\right)^{-\beta-1},\ y_s>x_r,\ d_1=0,1,...,r-1, d_2=r,r+1,...,n. \end{cases} \tag{2.5}$$

2.1.1 Point predictor

In this subsection, the point predictor of y_s is obtained by computing the values of α,β and y_s which maximize the logarithm of the LPF, which are denoted by $\hat{\alpha},\hat{\beta}$ and y_s^*, respectively. The values $\hat{\alpha}$ and $\hat{\beta}$ are called the predictive maximum likelihood estimates ($PMLE's$) and the value y_s^* are called the maximum likelihood predictor (MLP) of y_s.
To maximize the logarithm of the LPF, we will differentiate $log(g_1^*(y_s;\alpha,\beta,x))$ with respect to α,β and y_s, set the resulting derivatives to zero as follows:

$$\begin{aligned} \frac{\partial log(g_1^*(y_s;\alpha,\beta,x))}{\partial\alpha}&=0, \\ \frac{\partial log(g_1^*(y_s;\alpha,\beta,x))}{\partial\beta}&=0, \\ \frac{\partial log(g_1^*(y_s;\alpha,\beta,x))}{\partial y_s}&=0, \end{aligned} \tag{2.6}$$

and solve the resulting nonlinear equations. The solution of the resulting nonlinear equations will be $\hat{\alpha},\hat{\beta}$ and y_s^*.
By replacing α and β in the LPF (2.5) by $\hat{\alpha}$ and $\hat{\beta}$, respectively, a function called maximum likelihood predictive function ($MLPF$) is obtained in the form:

$$g_1^{**}(y_s;x)=A\,y_s^{\hat{\alpha}-1}\,(1+y_s^{\hat{\alpha}})^{-\hat{\beta}-1}\,(1+y_s^{\hat{\alpha}})^{-\hat{\beta}(R_D-s)}\left((1+T^{*\hat{\alpha}})^{-\hat{\beta}}-(1+y_s^{\hat{\alpha}})^{-\hat{\beta}}\right)^{s-1},\ y_s>T^*. \tag{2.7}$$

where the normalizing constant A is given by:

$$A=\frac{1}{\int_{T^*}^{\infty}\left(y_s^{\hat{\alpha}-1}\,(1+y_s^{\hat{\alpha}})^{-\hat{\beta}-1}\,(1+y_s^{\hat{\alpha}})^{-\hat{\beta}(R_D-s)}\,((1+T^{*\hat{\alpha}})^{-\hat{\beta}}-(1+y_s^{\hat{\alpha}})^{-\hat{\beta}})^{s-1}\right)dy_s}, \tag{2.8}$$

with

$$\begin{cases} D = d_1,\, y_s > T^*,\, T^* = T_1,\ R_D = R^*_{d_1},\ d_1 = r, r+1, ..., n, \\ D = d_2,\, y_s > T^*,\, T^* = T_2,\ R_D = R^*_{d_2},\ d_2 = 1, 2, ..., r-1, \\ D = r,\, y_s > T^*,\, T^* = x_r,\ R_D = R_r,\ d_1 = 0, 1, ..., r-1, d_2 = r, r+1, ..., n. \end{cases} \tag{2.9}$$

2.1.2 Interval prediction

A $(1-\tau) \times 100\%$ prediction interval (LM_1, UM_1) of the future failure time y_s can be obtained by solving the following two nonlinear equations for LM_1 and UM_1.

$$\begin{cases} \int_{LM_1}^{\infty} g_1^{**}(y_s; x)\, dy_s = 1 - \frac{\tau}{2}, \\ \int_{UM_1}^{\infty} g_1^{**}(y_s; x)\, dy_s = \frac{\tau}{2}. \end{cases} \tag{2.10}$$

Substitute (2.7) and (2.8) in (2.10), the two nonlinear equations in (2.10) can be rewritten in the form:

$$\begin{cases} A\, \int_{LM_1}^{\infty} \left(y_s^{\hat{\alpha}}\, (1+y_s^{\hat{\alpha}})^{-\hat{\beta}-1}\, (1+y_s^{\hat{\alpha}})^{-\hat{\beta}(R_D-s)}\, ((1+{T^*}^{\hat{\alpha}})^{-\hat{\beta}} - (1+y_s^{\hat{\alpha}})^{-\hat{\beta}})^{s-1} \right) dy_s = 1 - \frac{\tau}{2}, \\ A\, \int_{UM_1}^{\infty} \left(y_s^{\hat{\alpha}}\, (1+y_s^{\hat{\alpha}})^{-\hat{\beta}-1}\, (1+y_s^{\hat{\alpha}})^{-\hat{\beta}(R_D-s)}\, ((1+{T^*}^{\hat{\alpha}})^{-\hat{\beta}} - (1+y_s^{\hat{\alpha}})^{-\hat{\beta}})^{s-1} \right) dy_s = \frac{\tau}{2}. \end{cases} \tag{2.11}$$

2.2 Bayesian Prediction

Using the bivariate prior suggested by Ateya[9, 10] which of the form:

$$\pi(\alpha, \beta) \propto \alpha^{c_1+c_3-1}\, \beta^{c_3-1} e^{-\alpha\, (\beta+c_2)}, \alpha > 0,\ \beta > 0,\ (c_1 > 0,\ c_2 > 0,\ c_3 > 0), \tag{2.12}$$

where c_1, c_2 and c_3 are the prior parameters (also known as hyperparameters) and LF (1.1) after replacing $f(x_i; \theta)$ and $R(x_i; \theta)$ by its definitions from (1.2) and (1.4), the posterior PDF of α and β can be written as:

$$\pi^*(\alpha, \beta; x) \propto \begin{cases} \alpha^{c_1+c_3+d_1}\, \beta^{c_3+d_1} e^{-\alpha\, (\beta+c_2)} \left(\prod_{i=1}^{d_1} x_i \right)^{\alpha-1} \left(\prod_{i=1}^{d_1} (1+x_i^{\alpha}) \right)^{-\beta-1} \times \\ \left(\prod_{i=1}^{r-1} (1+x_i^{\alpha})^{R_i} \right)^{-\beta} T_1^{\alpha-1}\, (1+T_1^{\alpha})^{-\beta\, (R^*_{d_1}+1)-1}, \\ R^*_{d_1} = n - d_1 - \sum_{i=1}^{r-1} R_i,\ R_r = 0 \text{ for } d_1 \geq r, d_1 = r, r+1, ..., n, \\ \alpha^{c_1+c_3+d_2}\, \beta^{c_3+d_2} e^{-\alpha\, (\beta+c_2)} \left(\prod_{i=1}^{d_2} x_i \right)^{\alpha-1} \left(\prod_{i=1}^{d_2} (1+x_i^{\alpha}) \right)^{-\beta-1} \times \\ \left(\prod_{i=1}^{d_2} (1+x_i^{\alpha})^{R_i} \right)^{-\beta} T_2^{\alpha-1} (1+T_2^{\alpha})^{-\beta\, (R^*_{d_2}+1)-1}, \\ R^*_{d_2} = \sum_{i=d_2+1}^{r} R_i,\ d_2 = 1, 2, ..., r-1, \\ \alpha^{c_1+c_3+r}\, \beta^{c_3+r} e^{-\alpha\, (\beta+c_2)} \left(\prod_{i=1}^{r-1} x_i \right)^{\alpha-1} \left(\prod_{i=1}^{r-1} (1+x_i^{\alpha}) \right)^{-\beta-1} \times \\ \left(\prod_{i=1}^{r-1} (1+x_i^{\alpha})^{R_i} \right)^{-\beta} x_r^{\alpha-1} (1+x_r^{\alpha})^{-\beta-1} (1+x_r^{\alpha})^{-\beta\, (R_r+1)-1}, \\ d_1 = 0, 1, ..., r-1, d_2 = r, r+1, ..., n. \end{cases} \tag{2.13}$$

Using the previous posterior PDF and the conditional PDF of y_s given α and β, (2.2), after using the definition of $f(x_i; \theta)$ and $R(x_i; \theta)$ from (1.2) and (1.4), the Bayesian predictive PDF of y_s given x will be as follows:

$$h_1^*(y_s; x) = \int_0^{\infty} \int_0^{\infty} h_1(y_s; \alpha, \beta, x) d\beta\, d\alpha, \tag{2.14}$$

where

$$h_1(y_s;\alpha,\beta,x)=\pi^*(\alpha,\beta;x)\,g_1(y_s;\alpha,\beta)=$$

$$\begin{cases} A_1\,\alpha^{c_1+c_3+d_1}\,\beta^{c_3+d_1}e^{-\alpha\,(\beta+c_2)}\left(\prod_{i=1}^{d_1}x_i\right)^{\alpha-1}\left(\prod_{i=1}^{d_1}(1+x_i^{\alpha})\right)^{-\beta-1}\times \\ \left(\prod_{i=1}^{r-1}(1+x_i^{\alpha})^{R_i}\right)^{-\beta}\left(T_1\,y_s\right)^{\alpha-1}\left((1+T_1^{\alpha})(1+y_s^{\alpha})\right)^{-\beta-1}\times \\ \left((1+T_1^{\alpha})^{-\beta}-(1+y_s^{\alpha})^{-\beta}\right)^{s-1}(1+y_s^{\alpha})^{-\beta(R^*_{d_1}-s)},\ y_s>T_1, \\ R^*_{d_1}=n-d_1-\sum_{i=1}^{r-1}R_i,\ R_r=0 \text{ for } d_1\geq r, d_1=r,r+1,...,n, \\ A_2\,\alpha^{c_1+c_3+d_2}\,\beta^{c_3+d_2}e^{-\alpha\,(\beta+c_2)}\left(\prod_{i=1}^{d_2}x_i\right)^{\alpha-1}\left(\prod_{i=1}^{d_2}(1+x_i^{\alpha})\right)^{-\beta-1}\times \\ \left(\prod_{i=1}^{d_2}(1+x_i^{\alpha})^{R_i}\right)^{-\beta}\left(T_2\,y_s\right)^{\alpha-1}\left((1+T_2^{\alpha})(1+y_s^{\alpha})\right)^{-\beta-1}\times \\ \left((1+T_2^{\alpha})^{-\beta}-(1+y_s^{\alpha})^{-\beta}\right)^{s-1}(1+y_s^{\alpha})^{-\beta(R^*_{d_2}-s)},\ y_s>T_2, \\ R^*_{d_2}=\sum_{i=d_2+1}^{r}R_i,\ d_2=1,2,...,r-1, \\ A_3\,\alpha^{c_1+c_3+r}\,\beta^{c_3+r}e^{-\alpha\,(\beta+c_2)}\left(\prod_{i=1}^{r-1}x_i\right)^{\alpha-1}\left(\prod_{i=1}^{r-1}(1+x_i^{\alpha})\right)^{-\beta-1}\times \\ \left(\prod_{i=1}^{r-1}(1+x_i^{\alpha})^{R_i}\right)^{-\beta}\left(x_r\,y_s\right)^{\alpha-1}\left((1+x_r^{\alpha})(1+y_s^{\alpha})\right)^{-\beta-1}\times \\ \left((1+x_r^{\alpha})^{-\beta}-(1+y_s^{\alpha})^{-\beta}\right)^{s-1}(1+y_s^{\alpha})^{-\beta(R_r-s)},\ y_s>x_r, \\ d_1=0,1,...,r-1, d_2=r,r+1,...,n, \end{cases} \tag{2.15}$$

where $A_i, i=1,2,3$ are normalizing constants.
Therefore, the Bayesian predictor (BP) of y_s is given by:

$$y_s^{**}=E[Y_s]=\int_{T^*}^{\infty}y_s h_1^*(y_s;x)dy_s, \tag{2.16}$$

and the $(1-\tau)\times 100\%$ Bayesian prediction interval (BPI), (LB_1,UB_1), of y_s can be obtained by solving the following two nonlinear equations.

$$\begin{cases}\int_{LB_1}^{\infty}h_1^*(y_s;x)\,dy_s=1-\frac{\tau}{2}, \\ \int_{UB_1}^{\infty}h_1^*(y_s;x)\,dy_s=\frac{\tau}{2}.\end{cases} \tag{2.17}$$

Since the above system contains double integration in α and β, it is more convenient to apply Markov Chain Monte Carlo $(MCMC)$ method to generate a random sample $(\alpha^{(1)},\beta^{(1)}),(\alpha^{(2)},\beta^{(2)}),...,(\alpha^{(K)},\beta^{(K)})$ from the posterior PDF (2.13), then the system (2.17) will be of the form:

$$\begin{cases}\frac{\sum_{i=1}^{K}\int_{LB_1}^{\infty}h_1(y_s;\alpha^{(i)},\beta^{(i)},x)\,dy_s}{\sum_{i=1}^{K}\int_{T^*}^{\infty}h_1(y_s;\alpha^{(i)},\beta^{(i)},x)\,dy_s}=1-\frac{\tau}{2}, \\ \frac{\sum_{i=1}^{K}\int_{UB_1}^{\infty}h_1(y_s;\alpha^{(i)},\beta^{(i)},x)\,dy_s}{\sum_{i=1}^{K}\int_{T^*}^{\infty}h_1(y_s;\alpha^{(i)},\beta^{(i)},x)\,dy_s}=\frac{\tau}{2}.\end{cases} \tag{2.18}$$

For more details about the $MCMC$ method, e.g. Jaheen and Al-Harbi[11], Press[12], Upadhyaya and Gupta [13] and Upadhyaya et al.[14].

3 TWO-SAMPLE PREDICTION

Assume that $x=(x_1,x_2,...,x_D)$ and $z=(z_1,z_2,...,z_m)$ represent the informative sample which from the studied $GTIIPHCS$ and a future ordered sample of size m, respectively. It is assumed that the two samples are independent.
In this section, the point and interval predictors of the observation $z_s, s=1,2,...,m$ are obtained using the maximum likelihood and Bayesian methods.

3.1 Maximul Likelihood Two-Sample Prediction

The conditional PDF of the observation z_s given the vector of parameters θ is the PDF of the s^{th} ordered value from the m ordered values which can be written as:

$$g_2(z_s;\theta) \propto [1-R(z_s;\theta)]^{s-1}\,[R(z_s;\theta)]^{m-s}\,f(z_s;\theta),\; z_s>0, \tag{3.1}$$

and the corresponding LPF will be in the form:

$$g_2^*(z_s;\theta) \propto\; [1-R(z_s;\theta)]^{s-1}\,[R(z_s;\theta)]^{m-s}\,f(z_s;\theta)\,L(\theta;\,x),\; z_s>0, \tag{3.2}$$

Substitute (1.1), (1.2) and (1.4) in (3.1), and then differentiate $log(g_2^*(z_s;\alpha,\beta,x))$ with respect to α,β and z_s and finally equate the resulting derivatives to zero, a system of three nonlinear equation are derived in the form:

$$\begin{aligned}\frac{\partial log(g_2^*(z_s;\alpha,\beta,x))}{\partial\alpha}&=0,\\ \frac{\partial log(g_2^*(z_s;\alpha,\beta,x))}{\partial\beta}&=0,\\ \frac{\partial log(g_2^*(z_s;\alpha,\beta,x))}{\partial z_s}&=0,\end{aligned} \tag{3.3}$$

The solution of the resulting nonlinear equations will be $\hat{\alpha},\hat{\beta}$ and z_s^*, where z_s^* is the PP of z_s.
By replacing α and β in the LPF (3.2) by $\hat{\alpha}$ and $\hat{\beta}$, respectively, the $MLPF$ can be obtained in the form:

$$g_2^{**}(z_s;x) = B\; z_s^{\hat{\alpha}-1}\,(1+z_s^{\hat{\alpha}})^{-\hat{\beta}\,(m-s+1)-1}\;[1-(1+z_s^{\hat{\alpha}})^{-\hat{\beta}}]^{s-1},\; z_s>0, \tag{3.4}$$

where B is a normalizing constant has the value:

$$B=\frac{1}{\int_0^\infty \left(z_s^{\hat{\alpha}-1}\,(1+z_s^{\hat{\alpha}})^{-\hat{\beta}\,(m-s+1)-1}\;[1-(1+z_s^{\hat{\alpha}})^{-\hat{\beta}}]^{s-1}dz_s\right.}. \tag{3.5}$$

A $(1-\tau)\times 100\%$ prediction interval (LM_2, UM_2) of z_s can be obtained by solving the following two nonlinear equations which of the form:

$$\begin{cases}\int_{LM_2}^\infty g_2^{**}(z_s;x)\,dz_s = 1-\frac{\tau}{2},\\ \int_{UM_2}^\infty g_2^{**}(z_s;x)\,dz_s = \frac{\tau}{2}.\end{cases} \tag{3.6}$$

From (3.4) and (3.6) in (3.6), the two nonlinear equations in (3.6) can be rewritten to be of the form:

$$\begin{cases} B\;\int_{LM_2}^\infty\left(z_s^{\hat{\alpha}-1}\,(1+z_s^{\hat{\alpha}})^{-\hat{\beta}\,(m-s+1)-1}\;[1-(1+z_s^{\hat{\alpha}})^{-\hat{\beta}}]^{s-1}\right)dz_s = 1-\frac{\tau}{2},\\ B\;\int_{UM_2}^\infty\left(z_s^{\hat{\alpha}-1}\,(1+z_s^{\hat{\alpha}})^{-\hat{\beta}\,(m-s+1)-1}\;[1-(1+z_s^{\hat{\alpha}})^{-\hat{\beta}}]^{s-1}\right)dz_s = \frac{\tau}{2}.\end{cases} \tag{3.7}$$

By solving the previous system, (LM_2, UM_2), can be computed.

3.2 Baesian Two-Sample Prediction

Using the conditional PDF (3.1) and the same posterior PDF (2.13), the Bayesian predictive PDF of z_s given x will be in the form:

$$h_2^*(z_s;x) = \int_0^\infty\int_0^\infty h_2(z_s;\alpha,\beta,x)d\beta\,d\alpha, \tag{3.8}$$

where

$$h_2(z_s;\alpha,\beta,x)=\pi^*(\alpha,\beta;x)\,g_2(z_s;\alpha,\beta)=$$

$$\begin{cases} B_1\, z_s^{\alpha-1}\,(1+z_s^{\alpha})^{-\beta\,(m-s+1)-1}\,[1-(1+z_s^{\alpha})^{-\beta}]^{s-1}\alpha^{c_1+c_3+d_1+1}\,\beta^{c_3+d_1+1}e^{-\alpha\,(\beta+c_2)} \\ \left(\prod_{i=1}^{d_1}x_i\right)^{\alpha-1}\left(\prod_{i=1}^{d_1}(1+x_i^{\alpha})\right)^{-\beta-1}\left(\prod_{i=1}^{r-1}(1+x_i^{\alpha})^{R_i}\right)^{-\beta}\times \\ T_1^{\alpha-1}\,(1+T_1^{\alpha})^{-\beta(R_{d_1}^*+1)-1},\ z_s>0,\ R_{d_1}^*=n-d_1-\sum_{i=1}^{r-1}R_i, \\ R_r=0 \text{ for } d_1\geq r, d_1=r,r+1,...,n, \\ B_2\, z_s^{\alpha-1}\,(1+z_s^{\alpha})^{-\beta\,(m-s+1)-1}\,[1-(1+z_s^{\alpha})^{-\beta}]^{s-1}\alpha^{c_1+c_3+d_2+1}\,\beta^{c_3+d_2+1}e^{-\alpha\,(\beta+c_2)} \\ \left(\prod_{i=1}^{d_2}x_i\right)^{\alpha-1}\left(\prod_{i=1}^{d_2}(1+x_i^{\alpha})\right)^{-\beta-1}\left(\prod_{i=1}^{d_2}(1+x_i^{\alpha})^{R_i}\right)^{-\beta}\times \\ T_2^{\alpha-1}\,(1+T_2^{\alpha})^{-\beta(R_{d_2}^*+1)-1},\ z_s>0,\ R_{d_2}^*=\sum_{i=d_2+1}^{r}R_i,\ d_2=1,2,...,r-1, \\ B_3\, z_s^{\alpha-1}\,(1+z_s^{\alpha})^{-\beta\,(m-s+1)-1}\,[1-(1+z_s^{\alpha})^{-\beta}]^{s-1}\alpha^{c_1+c_3+r+1}\,\beta^{c_3+r+1}e^{-\alpha\,(\beta+c_2)} \\ \left(\prod_{i=1}^{r-1}x_i\right)^{\alpha-1}\left(\prod_{i=1}^{r-1}(1+x_i^{\alpha})\right)^{-\beta-1}\left(\prod_{i=1}^{r-1}(1+x_i^{\alpha})^{R_i}\right)^{-\beta}\times \\ x_r^{\alpha-1}\,(1+x_r^{\alpha})^{-\beta(R_r+1)-1},\ z_s>0,\ d_1=0,1,...,r-1, d_2=r,r+1,...,n, \end{cases} \tag{3.9}$$

where $B_i, i=1,2,3$ are normalizing constants.
The Bayesian PP of z_s will equal to:

$$z_s^{**}=E[Z_s]=\int_0^{\infty}z_s h_2^*(z_s;x)dz_s, \tag{3.10}$$

and the $(1-\tau)\times 100\%$ BPI, (LB_2, UB_2), of z_s can be obtained by solving the following two nonlinear equations which of the form:

$$\begin{cases}\int_{LB_2}^{\infty}h_2^*(z_s;x)\,dz_s=1-\frac{\tau}{2}, \\ \int_{UB_2}^{\infty}h_2^*(z_s;x)\,dz_s=\frac{\tau}{2}.\end{cases} \tag{3.11}$$

Using $(\alpha^{(1)},\beta^{(1)}),(\alpha^{(2)},\beta^{(2)}),...,(\alpha^{(K)},\beta^{(K)})$ which are generated from the posterior PDF (2.11), then the system (3.11) will be of the form:

$$\begin{cases}\frac{\sum_{i=1}^{K}\int_{LB_2}^{\infty}h_2(z_s;\alpha^{(i)},\beta^{(i)},x)\,dz_s}{\sum_{i=1}^{K}\int_0^{\infty}h_2(z_s;\alpha^{(i)},\beta^{(i)},x)\,dz_s}=1-\frac{\tau}{2}, \\ \frac{\sum_{i=1}^{K}\int_{UB_2}^{\infty}h_2(z_s;\alpha^{(i)},\beta^{(i)},x)\,dz_s}{\sum_{i=1}^{K}\int_0^{\infty}h_2(z_s;\alpha^{(i)},\beta^{(i)},x)\,dz_s}=\frac{\tau}{2}.\end{cases} \tag{3.12}$$

By solving this system, the BPI, (LB_2, UB_2), for z_s is obtained.

4 NUMERICAL RESULTS

4.1 Simulated Results

In this paper, point and interval predictors of future failure times are obtained in case of one and two-sample schemes, using the ML and $Bayesian$ methods based on a generated $GTIIPHCS$ informative sample for different values of r, T_1 and T_2 as follows:

1. For a given set of prior parameters c_1, c_2 and c_3, the population parameters α and β are generated from the joint prior (2.12).

2. Making use of α and β obtained in step 1, and for selected values of n, r and $M=(R_1,R_2,...,R_r)$, a progressive type-II censoring sample is generated from $Burr-XII$ distribution, see Balakrishnan, Aggrawala[1].

3. For selected values of T_1 and T_2, one case of $GTIIPHCS$, shown in (1.1) is chosen.

4. For different values of n, r, T_1 and T_2, the point and interval predictors of the future failure times are computed using the ML and $Bayesian$ methods in case of one-sample scheme as explained in section 2.

5. The same is done in case of two-sample scheme as explained in section 3.

6. Repeating steps 2-5 M times.

7. If $\hat{y^*_{si}}$ is the estimate of the predictor of the failure time $Y_s, s = 1, 2, ..., n-r$, over the sample i, then the average estimate of Y_s over the M samples is $\bar{\hat{y^*_s}} = \frac{1}{M}\sum_{i=1}^{M} \hat{y^*_{si}}$ and $MSE(\hat{y^*_s}) = \frac{1}{M}\sum_{i=1}^{M}(\hat{y^*_{si}} - \bar{\hat{y^*_s}})^2$.

8. If $\hat{z^*_{si}}$ is the estimate of the predictor of the failure time $Z_s, s = 1, 2, ..., n-r$, over the sample i, then the average estimate of Z_s over the M samples is $\bar{\hat{z^*_s}} = \frac{1}{M}\sum_{i=1}^{M} \hat{z^*_{si}}$ and $MSE(\hat{z^*_s}) = \frac{1}{M}\sum_{i=1}^{M}(\hat{z^*_{si}} - \bar{\hat{z^*_s}})^2$.

9. For each future failure time, and under 10000 samples, the the average estimate of the point predictor, MSE of the point predictor estimate, interval predictor, length of the interval predictor and the coverage percentage CP of the interval predictor are computed.

10. The results are summarized in Tables 1-4.

Table 1:- Average estimates and mean squared errors ($MSE's$) of the estimated point predictors of the future failure times $y_s, s = 1, 2, ..., R_D$ based on 10000 generated $GTIIPHCS$ informative samples. ($\alpha = 3.13598, \beta = 5.13166$), ($c_1 = 3.2, c_2 = 0.8, c_3 = 2.7$).

Values of (T_1, T_2)		(0.95, 2.5)			
(n, r, R_D)	$Method$	$\bar{\hat{y}}_1^*$ $MSE(\hat{y}_1^*)$	$\bar{\hat{y}}_2^*$ $MSE(\hat{y}_2^*)$	$\bar{\hat{y}}_3^*$ $MSE(\hat{y}_3^*)$	$\bar{\hat{y}}_4^*$ $MSE(\hat{y}_4^*)$
$M = (2,1,0,0,0,0,0,1,2,0,0,0,0,0,1,0,0,0,0,3)$					
(30, 20, 2)	ML	1.2134	1.5134		
		0.3109	0.3716		
	$Bayes$	1.1417	1.6119		
		0.2438	0.3001		
$M = (1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,3)$					
(30, 25, 2)	ML	1.1108	1.5516		
		0.2117	0.3167		
	$Bayes$	1.2142	1.6015		
		0.1906	0.2116		
Values of (T_1, T_2)		(0.18, 0.5)			
$M = (2,1,0,0,0,0,0,1,2,0,0,0,0,0,1,0,0,0,0,3)$					
(30, 20, 3)	ML	0.7703	0.9217	1.0246	
		0.2107	0.2617	0.3014	
	$Bayes$	0.7512	0.8917	0.9819	
		0.2011	0.2177	0.2997	
$M = (1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,3)$					
(30, 25, 3)	ML	0.8105	0.9332	1.1019	
		0.1927	0.2015	0.2517	
	$Bayes$	0.7815	0.1.0056	1.1521	
		0.1433	0.1697	0.2015	
Values of (T_1, T_2)		(0.18, 1.5)			
$M = (2,1,0,0,0,0,0,1,2,0,0,0,0,0,1,0,0,0,0,3)$					
(30, 20, 3)	ML	1.0173	1.8609	2.0114	
		0.5176	0.6115	0.7018	
	$Bayes$	0.9781	1.6901	1.9717	
		0.4331	0.5515	0.6132	
$M = (1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,3)$					
(30, 25, 3)	ML	1.1109	1.7003	1.82213	
		0.4416	0.4901	0.5118	
	$Bayes$	1.1098	1.8915	2.0016	
		0.2155	0.3018	0.4402	

Table 2:- Interval predictors of the future failure times $y_s, s = 1, 2, ..., R_D$ based on generated $GTIIPHCS$ informative sample.
($\alpha = 3.13598, \beta = 5.13166$), ($c_1 = 3.2, c_2 = 0.8,\ c_3 = 2.7$), $CP \equiv$ Coverage Probability.

Values of (T_1, T_2)		(0.95, 2.5)			
(n, r, R_D)	$Method$	IP of y_1 $Length$ CP	IP of y_2 $Length$ CP	IP of y_3 $Length$ CP	IP of y_4 $Length$ CP
$M = (2, 1, 0, 0, 0, 0, 0, 1, 2, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 3)$					
$(30, 20, 2)$	ML	(0.9528,1.3442) 0.3914 97.571%	(0.9878,1.9197) 0.9319 98.738%		
	$Bayes$	(0.9517,1.1955) 0.2438 95.479%	(0.9291,1.5239) 0.5948 98.431%		
$M = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3)$					
$(30, 25, 2)$	ML	(0.9518,1.1971) 0.2453 95.443%	(0.9284,1.5202) 0.5918 97.876%		
	$Bayes$	(0.9516,1.1759) 0.2243 95.018%	(0.9306,1.4721) 0.5415 97.555%		
Values of (T_1, T_2)		(0.18, 0.5)			
$M = (2, 1, 0, 0, 0, 0, 0, 1, 2, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 3)$					
$(30, 20, 3)$	ML	(0.5023,0.7597) 0.2574 95.344%	(0.4746,0.9207) 0.4461 98.324%	(0.5845,1.2591) 0.6746 98.992%	
	$Bayes$	(0.5023,0.7481) 0.2458 94.818%	(0.4749,0.9112) 0.4363 97.214%	(0.5856,1.2103) 0.6247 97.931%	
$M = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3)$					
$(30, 25, 3)$	ML	(0.5022,0.7463) 0.2441 94.543%	(0.4761,0.9078) 0.4317 97.914%	(0.5798,1.2154) 0.6356 98.654%	
	$Bayes$	(0.5023,0.7434) 0.2411 94.136%	(0.4757,0.9050) 0.4293 96.974%	(0.5834,1.2018) 0.6184 97.296%	
Values of (T_1, T_2)		(0.18, 1.5)			
$M = (2, 1, 0, 0, 0, 0, 0, 1, 2, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 3)$					
$(30, 20, 3)$	ML	(0.7682,0.9533) 0.1851 95.563%	(0.7531,1.0971) 0.3440 97.966%	(0.8200,1.3892) 0.5692 98.191%	
	$Bayes$	(0.7682,0.9433) 0.1751 95.155%	(0.7531,1.0866) 0.3335 97.74%	(0.8201,1.3771) 0.5570 97.947%	
$M = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3)$					
$(30, 25, 3)$	ML	(0.8130,0.9926) 0.1796 95.195%	(0.8005,1.1375) 0.3370 97.612%	(0.8568,1.4106) 0.5538 97.97%	
	$Bayes$	(0.8132,0.9792) 0.1660 95.074%	(0.7992,1.1181) 0.3189 97.593%	(0.8618,1.4035) 0.5417 97.798%	

Table 3:- Average estimates and mean squared errors ($MSE's$) of the estimated point predictors of the future failure times $z_s, s = 1, 2, ..., m$ based on 10000 generated $GTIIPHCS$ informative samples.
($\alpha = 3.13598, \beta = 5.13166$), ($c_1 = 3.2, c_2 = 0.8, c_3 = 2.7$).

Values of (T_1, T_2)		(0.95, 2.5)			
(n, r, m)	$Method$	$\bar{\hat{z}}_1^*$ $MSE(\hat{z}_1^*)$	$\bar{\hat{z}}_2^*$ $MSE(\hat{z}_2^*)$	$\bar{\hat{z}}_3^*$ $MSE(\hat{z}_3^*)$	$\bar{\hat{z}}_4^*$ $MSE(\hat{z}_4^*)$
$M = (2,1,0,0,0,0,0,1,2,0,0,0,0,0,1,0,0,0,0,3)$					
(30, 20, 4)	ML	0.2811	0.4219	0.5113	0.6714
		0.1132	0.1612	0.2417	0.2601
	$Bayes$	0.2732	0.4711	0.5203	0.7017
		0.1029	0.1210	0.2188	0.2413
$M = (1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,3)$					
(30, 25, 4)	ML	0.3017	0.4123	0.5211	0.7106
		0.0812	0.1105	0.1221	0.1715
	$Bayes$	0.3114	0.4213	0.5007	0.6901
		0.0615	0.1091	0.1153	0.1455
Values of (T_1, T_2)		(0.18, 0.5)			
$M = (2,1,0,0,0,0,0,1,2,0,0,0,0,0,1,0,0,0,0,3)$					
(30, 20, 4)	ML	0.3221	0.4312	0.4891	0.6712
		0.13216	0.2014	0.2716	0.2901
	$Bayes$	0.2819	0.4011	0.4998	0.6324
		0.1214	0.1735	0.2442	0.2515
$M = (1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,3)$					
(30, 25, 4)	ML	0.3109	0.4082	0.4818	0.6817
		0.1109	0.1682	0.2218	0.2417
	$Bayes$	0.3442	0.4211	0.5021	0.6901
		0.0919	0.1219	0.1905	0.2106
Values of (T_1, T_2)		(0.18, 1.5)			
$M = (2,1,0,0,0,0,0,1,2,0,0,0,0,0,1,0,0,0,0,3)$					
(30, 20, 4)	ML	0.3091	0.4218	0.4897	0.7016
		0.1466	0.2017	0.2514	0.3015
	$Bayes$	0.2908	0.3817	0.4895	0.6892
		0.1421	0.1829	0.2107	0.2716
$M = (1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,3)$					
(30, 25, 4)	ML	0.3003	0.4279	0.5106	0.6717
		0.1012	0.1331	0.1902	0.2106
	$Bayes$	0.3142	0.4816	0.5891	0.7121
		0.0812	0.1026	0.1142	0.1905

Table 4:- Interval predictors of the future failure time $z_s, s = 1, 2, ..., m$ based on generated $GTIIPHCS$ informative sample.
$(\alpha = 3.13598, \beta = 5.13166)$, $(c_1 = 3.2, c_2 = 0.8,\ c_3 = 2.7)$, $CP \equiv$ Coverage Probability.

Values of (T_1, T_2)		$(0.95, 2.5)$			
(n, r, m)	$Method$	IP of z_1 $Length$ CP	IP of z_2 $Length$ CP	IP of z_3 $Length$ CP	IP of z_4 $Length$ CP
$M = (2, 1, 0, 0, 0, 0, 0, 1, 2, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 3)$					
$(30, 20, 4)$	ML	(0.1177,0.6321) 0.5144 96.412%	(0.2579,0.7751) 0.5172 96.498%	(0.3695,0.9225) 0.5530 96.736%	(0.4764,1.1244) 0.6480 97.559%
	$Bayes$	(0.1062,0.5516) 0.4454 95.236%	(0.2303,0.6785) 0.4482 95.422%	(0.3279,0.7845) 0.4566 95.49%	(0.4202,0.9351) 0.5149 95.564%
$M = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3)$					
$(30, 25, 4)$	ML	(0.0942,0.5703) 0.4761 96.758%	(0.2189,0.7067) 0.4878 97.256%	(0.3219,0.8471) 0.5252 97.365%	(0.4224,1.0367) 0.6143 97.839%
	$Bayes$	(0.1221,0.5521) 0.4300 94.84%	(0.2523,0.6850) 0.4327 95.05%	(0.3513,0.7958) 0.4445 95.33%	(0.4433,0.9381) 0.4948 95.421%
Values of (T_1, T_2)		$(0.18, 0.5)$			
$M = (2, 1, 0, 0, 0, 0, 0, 1, 2, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 3)$					
$(30, 20, 4)$	ML	(0.0329,0.5145) 0.4816 95.193%	(0.1082,0.6078) 0.4996 96.17%	(0.1861,0.7149) 0.5288 96.952%	(0.2726,0.9395) 0.6669 98.061%
	$Bayes$	(0.0934,0.5252) 0.4318 94.464%	(0.2104,0.6418) 0.4314 94.867%	(0.3046,0.7581) 0.4535 95.281%	(0.3950,0.9094) 0.5144 95.713%
$M = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3)$					
$(30, 25, 4)$	ML	(0.0718,0.4981) 0.4263 94.963%	(0.1784,0.6244) 0.4460 95.278%	(0.2702,0.7537) 0.4835 95.834%	(0.3617,0.9266) 0.5649 97.131%
	$Bayes$	(0.0949,0.5221) 0.4272 94.318%	(0.2117,0.6361) 0.4244 94.451%	(0.3051,0.7491) 0.4440 94.705%	(0.3943,0.8953) 0.5010 95.099%
Values of (T_1, T_2)		$(0.18, 1.5)$			
$M = (2, 1, 0, 0, 0, 0, 0, 1, 2, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 3)$					
$(30, 20, 4)$	ML	(0.1199,0.5756) 0.4557 95.786%	(0.2486,0.7175) 0.4689 95.914%	(0.3467,0.8272) 0.4805 96.344%	(0.4378,1.1977) 0.7599 97.176%
	$Bayes$	(0.1081,0.5407) 0.4326 95.072%	(0.2322,0.6657) 0.4335 95.187%	(0.3293,0.7794) 0.4501 95.222%	(0.4208,0.9264) 0.5056 95.251%
$M = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 3)$					
$(30, 25, 4)$	ML	(0.1299,0.5646) 0.4347 95.22%	(0.2613,0.7030) 0.4417 95.633%	(0.3594,0.8183) 0.4589 95.185%	(0.4496,0.9921) 0.5425 96.054%
	$Bayes$	(0.1094,0.5314) 0.4220 95.008%	(0.2339,0.6656) 0.4317 95.08%	(0.3309,0.7784) 0.4475 95.103%	(0.4222,0.9238) 0.5016 95.176%

From tables 1-4, observe the following:

1. In all cases, the $MSE's$ of the $PP's$ estimates increase by increasing s and decreasing by increasing r.
2. In all cases, the *Bayesian* $MSE's$ of the $PP's$ estimates are less than that computed in case of the ML method which means the *Bayesian* method is better than the ML method.
3. For fixed r, n, T_1 and T_2, the lengths and the $CP's$ of the $IP's$ increase by increasing s.
4. For fixed s, n, T_1 and T_2, the lengths and the $CP's$ of the $IP's$ decrease by increasing r.
5. In all cases, the lengths of the $IP's$ are shorter in case of the *Bayesian* method than that computed by the ML method which means the *Bayesian* method is better than the ML.
6. In all cases, the *Bayesian* $CP's$ are less than that computed by the ML method.

4.2 Data Analysis

In this section, two real data sets from Burr-XII distribution are introduced and analyzed. The studied real data sets from Lio and Tzong-Ru Tsai[15]. These real data sets are:

Data I: 0.5926, 0.8004, 0.9646, 1.1105, 1.2412, 1.3890, 1.7636, 2.0045, 3.2858, 4.2851, 4.5854, 5.0385, 5.2283, 6.1484 and 14.4621.

Data II: 0.4909, 0.6954, 0.9359, 0.9522, 1.0001, 1.0181, 1.0272, 1.0768, 1.2710, 1.4043, 1.7208, 2.0806, 2.1688, 2.4537 and 6.7067.

In tables 5 and 6, $MLE's$ of the parameters α and β and the corresponding Kolmogorov Smirnov $(K-S)$ test statistic are computed under Burr-XII model.

Table 5:-$MLE's$ of the parameters and the associated $K-S$ based on the real data set I.

Model	$MLE's$	$K-S$
$Burr-XII(\alpha,\beta)$	$\hat{\alpha}=0.275213, \hat{\beta}=3.76049$	0.111789

Table 6:-$MLE's$ of the parameters and the associated $K-S$ based on the real data set II.

Model	$MLE's$	$K-S$
$Burr-XII(\alpha,\beta)$	$\hat{\alpha}=0.516174, \hat{\beta}=4.41241$	0.171666

Under significance level (0.05) and using Kolmogorov Smirnov table, the critical value for $K-S$ test statistic is 0.33760 which is greater than the computed $K-S$ test statistics for the two real data sets under $Burr-XII$ model. This means that the $Burr-XII$ fits the two real data sets well.
$PP's$ and its $RE's$ of the remaining future failure times $(y_s, s=1,2,...,R_D)$ are computed based on a generated $GTIIPHCS$ informative sample from real data sets I and II in table 7.
Interval predictors of the remaining future failure times $(y_s, s=1,2,...,R_D)$ based on a generated $GTIIPHCS$ informative sample from the given real data sets are computed and summarized in tables 8.
The same is done for the observations $(z_s, s=1,2,3,4)$ of an independent ordered sample based on a generated $GTIIPHCS$ informative sample from the given real data sets are computed and summarized in tables 9 and 10.

Table 7:- Actual values $(AV's)$ and $PP's$ of the future failure time $y_s, s = 1, 2, ..., R_D$ and the corresponding relative errors $RE's$ based on a generated $GTIIPHCS$ informative sample from real data sets I and II.

Values of (T_1, T_2)		$(2.5, 4.0)(I)$			
(n, r, R_D)	$Method$	AV of y_1 PP of y_1 RE	AV of y_2 PP of y_2 RE	AV of y_3 PP of y_3 RE	AV of y_4 PP of y_4 RE
$M = (1, 0, 0, 0, 1, 0, 0, 1, 0, 2)$					
$(15, 10, 3)$	ML	4.2851 4.4214 0.0318	4.5854 5.0918 0.1105	5.0381 5.7062 0.1326	
	$Bayes$	4.2851 4.3331 0.0112	4.5854 5.0054 0.0916	5.0381 4.4819 0.1104	
Values of (T_1, T_2)		$(2.2, 10.0)(II)$			
$M = (1, 0, 0, 0, 1, 0, 0, 1, 0, 2)$					
$(15, 10, 2)$	ML	2.4537 2.7042 0.1021	6.7067 7.5235 0.1218		
	$Bayes$	2.4537 2.5796 0.0513	6.7067 6.0152 0.1031		

Table 8:- Interval predictors of the future failure time $y_s, s = 1, 2, ..., R_D$ based on a generated $GTIIPHCS$ informative sample from real data sets I and II.

Values of (T_1, T_2)		$(2.5, 4.0)(I)$			
(n, r, R_D)	$Method$	IP of y_1 $Length$	IP of y_2 $Length$	IP of y_3 $Length$	IP of y_4 $Length$
$M = (1, 0, 0, 0, 1, 0, 0, 1, 0, 2)$					
$(15, 10, 3)$	ML	(3.2116, 7.3152) 4.1036	(3.6152, 8.5517) 4.9365	(4.0165, 10.6614) 6.6449	
	$Bayes$	(3.2213, 6.4931) 3.2718	(3.7769, 7.8085) 4.0316	(4.2136, 10.2033) 5.9897	
Values of (T_1, T_2)		$(2.2, 10.0)(II)$			
$M = (1, 0, 0, 0, 1, 0, 0, 1, 0, 2)$					
$(15, 10, 2)$	ML	(1.9155, 4.0327) 2.1172	(4.1803, 7.9854) 3.8051		
	$Bayes$	(1.5409, 3.1458) 1.6049	(4.4431, 7.4468) 3.0037		

Table 9:- Generated values ($GV's$) and $PP's$ of the future failure time $z_s, s = 1, 2, 3, 4$ and the corresponding relative errors $RE's$ based on a generated $GTIIPHCS$ informative sample from real data sets I and II.

Values of (T_1, T_2)		$(2.5, 4.0)(I)$			
(n, r, m)	$Method$	GV of z_1 PP of z_1 RE	GV of z_2 PP of z_2 RE	GV of z_3 PP of z_3 RE	GV of z_4 PP of z_4 RE
$M = (1, 0, 0, 0, 1, 0, 0, 1, 0, 2)$					
$(15, 10, 4)$	ML	0.50751 0.34585 0.31854	1.61206 0.98231 0.39065	2.00089 1.19707 0.40173	3.54681 1.90914 0.46173
	$Bayes$	0.50751 0.65552 0.29163	1.61206 2.13103 0.32193	2.00089 2.78435 0.39173	3.54681 2.05066 0.42183
Values of (T_1, T_2)		$(2.2, 10.0)(II)$			
$M = (1, 0, 0, 0, 1, 0, 0, 1, 0, 2)$					
$(15, 10, 4)$	ML	0.50751 0.68062 0.34109	1.61206 2.21196 0.37213	2.00089 2.80195 0.40035	3.54681 5.13670 0.44826
	$Bayes$	0.50751 0.63511 0.25142	1.61206 2.06653 0.28192	2.00089 2.60439 0.30162	3.54681 4.72502 0.33219

Table 10:- Interval predictors of the future failure time $z_s, s = 1, 2, ..., m$ based on a generated $GTIIPHCS$ informative sample from real data sets I and II.

Values of (T_1, T_2)		$(2.5, 4.0)(I)$			
(n, r, m)	$Method$	IP of z_1 $Length$	IP of z_2 $Length$	IP of z_3 $Length$	IP of z_4 $Length$
$M = (1, 0, 0, 0, 1, 0, 0, 1, 0, 2)$					
$(15, 10, 4)$	ML	(0.17722, 1.27068) 1.09346	(0.21318, 3.12658) 2.91340	(0.50098, 5.67804) 5.17706	(0.83622, 6.43128) 5.59506
	$Bayes$	(0.07967, 0.48715) 0.40748	(0.16236, 1.20898) 1.04662	(0.39734, 1.66735) 1.27001	(0.63953, 2.45266) 1.81313
Values of (T_1, T_2)		$(2.2, 10.0)(II)$			
$M = (1, 0, 0, 0, 1, 0, 0, 1, 0, 2)$					
$(15, 10, 4)$	ML	(0.38858, 1.48343) 1.09485	(0.66326, 2.06095) 1.39769	(0.86655, 3.04177) 2.17522	(1.07911, 5.51196) 4.43285
	$Bayes$	(0.41452, 1.11386) 0.69934	(0.83596, 2.11438) 1.27833	(0.92501, 3.01415) 2.08914	(0.95282, 4.76923) 3.81647

From tables 5-10, observe the following:

1. Using significance level (0.05), the computed $K-S$ test statistics is less than the critical value for $K-S$ test statistic which is 0.33760 which for the two real data sets under $Burr-XII$ model which means that the $Burr-XII$ is a good distribution for fitting the two real data sets.
2. In all cases, the $RE's$ of the $PP's$ estimates increase by increasing s.
3. In all cases, the $Bayesian$ $RE's$ of the $PP's$ estimates are less than that computed in case of the ML method which means the $Bayesian$ method is better than the ML method.
4. In all cases, the lengths of the $IP's$ are shorter in case of the $Bayesian$ method than that computed by the ML method which means the $Bayesian$ method is better than the ML.

5 CONCLUSIONS

In this paper, the point and interval predictors of the future failure times from Burr-XII are obtained based on $GTIIPHCS$ informative sample using different values of n, r, T_1 and T_2. Two real data sets are introduced and analyzed using Burr-XII model to emphasize that the Burr-XII fits the given real data sets well. Based on a generated $GTIIPHCS$ informative sample from the given real data sets, the point and interval predictors of the future failure times using one- and two-sample schemes are computed. A comparative study is carried out between the ML and $Bayesian$ methods.

Acknowledgements. The authors are very grateful to the anonymous reviewers for the valuable comments and suggestions which have improved the manuscript.

References

[1] N. Balakrishnan, R. Aggrawala, *Progessive censoring, theory, methods and applications*, Birkhäuser, Boston, 2000.

[2] D. Kundu, A. Joarder, Analysis of Type-II progressively hybrid censored data, Comput. Stat. & Data Analy., **50**(2006) 2509 - 2528.

[3] A. Childs, B.Chandrasekar, N. Balakrishnan, D. Kundu, Exact likelihood inference based on Type- I and Type-II hybrid censored samples from the exponential distribution. Ann. of the Instit. of Statist. Mathem., **55**(2003) 319-330.

[4] K. Lee, H. Sun, Y. Cho, Exact likelihood inference of the exponential parameter under generalized Type-II progressive hybrid censoring, J. Korean Statist. Soc., **45**(2016) 123-136.

[5] M. Amein, Estimation for unknown parameters of the Burr type-XII distribution based on an adaptive progressive type-II censoring scheme, J. of Math. and Statist., **12**(2016) 119-126.

[6] N. Balakrishnan, E. Cramer, *The Art of Progressive Censoring: Applications to Reliability and Quality*, Birkhäuser, 2014.

[7] S., Geisser, *Predictive inference: An Introduction*, Chapman and Hall, London, 1993.

[8] K.S. Kaminsky, L.S., Rhodin, Maximum likelihood prediction. *Ann. Instit. Statist. Math.*, 37(1985), 507-517.

[9] S.F. Ateya, Estimation under modified Weibull distribution based on right censored generalized order statistics, J. Appl. Stat., **40**(2013) 2720-2734.

[10] S.F. Ateya, Estimation under Inverse Weibull Distribution based on Balakrishnan's Unified Hybrid Censored Scheme, Commun. Stat. Simul. Comput., **46**(2017) 3645 - 3666.

[11] Z.F. Jaheen, M.M. Al Harbi, Bayesian estimation for the exponentiated Weibull model via Markov chain Monte Carlo simulation, Commun. Stat. Simul. Comput., **40**(2011) 532-543.

[12] S.J. Press, *Subjective and Objective Bayesian Statistics: Principles, Models and Applications*, Wiley, New York, 2003.

[13] S.K. Upadhyaya, A. Gupta, A Bayes analysis of modified Weibull distribution via Markov chain Monte Carlo simulation, J. Stat. Comput. Simul., **80**(2010) 241-254.

[14] S.K. Upadhyaya, N. Vasishta, A.F.M. Smith, Bayes inference in life testing and reliability via Markov chain Monte Carlo simulation, Sankhya A, **63**(2001) 15-40.

[15] Y.L. Lio, Tzong-Ru Tsai, Estimation of $\delta = P(X < Y)$ for Burr XII distribution based on the progressively first failure-censored samples, J. Appl. Stat., **39**(2012) 309-322.