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EXISTENCE OF POSITIVE SOLUTIONS FOR A SINGULAR FRACTIONAL NONLINEAR DIFFERENTIAL EQUATION WITH FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

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Abstract

In this paper, we investigate the appropriate conditions for the presence of positive solutions for a singular fractional differential equation (SFDE) with fractional integral boundary conditions (FIBCs). Our nonlinear function may has a singularity in its dependent variable. Our investigation is based on the fixed point theorem of Krasnosel'skii on a cone and via regularization and sequential techniques (RASTs).

Keywords and phrases. Fractional differential problem; positive solution; space singularity; integral boundary conditions; regularization; Sequential technique.

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1 Introduction

In this article, we establish the existence of at least one positive solution for the SFDE:

$$D^{\alpha}u(t) + f(t, u(t), D^{\delta}u(t)) = 0, \quad \alpha \in (1, 2),\ \delta > 0,\ \alpha - \delta \geq 1, \tag{1.1}$$

with the FIBCs:

$$u(1) = \lambda I^{\gamma}u(1), \quad I^{\beta}u(t)|_{t=0} = 0, \tag{1.2}$$

where D^{α} is the Riemann-Liouville fractional derivative of order α, $\gamma \in (0, 1)$, $\beta \in [0, 2-\alpha)$, $\lambda \geq 0$, f is positive and satisfies the Carathéodory conditions on $[0, 1] \times (0, \infty) \times R$ and $f(t, u, v)$ may be become singular at $u = 0$.

Here the function $f : [0, 1] \times (0, \infty) \times R$ satisfies the local Carathéodory conditions on the set $[0, 1] \times \mathcal{M}$, $\mathcal{M} = (0, \infty) \times R$ ($f \in Car([0, 1] \times \mathcal{M})$), if $f(., u, v)$ is measurable in t $\forall\, (u, v) \in (0, \infty) \times R$ and $f(t, ., .)$ is continuous in (u, v) for $a.e.$ $t \in [0, 1]$ and further for each compact set $\mathcal{K} \subset \mathcal{M} = (0, \infty) \times R$ there exists $\phi_{\mathcal{K}} \in L^1[0, \infty)$ such that

$$|f(t, u, v)| \leq \phi_{\mathcal{K}}(t), \quad \text{for a.e. } t \in [0, 1],\ \forall\, (u, v) \in \mathcal{K}.$$

The positive solutions for boundary value problems (BVPs) have received considerable attention owing to their importance in the mathematic theory and application (For details and examples, see Ahmad and Ntouyas [1], Bai [2], Baleanu et al. [3], Cabada, and Wang [4], El-Sayed et al. [5,6], Sabatier [7], Wei et al. [8], Zhao et al. [9], and the reference therein).

The investigation of singular BVPs for ordinary differential problems is moderately new, Indeed it was just in the center 1970s that scientists understood that large numbers of applications in the investigation of nonlinear phenomena gave rise to singular BVPs, however, as we would like to think, it was the 1979 Taliafcrro paper [10] that generated the interest of numerous researchers in singular problems in the 1980's and 1990's.

Singular boundary value problems (SBVPs) arise frequently in the study of nonlinear phenomena, for instance in non-Newtonian theory of fluid [10], Permeable Clarendon [11], appear in boundary layer theory [12], Electrostatic and gravitational forces. During the last few decades, the study of the existence of positive solutions for SBVPs has deserved the consideration of numerous researchers: with time singularities have been discussed by: Qui and Bai [13], Bai and Qui [14], Tian and Chen [15] and the reference therein. However, there are papers researching singular fractional BVPs which having singularities in dependent variables (see Agarwal et al. [16], Bai and Sun [17], Lyons and Neugebauer [18], Qiao and Zhou [19], Staněk [20], Xu et al. [21], Yuan et al. [22]).

Much consideration has been focused around the investigation of integral boundary conditions (IBCs), which are associated in various fields, for example, populations dynamics, chemical engineering and underground water flow, for a detailed description for FDEs with IBCs, we refer to some recent papers (see Cabada and Hamdi [23], Li et al. [24], Nanware and Dhaigude [25], Sun and Zhao [26], Zhang et al. [27], Wang and Xie [28]).

In [16], Agarwal et al. investigate the positivity and multiple solutions of the SFDE:

$$D^{\alpha}u(t) + f(t, u(t), D^{\mu}u(t)) = 0, \quad \alpha \in (1,2), \quad \mu > 0, \quad \alpha - \mu \geq 1$$

with $$u(0) = u(1) = 0$$

where $f > 0$ satisfies the Carathédory conditions on $[0,1] \times (0,\infty) \times R$ and singular at $x = 0$. The same last SFDE is considered by Qiao and Zhou [19], yet rather than the previous boundary conditions, they utilize the boundary conditions, $x(0) = D^{\beta}x(1) = 0$ with $\beta \in (0, \alpha-1)$, $\mu \in (0, \alpha-1)$ under similar conditions on f.

This paper is orderly as follows. In Section 2, we provided several preliminaries. In Sections 3, we convert SFDE with FIBCs (1.1)-(1.2) to an equivalent integral form and afterward acquire the Green's function and demonstrate a several properties of it. Existence result for the SFDE with FIBCs (1.1)-(1.2) is demonstrate by RASTs. At first we define a sequence of auxiliary regular FDE to the SFDE (1.1) and prove that the sequence of the auxiliary regular FDEs with FIBCs (4.4), (1.2) has a sequence $\{u_m\}$ of positive solutions which is relative compact (Lemmas 4.2 and 4.3). At last, in Section 5, we construct a convergent subsequence of $\{u_{n_m}\}$ in $C[0,1]$, and afterward the Lebesgue dominated convergence theorem demonstrates that its limit u is a solution of the SFDE with IBC (1.1)-(1.2) (Theorem 5.1).

2 Preliminaries

Here we presence definitions, preliminary facts, notations which are utilized all through this paper.
Let $||u|| = \max\{|u(t)| : t \in [0,1]\}$ be the norm in the space $C[0,1]$ and $||u||_{L^1} = \int_0^1 |u(t)|dt$ be the norm in $L^1[0,1]$, while $||u||_0 = \max\{||u||, ||D^{\delta}u||\}$.

Definition 2.1. (see [29]) The Riemann-Liouville fractional integral of the function $u(t) \in L^1[0,1]$ is identified as

$$I^{\beta}u(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}u(s)ds,$$

and the Riemann-Liouville fractional derivative of order $n-1 < \beta \leq n$ for the function $u(t)$ is identified as

$$D^{\beta}u(t) = \frac{d^n}{dt^n}I^{n-\beta}u(t) = \frac{d^n}{dt^n}\int_0^t \frac{(t-s)^{n-\beta-1}}{\Gamma(n-\beta)}u(s)ds.$$

Theorem 2.1. (Krasnosel'skii fixed point theorem [30]). Let $\mathcal{E}$ be a Banach space, and $\mathcal{Q} \subset \mathcal{E}$ be a cone in $\mathcal{E}$. Let Ω_1, Ω_2 be bounded open balls of $\mathcal{E}$ centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $P : \mathcal{Q} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{Q}$ is a complete continuous operator such that either

(a) $||Pu|| \geq ||u||$ for $u \in \mathcal{Q} \cap \partial\Omega_1$, and $||Pu|| \leq ||u||$ for $u \in \mathcal{Q} \cap \partial\Omega_2$, or
(b) $||Pu|| \leq ||u||$ for $u \in \mathcal{Q} \cap \partial\Omega_1$, and $||Pu|| \geq ||u||$ for $u \in \mathcal{Q} \cap \partial\Omega_2$

hold. Then P has a fixed point in $\mathcal{Q} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

We list below assumptions to be used later in the paper.

(H_1) The function $f \in Car([0,1] \times \mathcal{M})$, for all $v \in R$, a.e. $t \in [0,1]$, and $\mathcal{M} = (0,\infty) \times R$

$$\lim_{u \to 0^+} f(t,u,v) = \infty.$$

(H_2) for a.e. $t \in [0,1]$ and all $(u,v) \in \mathcal{M}$ there exists a constant $\rho > 0$ such that .

$$f(t,u,v) \geq \rho(1-t)^{2-2\alpha}$$

(H_3) f satisfies the estimate

$$f(t,u,v) \leq p_0(t)\,(p_1(t) + q(u) + r(u) + w(|v|)), \quad \text{for a.e. } t \in [0,1], \text{ and } (u,v) \in \mathcal{M},$$

where p_0, $p_1 \in L^1[0,1]$, $q \in C(0,\infty)$ and non increasing function, r, $w \in C[0,\infty)$ are positive functions and nondecreasing,

$$W = \int_0^1 p_0(t)\, q(\frac{\rho\, t\,(1-t)^2}{2\Gamma(\alpha)})\, dt < \infty, \tag{2.1}$$

and

$$\lim_{u \to \infty} \frac{r(u) + w(u)}{u} = 0.$$

(H_4) $\frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\gamma)} < 1.$

Remark 2.1. Under the assumption (H_1), it follows that $\lim_{u \to 0^+} q(u) = \infty$.

3 Green's function and its properties

Lemma 3.1. [29] Let $\alpha \in (1,2]$. Then the following equality holds for $u \in L^1[0,1]$,

$$I^\alpha\, D^\alpha\, u(t) \;=\; u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \quad c_i \in R,\ i = 1,2.$$

Lemma 3.2. Let $h(t) \in L^1[0,1]$, then the unique solution of the problem

$$D^\alpha u(t) + h(t) = 0, \qquad \alpha \in (1,2), \tag{3.1}$$
$$u(1) = \lambda I^\gamma u(1), \qquad I^\beta u(t)|_{t=0} = 0,$$

is obtained by

$$u(t) = \int_0^1 G(t,s) h(s) ds,$$

where

$$G(t,s) = \begin{cases} At^{\alpha-1}\left[\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\lambda(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\right] - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1; \\ At^{\alpha-1}\left[\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\lambda(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\right], & 0 \leq t \leq s \leq 1, \end{cases} \tag{3.2}$$

and $A = (1 - \frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\gamma)})^{-1}$.

Proof. We have the solution of (3.1) from Lemma 3.1 is given by

$$\begin{aligned} u(t) &= -I^\alpha\, h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \\ I^\beta u(t) &= -I^{\alpha+\beta}\, h(t) + \frac{c_1\Gamma(\alpha)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1} + \frac{c_2\Gamma(\alpha-1)}{\Gamma(\alpha+\beta-1)} t^{\alpha+\beta-2}, \end{aligned}$$

and from $I^\beta u(t)|_{t=0} = 0$, we have $c_2 = 0$. Then

$$u(t) \;=\; -I^\alpha\, h(t) + c_1 t^{\alpha-1},$$

and

$$I^{\gamma}u(t) \;=\; -I^{\alpha+\gamma}\; h(t) + \frac{c_1\Gamma(\alpha)}{\Gamma(\alpha+\gamma)}t^{\alpha+\gamma-1}.$$

Therefore, we get

$$I^{\gamma}u(1) \;=\; -\int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\; h(s)ds + \frac{c_1\Gamma(\alpha)}{\Gamma(\alpha+\gamma)}.$$

From $u(1) = \lambda I^{\gamma}u(1)$, we have

$$c_1 - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\; h(s)ds \;=\; \frac{c_1\lambda\Gamma(\alpha)}{\Gamma(\alpha+\gamma)} - \lambda\int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\; h(s)ds.$$

Hence

$$c_1 = A\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\; h(s)ds \;-\; \lambda\; \int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\; h(s)ds\right].$$

We have that the solution of (3.1) can be expressed by the formula

$$\begin{aligned} u(t) \;&=\; A\,t^{\alpha-1}\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\, h(s)ds - \lambda\int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\, h(s)ds\right] \\ &\quad - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, h(s)ds. \end{aligned} \tag{3.3}$$

Lemma 3.3. Assume that (H_4) holds, then the function $G(t,s)$ defined by (3.2) has the following properties:
(i) $G(t,s),\;t\in[0,1]$ is uniformly continuous;
(ii) $G(t,s)\geq 0\;\forall\;(t,s)\in[0,1]\times[0,1]$;
(iii) $G(t,s)\leq \frac{A}{\Gamma(\alpha)}$ for all $(t,s)\in[0,1]\times[0,1]$.

Proof. It is easy to verify property (i), to show that (ii) holds.

Define $\quad g_1(t,s) = At^{\alpha-1}\left[\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\lambda(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\right] - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$ for $0\leq s\leq t\leq 1$, and

$$g_2(t,s) = At^{\alpha-1}\left[\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\lambda(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\right] \text{ for } 0\leq t\leq s\leq 1.$$

Then for $0\leq s\leq t\leq 1$, we have that

$$\begin{aligned} g_1(t,s) \;&=\; \frac{At^{\alpha-1}}{\Gamma(\alpha)}\left[(1-s)^{\alpha-1} - \frac{\lambda\Gamma(\alpha)(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} - \frac{1}{A}(1-\frac{s}{t})^{\alpha-1}\right], \\ &\geq\; \frac{At^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[1 - \frac{\lambda\Gamma(\alpha)(1-s)^{\gamma}}{\Gamma(\alpha+\gamma)} - \frac{1}{A}\right], \\ &\geq\; \frac{At^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[1 - \frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\gamma)} - \frac{1}{A}\right] = \frac{At^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\frac{1}{A} - \frac{1}{A}\right] = 0. \end{aligned} \tag{3.4}$$

For $0\leq t\leq s\leq 1$, we get that

$$\begin{aligned} g_2(t,s) \;&=\; At^{\alpha-1}\left[\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\lambda(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\right], \\ &\geq\; \frac{At^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[1 - \frac{\lambda(1-s)^{\gamma}\Gamma(\alpha)}{\Gamma(\alpha+\gamma)}\right], \\ &\geq\; \frac{At^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[1 - \frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\gamma)}\right] = \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} \geq 0. \end{aligned} \tag{3.5}$$

Therefore, (ii) holds.
(iii) Since $\alpha>1,\;(t,s)\in[0,1]\times[0,1]$, then $t^{\alpha-1}$and $(1-s)^{\alpha-1}\leq 1$.

So from (3.2), $G(t,s) \le \frac{A}{\Gamma(\alpha)}$ for $(t,s) \in [0,1] \times [0,1]$.

Lemma 3.4. Suppose that $h(t) \in L^1[0,1]$, let $h(t) \ge \rho(1-t)^{2-2\alpha}$ for a.e. $t \in [0,1]$ with ρ is a positive constant, then
$$\int_0^1 G(t,s)h(s)ds \ge \frac{\rho\, t\, (1-t)^2}{2\Gamma(\alpha)}.$$

Proof. Firstly, for $0 \le s \le t \le 1$, using (3.4), we have
$$\begin{aligned}
g_1(t,s) &\ge \frac{At^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[1 - \frac{\lambda\Gamma(\alpha)(1-s)^{\gamma}}{\Gamma(\alpha+\gamma)} - \frac{1}{A}\right],\\
&\ge \frac{At^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[1 - \frac{\lambda\Gamma(\alpha)(1-s)^{\gamma}}{\Gamma(\alpha+\gamma)} - 1 + \frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\gamma)}\right],\\
&\ge \frac{A\,\lambda\, t^{\alpha-1}\,(1-s)^{\alpha-1}}{\Gamma(\alpha+\gamma)}\,[1-(1-s)^{\gamma}] \quad \text{(using Lagrange mean value theorem)},\\
&\ge \frac{A\,\lambda\,\gamma\, t^{\alpha-1}\,(1-s)^{\alpha-1} s}{\Gamma(\alpha+\gamma)}.
\end{aligned}$$

Consequently, for $t \in [0,1]$ and $\gamma \in (0,1)$, and by using (3.5)
$$\begin{aligned}
\int_0^1 G(t,s)h(s)ds &= \int_0^t g_1(t,s)h(s)ds + \int_t^1 g_2(t,s)h(s)ds\\
&\ge \int_0^t \frac{A\,\rho\,\lambda\,\gamma\, t^{\alpha-1}\,(1-s)^{\alpha-1}\,(1-s)^{2-2\alpha} s}{\Gamma(\alpha+\gamma)}\\
&\quad +\rho\int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}(1-s)^{2-2\alpha}}{\Gamma(\alpha)}ds\\
&\ge \frac{A\,\rho\,\lambda\,\gamma\, t^{\alpha-1}}{\Gamma(\alpha+\gamma)}\int_0^t (1-s)^{1-\alpha}\, s\, ds + \frac{\rho\, t^{\alpha-1}}{\Gamma(\alpha)}\int_t^1 (1-s)^{1-\alpha}\, ds\\
&\ge \frac{A\,\rho\,\lambda\,\gamma\, t^{\alpha-1}}{\Gamma(\alpha+\gamma)}\int_0^t (1-s)s\, ds + \frac{\rho\, t^{\alpha-1}}{\Gamma(\alpha)}\int_t^1 (1-s)\, ds\\
&\ge \frac{\rho\, t^{\alpha-1}\,(1-t)^2}{2\Gamma(\alpha)} \ge \frac{\rho\, t\,(1-t)^2}{2\Gamma(\alpha)}, \qquad t^{\alpha-1} \ge t, \text{ for } t \in [0,1].
\end{aligned}$$

Define an operator P_m by the formula
$$\begin{aligned}
(P_m u)(t) &= \int_0^1 G(t,s)\, f_m(s,u(s),D^{\delta}u(s))ds, \qquad 0 \le t \le 1, \qquad (3.6)\\
&= A\, t^{\alpha-1}\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_m(s,u(s),D^{\delta}u(s))\, ds\right.\\
&\quad \left.- \lambda\int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} f_m(s,u(s),D^{\delta}u(s))ds\right] - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_m(s,u(s),D^{\delta}u(s))ds.
\end{aligned}$$

Lemma 3.5. Suppose that (H_4) holds, then we have
$$\begin{aligned}
(D^{\delta}P_m u)(t) &= \frac{A\,\Gamma(\alpha)t^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_m(s,u(s),D^{\delta}u(s))ds\right.\\
&\quad \left.- \lambda\int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\, f_m(s,u(s),D^{\delta}u(s))ds\right]\\
&\quad - \int_0^t \frac{(t-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}\, f_m(s,u(s),D^{\delta}u(s))ds,\\
&= \int_0^1 G^{*}(t,s)f_m(s,u(s),D^{\delta}u(s))ds \in C[0,1], \qquad 0 \le t \le 1. \qquad (3.7)
\end{aligned}$$

Where $G^*(t,s)$ is the Green function defined by:

$$G^*(t,s)=\begin{cases}\frac{A\,\Gamma(\alpha)\,t^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}\left[\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}-\frac{\lambda(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\right]-\frac{(t-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}, & 0\le s\le t\le 1;\\ \frac{A\,\Gamma(\alpha)\,t^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}\left[\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}-\frac{\lambda(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\right], & 0\le t\le s\le 1,\end{cases} \tag{3.8}$$

and has the following properties:
(1) $G^*(t,s)\le\frac{A}{\Gamma(\alpha-\delta)}$ for all $(t,s)\in[0,1]\times[0,1]$;
(2) $G^*(t,s)\in C([0,1]\times[0,1])$.
Proof. From Definition 2.1 and by using equation (3.6), let $h(r)=f_m(r,u(r),D^\delta u(r))$ we have

$$\begin{aligned}(D^\delta P_m u)(t) &= \frac{d}{dt}\int_0^t\frac{(t-s)^{-\delta}}{\Gamma(1-\delta)}(P_m u)(s)\,ds,\\ &= \frac{d}{dt}\int_0^t\frac{(t-s)^{-\delta}}{\Gamma(1-\delta)}\left(\int_0^1 G(s,r)\,h(r)dr\right)ds,\\ &= \frac{d}{dt}\int_0^t\frac{(t-s)^{-\delta}}{\Gamma(1-\delta)}\left[A\,s^{\alpha-1}\left(\int_0^1\frac{(1-r)^{\alpha-1}}{\Gamma(\alpha)}h(r)dr\right.\right.\\ &\quad\left.\left.-\lambda\int_0^1\frac{(1-r)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}h(r)dr\right)-\int_0^s\frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)}h(r)dr\right]ds,\\ &= A\frac{d}{dt}\int_0^t\frac{(t-s)^{-\delta}\,s^{\alpha-1}}{\Gamma(1-\delta)}ds\left(\int_0^1\frac{(1-r)^{\alpha-1}}{\Gamma(\alpha)}h(r)dr\right.\\ &\quad\left.-\lambda\int_0^1\frac{(1-r)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}h(r)dr\right)-\frac{d}{dt}\int_0^t\frac{(t-s)^{-\delta}}{\Gamma(1-\delta)}ds\int_0^s\frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)}h(r)dr,\\ &= \frac{d}{dt}\frac{A\,\Gamma(\alpha)\,t^{\alpha-\delta}}{\Gamma(1+\alpha-\delta)}\left[\int_0^1\frac{(1-r)^{\alpha-1}}{\Gamma(\alpha)}h(r)dr-\lambda\int_0^1\frac{(1-r)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}h(r)dr\right]\\ &\quad-\frac{d}{dt}\int_0^t\frac{(t-s)^{\alpha-\delta}}{\Gamma(1+\alpha-\delta)}h(s)ds,\\ &= \frac{A\,\Gamma(\alpha)\,t^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}\left[\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds-\lambda\int_0^1\frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}h(s)ds\right]\\ &\quad-\int_0^t\frac{(t-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}h(s)ds,\\ &= \int_0^1 G^*(t,s)h(s)ds,\quad 0\le t\le 1.\end{aligned}$$

We now show that (1) and (2) are true.
(1) It is easy to prove $G^*(t,s)\le\frac{A}{\Gamma(\alpha-\delta)}$ for all $(t,s)\in[0,1]\times[0,1]$.
(2) Since $\alpha-\delta\ge 1$. Hence $G^*(t,s)\in C([0,1]\times[0,1])$ and $D^\delta P_m u\in C[0,1]$ (notice that $\int_0^t\frac{(t-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}h(s)ds\in C[0,1]$).

4 Auxiliary regular FDE

If assumption (H_1) is satisfied, then equation (1.1) is singular equation and we apply RASTs for the existence of positive solutions of (1.1)-(1.2).
Let $\mathcal{E}$=$\{u\in C[0,1]: D^\delta u\in C[0,1]\}$ endowed with the norm $||u||_0=\max\{||u||,\,||D^\delta u||\}$, then $\mathcal{E}$ is a Banach space (see [31]).
Let $\mathcal{Q}$= $\{u\in\mathcal{E}:\ u(t)\ge 0\ \text{ for }\ t\in[0,1]\}$, then $\mathcal{Q}$ is a cone in $\mathcal{E}$.
Define the function f_m, $m\in N$, by the formula

$$f_m(t,u,v)=\begin{cases}f(t,u,v) & \text{if}\ \ u\ge\frac{1}{m},\\ f(t,\frac{1}{m},v) & \text{if}\ \ 0\le u<\frac{1}{m}.\end{cases}$$

Then $f_m \in Car([0,1]\times\mathcal{M}^*)$, $\mathcal{M}^* = [0,\infty)\times R$, and assumptions (H_2) and (H_3) give

$$f_m(t,u,v) \geq \rho(1-t)^{2-2\alpha} \quad \text{for a.e. } t\in[0,1] \text{ and all } (u,v)\in\mathcal{M}^*. \tag{4.1}$$

$$f_m(t,u,v) \leq p_0(t)\,(p_1(t) + q(\frac{1}{m}) + r(u) + r(1) + w(|v|)) \tag{4.2}$$
$$\text{for a.e. } t\in[0,1] \text{ and all } (u,v)\in\mathcal{M}^*, and$$

$$f_m(t,u,v) \leq p_0(t)\,(p_1(t) + q(u) + r(u) + r(1) + w(|v|)) \tag{4.3}$$
$$\text{for a.e. } t\in[0,1] \text{ and all } (u,v)\in\mathcal{M}=(0,\infty)\times R.$$

Consider the family of regular differential equations

$$D^\alpha u(t) + f_m(t,u(t),D^\delta u(t)) = 0, \quad \alpha\in(1,2),\, 0<\delta<1,\, \alpha-\delta\geq 1, \tag{4.4}$$

with fractional integral boundary condition (1.2).

Definition 4.1. A function $u\in C[0,1]$ is a positive solution to (4.4), (1.2) if $u>0$ on $[0,1]$, $D^\delta u(t)\in C[0,1]$, $D^\alpha u(t)\in L^1[0,1]$ and (4.4), (1.2) hold a.e. on $[0,1]$.

Lemma 4.1. Suppose that $(H_1)-(H_4)$ hold. Then $P_m:\mathcal{Q}\to\mathcal{Q}$ and P_m is a completely continuous operator.
Proof. Let $u\in\mathcal{Q}$ and let $h(t)=f_m(t,u(t),D^\delta u(t))$ for a.e. $0\leq t\leq 1$. Then $h(t)\geq 0$ and $h(t)\in L^1[0,1]$. In order to prove that $P_m:\mathcal{Q}\to\mathcal{Q}$, we have the equality (cf. (3.6))

$$\begin{aligned}(P_m u)(t) &= \int_0^1 G(t,s)\,h(s)ds,\\ &= A\,t^{\alpha-1}\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds - \lambda\int_0^1\frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}h(s)ds\right]\\ &\quad -\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds.\end{aligned}$$

Notice from $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\in C[0,1]$ and from $G\geq 0$ by Lemma 3.3(ii) that, for $u\in\mathcal{Q}$ we have $P_m u\in C[0,1]$ and $(P_m u)(t)\geq 0$ on $[0,1]$. Likewise, from Lemma 3.5 we have $(D^\delta P_m u)(t)\in C[0,1]$ for $t\in[0,1]$. Consequently, $P_m:\mathcal{Q}\to\mathcal{Q}$.

Secondly, we prove that P_m is bounded on any bounded set from $\mathcal{Q}$. Let $\mathcal{K}\subset\mathcal{Q}$ ba a bounded subset in $\mathcal{Q}$. Since $f_m\in Car([0,1]\times\mathcal{M}^*)$, there exists $\phi\in L^1[0,1]$ such that

$$0<f_m(t,u(t),D^\delta u(t))\leq\phi(t) \;\; \forall\, u\in\mathcal{K} \text{ and a.e. } t\in[0,1]. \tag{4.5}$$

From Lemma 3.3, we have

$$\begin{aligned}|(P_m u)(t)| &= \left|\int_0^1 G(t,s)f_m(s,u(s),D^\delta u(s))ds\right|,\\ &\leq \frac{A}{\Gamma(\alpha)}\int_0^1 f_m(s,u(s),D^\delta u(s))ds \leq \frac{A}{\Gamma(\alpha)}\;||\phi||_{L^1}.\end{aligned}$$

Also from Lemma 3.5, we obtain

$$|(D^\delta P_m u)(t)| = \left|\int_0^1 G^*(t,s)f_m(s,u(s),D^\delta u(s))ds\right| \leq \frac{A}{\Gamma(\alpha-\delta)}\;||\phi||_{L^1}.$$

Hence $P_m(\mathcal{K})$ is bounded in $\mathcal{Q}$.

Thirdly, we show that P_m is a continuous operator. Let $\{u_k\}\subset\mathcal{Q}$ be a convergent sequence to u in $\mathcal{Q}$, then $\lim\limits_{k\to\infty}||u_k-u||_0=0$ and then $u\in\mathcal{Q}$.
Also by Lemma 3.3(iii), we have

$$|(P_m u_k)(t)-(P_m u)(t)| \leq \frac{A}{\Gamma(\alpha)}\int_0^1\left|f_m(s,u_k(s),D^\delta u_k(s)) - f_m(s,u(s),D^\delta u(s))\right|ds.$$

Since $f_m \in Car([0,1]\times\mathcal{M}^*)$, then we have

$$\lim_{k\to\infty} f_m(t,u_k(t),D^\delta u_k(t)) = f_m(t,u(t),D^\delta u(t)).$$

By applying Lebesgue dominated convergence theorem and using (4.2) we obtain,

$$\lim_{k\to\infty} \int_0^1 \left|f_m(t,u_k(t),D^\delta u_k(t)) - f_m(t,u(t),D^\delta u(t))\right| = 0.$$

Hence we get

$$|(P_m u_k)(t) - (P_m u)(t)| \quad \to 0 \quad \text{as} \quad k\to\infty.$$

Also from (3.7) we have

$$\begin{aligned}
&\left|(D^\delta P_m u_k)(t) - (D^\delta P_m u)(t)\right| \\
\leq\ & \frac{A\,\Gamma(\alpha)\,t^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f_m(s,u_k(s),D^\delta u_k(s)) - f_m(s,u(s),D^\delta u(s))|ds \\
&+\frac{A\,\lambda\,\Gamma(\alpha)\,t^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}\int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}|f_m(s,u_k(s),D^\delta u_k(s)) - f_m(s,u(s),D^\delta u(s))|ds \\
&+\int_0^t \frac{(t-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}|f_m(s,u_k(s),D^\delta u_k(s)) - f_m(s,u(s),D^\delta u(s))|ds, \\
\leq\ & \frac{A}{\Gamma(\alpha-\delta)}\int_0^1 |f_m(s,u_k(s),D^\delta u_k(s)) - f_m(s,u(s),D^\delta u(s))|ds \\
&+\frac{A\,\lambda\,\Gamma(\alpha)}{\Gamma(\alpha-\delta)\,\Gamma(\alpha+\gamma)}\int_0^1 |f_m(s,u_k(s),D^\delta u_k(s)) - f_m(s,u(s),D^\delta u(s))|ds \\
&+\frac{1}{\Gamma(\alpha-\delta)}\int_0^1 |f_m(s,u_k(s),D^\delta u_k(s)) - f_m(s,u(s),D^\delta u(s))|ds, \\
&\to 0 \quad \text{as} \quad k\to\infty.
\end{aligned}$$

Therefor $\lim_{m\to\infty} ||(P_m u_k)(t) - (P_m u)(t)||_0 = 0$, that is P_m is a continuous operator.

Finally, to prove P_m is an equicontinuous operator. Let $0 \leq t_1 < t_2 \leq 1$, then using (4.5) we obtain

$$\begin{aligned}
|(P_m u)(t_2) - (P_m u)(t_1)| \leq\ & A\,(t_2^{\alpha-1} - t_1^{\alpha-1})\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f_m(s,u(s),D^\delta u(s))ds\right. \\
&\left. + \lambda\int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} f_m(s,u(s),D^\delta u(s))ds\right] \\
&+\int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} f_m(s,u(s),D^\delta u(s))ds \\
&+\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f_m(s,u(s),D^\delta u(s))ds, \\
\leq\ & \frac{A}{\Gamma(\alpha)}\,(1+\frac{\lambda\,\Gamma(\alpha)}{\Gamma(\alpha+\gamma)})\ (t_2^{\alpha-1} - t_1^{\alpha-1})\ ||\phi||_{L^1} \\
&+\frac{(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}\int_{t_1}^{t_2}\phi(s)ds + \int_0^{t_1}\frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s)ds.
\end{aligned}$$

Also from (3.7) we have

$$\begin{aligned}
&|(D^\delta P_m u)(t_2) - (D^\delta P_m u)(t_1)| \\
&\le \frac{A\,\Gamma(\alpha)\,(t_2^{\alpha-\delta-1} - t_1^{\alpha-\delta-1})}{\Gamma(\alpha-\delta)} \left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f_m(s,u(s),D^\delta u(s))ds \right.\\
&\left. + \lambda \int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} f_m(s,u(s),D^\delta u(s))ds \right]\\
&+ \int_0^{t_1} \frac{(t_2-s)^{\alpha-\delta-1} - (t_1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} f_m(s,u(s),D^\delta u(s))ds\\
&+ \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} f_m(s,u(s),D^\delta u(s))ds,\\
&\le \frac{A\,(t_2^{\alpha-\delta-1} - t_1^{\alpha-\delta-1})}{\Gamma(\alpha-\delta)} (1 + \frac{\lambda\,\Gamma(\alpha)}{\Gamma(\alpha+\gamma)})\,||\phi||_{L^1}\\
&+ \int_0^{t_1} \frac{(t_2-s)^{\alpha-\delta-1} - (t_1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} \phi(s)ds + \frac{(t_2-t_1)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} \int_{t_1}^{t_2} \phi(s)ds, \qquad \text{if } \ \alpha-\delta > 1,
\end{aligned}$$

and

$$|(D^\delta P_m u)(t_2) - (D^\delta P_m u)(t_1)| \le \int_{t_1}^{t_2} \phi(s)ds, \qquad \text{if } \ \alpha-\delta = 1.$$

As $t_2 \to t_1$, the right side of the above inequalities tend to zero. Therefore $\{P_m u : \ u \in \mathcal{K}\}$ and $\{D^\delta P_m u : \ u \in \mathcal{K}\}$ are bounded in $C[0,1]$ and equicontinuous on $[0,1]$, and by the Arzelá-Ascoli theorem $P_m(\mathcal{K})$ is relative compact and we have P_m is a completely continuous operator.

We now apply Krasnosel'skii fixed point theorem to determine that P_m has a fixed point theorem in $\mathcal{Q} \cap (\overline{\Omega_2} \backslash \Omega_1)$. Accordingly, problem (4.4), (1.2) has a multiple solutions.

Lemma 4.2. Assume that $(H_1) - (H_4)$ hold. Then the auxiliary regular FDE (4.4) with FIBCs (1.2) has multiple positive solutions.
Proof. By Lemma 4.1 $P_m : \mathcal{Q} \ \to \ \mathcal{Q}$ is completely continuous operator.
Define $\Omega_1 = \{u \in \mathcal{E} : \ ||u||_0 < \frac{2\rho}{27\,\Gamma(\alpha)}\}$.
Since $||P_m u||_0 = \max\{||P_m u||, ||D^\delta P_m u||\}$, hence $||P_m u||_0 \ge ||P_m u|| = \{\max |P_m u(t)| : t \in [0,1]\}$.
From Lemma 3.4 $(P_m u)(t) \ge \frac{\rho\, t\,(1-t)^2}{2\Gamma(\alpha)}$, $u \in \mathcal{Q}$ and we have $||P_m u||_0 \ge \max\limits_{t\in[0,1]} \frac{\rho\, t\,(1-t)^2}{2\Gamma(\alpha)}$. Hence $||P_m u||_0 \ge \frac{2\rho}{27\,\Gamma(\alpha)}$ and therefore

$$||P_m u||_0 \ge ||u||_0 \ \text{ for } \ u \in \mathcal{Q} \cap \partial\Omega_1. \tag{4.6}$$

Next, from inequality (4.2) and Lemma 3.3(iii)

$$\begin{aligned}
|(P_m u)(t)| &\le \frac{A}{\Gamma(\alpha)} \int_0^1 p_0(s)\,(p_1(s) \,+\, q(\frac{1}{m}) \,+\, r(u(s)) \,+\, r(1) \,+\, w(|D^\delta u(s)|))ds,\\
&\le \frac{A}{\Gamma(\alpha)} \int_0^1 p_0(s)\,(p_1(s) \,+\, q(\frac{1}{m}) \,+\, r(||u||) \,+\, r(1) \,+\, w(||D^\delta u||))ds,\\
&\le \frac{A}{\Gamma(\alpha)} \left[||p_0 p_1||_{L^1} + \left(q(\frac{1}{m}) \,+\, r(||u||) \,+\, r(1) \,+\, w(||D^\delta u||) \right) ||p_0||_{L^1} \right].
\end{aligned}$$

Also from Lemma 3.5 and inequality (4.2), we have

$$|(D^\delta P_m u)(t)| \le \frac{A}{\Gamma(\alpha-\delta)} \left[||p_0 p_1||_{L^1} + \left(q(\frac{1}{m}) \,+\, r(||u||) \,+\, r(1) \,+\, w(||D^\delta u||) \right) ||p_0||_{L^1} \right],$$

because r, w are nondecreasing by (H_3). Let $\Upsilon = \max\{\frac{A}{\Gamma(\alpha)}, \frac{A}{\Gamma(\alpha-\delta)}\}$. Hence for $u \in \mathcal{E}$, we have

$$||P_m u||_0 \le \Upsilon \left[||p_0 p_1||_{L^1} + \left(q(\frac{1}{m}) \,+\, r(||u||_0) \,+\, r(1) \,+\, w(||u||_0) \right) ||p_0||_{L^1} \right].$$

Now from condition (H_3), since $\lim_{\vartheta\to\infty} \frac{r(\vartheta)+w(\vartheta)}{\vartheta} = 0$, then there exists an adequately large number $S > 0$ such that

$$\Upsilon\left[||p_0p_1||_{L^1} + \left(q(\frac{1}{m}) + r(S) + r(1) + w(S)\right)||p_0||_{L^1}\right] < S.$$

Let $\Omega_2 = \{u \in \mathcal{E} : \ ||u||_0 < S\}$. Then for $u \in \mathcal{Q} \cap \partial\Omega_2$ and $t \in [0,1]$, it follows from the last two inequalities that,

$$||P_m u||_0 \le ||u||_0 \quad \text{for} \quad u \in \mathcal{Q} \cap \partial\Omega_2. \tag{4.7}$$

Applying Theorem 2.1, we deduce from (4.6) and (4.7) that P_m has a fixed point in $\mathcal{Q}\cap(\overline{\Omega_2}\backslash\Omega_1)$, thus auxiliary regular FDE (4.4) with FIBCs (1.2) has multiple positive solutions.

Now, we deal with a sequence of solutions of regular FDE (4.4) with FIBCs (1.2).

Lemma 4.3. Suppose that (H_1)-(H_4) hold. Let u_m be a solution of (4.4), (1.2) defined by

$$u_m(t) \ = \ \int_0^1 G(t,s)\, f_m(s, u_m(s), D^\delta u_m(s))ds, \ m \in N, \ t \in [0,1]. \tag{4.8}$$

Then the sequences $\{u_m\}$ and $\{D^\delta u_m\}$ are relatively compact in $C[0,1]$.

Proof. We prove this lemma by a same route as the proof of Lemma 4.1, for $t \in [0,1], \ m \in N$, we have (cf. (3.7))

$$\begin{aligned} D^\delta u_m(t) = &\frac{A\,\Gamma(\alpha)\,t^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f_m(s,u_m(s),D^\delta u_m(s))ds\right.\\ &\left.-\lambda\int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\, f_m(s,u_m(s),D^\delta u_m(s))ds\right]\\ &-\int_0^t \frac{(t-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}\, f_m(s,u_m(s),D^\delta u_m(s))ds. \end{aligned}$$

It follow from (4.1) and Lemma 3.4 that:

$$u_m(t) \ge \frac{\rho\, t\,(1-t)^2}{2\Gamma(\alpha)}, \tag{4.9}$$

but from (4.3) and (H_3), we have

$$\begin{aligned} f_m(t,u_m,D^\delta u_m(t)) &\le p_0(t)\,(p_1(t) + q(u_m(t)) + r(u_m(t)) + r(1) + w(|D^\delta u_m(t)|)),\\ &\le p_0(t)\,(p_1(t) + q(\frac{\rho\,t\,(1-t)^2}{2\Gamma(\alpha)}) + r(||u_m||_0) + r(1) + w(||u_m||_0). \end{aligned} \tag{4.10}$$

This implies (cf. (2.1), (4.8), (4.9) and (4.10))

$$\begin{aligned} |u_m(t)| &\le \frac{A}{\Gamma(\alpha)}\left[\,||p_0p_1||_{L^1} + \int_0^1 p_0(t)\,q(\frac{\rho\,t\,(1-t)^2}{2\Gamma(\alpha)})dt + \Big(r(||u_m||_0) + r(1)\right.\\ &\qquad\left. + w(||u_m||_0)\Big)\,||p_0||_{L^1}\right],\\ &\le \frac{A}{\Gamma(\alpha)}\left[\,||p_0p_1||_{L^1} + W + (r(||u_m||_0) + r(1) + w(||u_m||_0))\,||p_0||_{L^1}\right]. \end{aligned}$$

Also from Lemma 3.5, we obtain

$$|D^\delta u_m(t)| \le \frac{A}{\Gamma(\alpha-\delta)}\left[\,||p_0p_1||_{L^1} + W + (r(||u_m||_0) + r(1) + w(||u_m||_0))\,||p_0||_{L^1}\right].$$

Consequently,

$$||u_m||_0 \le \Upsilon\left[\,||p_0p_1||_{L^1} + W + (r(||u_m||_0) + r(1) + w(||u_m||_0))\,||p_0||_{L^1}\right].$$

Now from (H_3), since $\lim_{\vartheta\to\infty} \frac{r(\vartheta)+w(\vartheta)}{\vartheta} = 0$, then there exists an adequately large number $S > 0$ such that

$$\Upsilon \left[||p_0 p_1||_{L^1} + W + (r(\vartheta) + r(1) + w(\vartheta))||p_0||_{L^1} \right] < \vartheta \quad \text{for each} \quad \vartheta \geq S,$$

therefore,

$$||u_m||_0 < S \quad \text{for} \quad m \in N. \tag{4.11}$$

Hence, the sequences $\{u_m\}$ and $\{D^\delta u_m\}$ are uniformly bounded in $C[0,1]$.

Now, we demonstrate that the sequences $\{u_m\}$ and $\{D^\delta u_m\}$ are equicontinuous in $C[0,1]$. As in Lemma 4.1, let $0 \leq t_1 < t_2 \leq 1$, then

$$\begin{aligned}
|u_m(t_2) - u_m(t_1)| &= \left| \int_0^1 (G(t_2,s) - G(t_1,s)) f_m(t, u_m(t), D^\delta u_m(t)) \right|, \\
&= \Bigg| A\,(t_2^{\alpha-1} - t_1^{\alpha-1}) \left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_m(s, u_m(s), D^\delta u_m(s)) ds \right. \\
&\quad \left. -\lambda \int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\, f_m(s, u_m(s), D^\delta u_m(s)) ds \right] \\
&\quad + \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_m(s, u_m(s), D^\delta u_m(s)) ds \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_m(s, u_m(s), D^\delta u_m(s)) ds \Bigg|.
\end{aligned}$$

It follow from (4.10) and (4.11) that

$$\begin{aligned}
&|u_m(t_2) - u_m(t_1)| \\
&\leq A \left| t_2^{\alpha-1} - t_1^{\alpha-1} \right| \left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p_0(s) \left(p_1(s) + q(\frac{\rho s(1-s)^2}{2\Gamma(\alpha)}) + r(S) + r(1) + w(S) \right) ds \right. \\
&\quad \left. + \lambda \int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} p_0(s) \left(p_1(s) + q(\frac{\rho s(1-s)^2}{2\Gamma(\alpha)}) + r(S) + r(1) + w(S) \right) ds \right] \\
&\quad + \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} p_0(s) \left(p_1(s) + q(\frac{\rho s(1-s)^2}{2\Gamma(\alpha)}) + r(S) + r(1) + w(S) \right) ds \\
&\quad + \frac{(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)} \int_{t_1}^{t_2} p_0(s) \left(p_1(s) + q(\frac{\rho s(1-s)^2}{2\Gamma(\alpha)}) + r(S) + r(1) + w(S) \right) ds.
\end{aligned}$$

Also we have

$$\begin{aligned}
&|D^\delta u_m(t_2) - D^\delta u_m(t_1)| \\
&\leq \frac{A\,\Gamma(\alpha)\,|t_2^{\alpha-\delta-1} - t_1^{\alpha-\delta-1}|}{\Gamma(\alpha-\delta)} \left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\, f_m(t, u_m(t), D^\delta u_m(t)) ds \right. \\
&\quad \left. + \lambda \int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\, f_m(s, u_m(s), D^\delta u_m(s)) ds \right] \\
&\quad + \int_0^{t_1} \frac{(t_2-s)^{\alpha-\delta-1} - (t_1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}\, f_m(s, u_m(s), D^\delta u_m(s)) ds, \\
&\quad + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}\, f_m(s, u_m(s), D^\delta u_m(s)) ds \qquad \text{if} \quad \alpha-\delta > 1,
\end{aligned}$$

$$|D^\delta u_m(t_2) - D^\delta u_m(t_1)| \leq \int_{t_1}^{t_2} f_m(s, u_m(s), D^\delta u_m(s)) ds, \qquad \text{if} \quad \alpha-\delta = 1.$$

Hence

$$
\begin{aligned}
&|D^\delta u_m(t_2) - D^\delta u_m(t_1)| \\
&\le \frac{A\Gamma(\alpha)|t_2^{\alpha-\delta-1} - t_1^{\alpha-\delta-1}|}{\Gamma(\alpha-\delta)} \Bigg[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p_0(s) \\
&\quad \times \left(p_1(s) + q(\frac{\rho s(1-s)^2}{2\Gamma(\alpha)}) + r(S) + r(1) + w(S)\right) ds \\
&\quad + \lambda \int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} p_0(s) \left(p_1(s) + q(\frac{\rho s(1-s)^2}{2\Gamma(\alpha)}) + r(S) + r(1) + w(S)\right) ds\Bigg] \\
&\quad + \int_0^{t_1} \frac{(t_2-s)^{\alpha-\delta-1} - (t_1-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} p_0(s) \\
&\quad \times \left(p_1(s) + q(\frac{\rho s(1-s)^2}{2\Gamma(\alpha)}) + r(S) + r(1) + w(S)\right) ds \\
&\quad + \frac{(t_2-t_1)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)} \int_{t_1}^{t_2} p_0(s) \left(p_1(s) + q(\frac{\rho s(1-s)^2}{2\Gamma(\alpha)}) + r(S) + r(1) + w(S)\right) ds \\
&\qquad\qquad \text{if } \alpha - \delta > 1,
\end{aligned}
$$

and

$$
|D^\delta u_m(t_2) - D^\delta u_m(t_1)| \le \int_{t_1}^{t_2} p_0(s) \left(p_1(s) + q(\frac{\rho s(1-s)^2}{2\Gamma(\alpha)}) + r(S) + r(1) + w(S)\right) ds \quad \text{if } \alpha - \delta = 1.
$$

Now, choosing an arbitrary $\epsilon > 0$, there exists a positive number $\delta(\epsilon)$ such that when $|t_2 - t_1| < \delta$, then $|u_m(t_2) - u_m(t_1)| < \epsilon$ and $|D^\delta u_m(t_2) - D^\delta u_m(t_1)| < \epsilon$. Consequently, $\{u_m\}$ and $\{D^\delta u_m\}$ are equicontinuous in $C[0,1]$. Thus by Arzelà-Ascoli theorem $\{u_m\}$ and $\{D^\delta u_m\}$ are relatively compact in $\mathcal{E}$.

5 Positive solutions of SFDE with FIBCs (1.1)-(1.2)

Theorem 5.1. Let $(H_1) - (H_4)$ hold. Then SFDE (1.1) with FIBCs (1.2) has a multiple positive solution u and

$$
u(t) \ge \frac{\rho\, t\,(1-t)^2}{2\Gamma(\alpha)}, \quad t \in [0,1] \tag{5.1}
$$

Proof. Lemmas 4.2 and 4.3 guarantee that regular FDE (4.4) with FIBCs (1.2) has a positive solution $u_m,\ m \in N$ satisfying (4.9) and the sequences $\{u_m\}$ and $\{D^\delta u_m\}$ are relatively compact in $\mathcal{E}$. Thus there exist $u \in \mathcal{E}$ and a subsequence $\{u_{n_m}\}$ of $\{u_m\}$ such that $\lim_{m\to\infty} u_{n_m} = u$ in $\mathcal{E}$. Consequently, $u \in \mathcal{Q}$, u fulfills (5.1) and

$$
\lim_{m\to\infty} f_{n_m}(t, u_{n_m}(t), D^\delta u_{n_m}(t)) = f(t, u(t), D^\delta u(t)), \quad a.e.\ t \in [0,1].
$$

For a.e. $s \in [0,1]$ and all $t \in [0,1]$, $m \in N$, using inequalities (4.3), (4.9), (4.11) and Lemma 3.3(iii), we have

$$
0 \le G(t,s)\, f_m(s, u_m(s), D^\delta u_m(s)) \le \frac{A}{\Gamma(\alpha)} p_0(s)\,(p_1(s) + q(\frac{\rho\, s\,(1-s)^2}{2\Gamma(\alpha)}) + r(S) + r(1) + w(S))
$$

holds. Hence by the Lebesgue dominated convergence theorem

$$
\lim_{m\to\infty} \int_0^1 G(t,s) f_{n_m}(s, u_{n_m}(s), D^\delta u_{n_m}(s))\, ds = \int_0^1 G(t,s) f(s, u(s), D^\delta u(s)) ds.
$$

Since

$$
u_{n_m}(t) = \int_0^1 G(t,s)\, f_{n_m}(s, u_{n_m}(s), D^\delta u_{n_m}(s)) ds, \quad t \in [0,1],
$$

as $m \to \infty$ in the last equality, at that point we have

$$u(t) = \int_0^1 G(t,s)\, f(s, u(s), D^{\delta} u(s)) ds, \quad t \in [0,1].$$

Consequently, u is a positive solution of SFDE (1.1) with FIBCs (1.2).

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