Journal of the Egyptian Mathematical Society
Volume (26) - Issue (3) - 2018
ISSN:1110-256X
DOI:10.21608/joems.2018.2884.1043

SOME FAMILIES OF MEROMORPHIC $p-$ VALENT FUNCTIONS INVOLVINGA NEW OPERATOR DEFINED BY GENERALIZED MITTAG-LEFFLER FUNCTION

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Receive 10/2/2018, Revised , Accepted 21/5/2018

Abstract. In this paper, we introduce some classes of meromorphic multivalent functions which are defined by a new linear operator containing the generalized Mittag-Leffler function. Using convolution results obtained by Sarkar et al . [1] (see also Mostafa and Aouf [2]), we investigate some properties and inclusion relations for functions in these classes.
2010 Mathematics Subject Classification. 30C45, 30C80, 33E12.
Key Words and Phrases. subordination, meromorphic $p-$ valent functions, Mittag-Leffler function, convolution, linear operator.

1 Introduction

Let $\sum_p$ be the class of functions of the form:

$$f(z)=z^{-p}+\sum_{n=1}^{\infty}a_n z^{n-p} \quad (p\in\mathbb{N}=\{1,2,...\}), \tag{1.1}$$

which are analytic and $p-$valent in the punctured unit disk $\mathbb{U}^*=\mathbb{U}\backslash\{0\}$, where $\mathbb{U}=\{z:z\in\mathbb{C},|z|<1\}$. If f and g are analytic functions in $\mathbb{U}$, we say that f is subordinate to g, written $f\prec g$ if there exists a Schwarz function w, which (by definition) is analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1$ for all $z\in\mathbb{U}$, such that $f(z)=g(w(z)),\ z\in\mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence (see [3] and [4]):

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0) \text{ and } f(U)\subset g(U).$$

For functions f given by (1.1) and $g\in\sum_p$ given by

$$g(z)=z^{-p}+\sum_{n=1}^{\infty}b_n z^{n-p}, \tag{1.2}$$

the Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z)=z^{-p}+\sum_{n=1}^{\infty}a_n b_n z^{n-p}=(g*f)(z).$$

The Mittag-Leffler function $E_\alpha(z)\,(z\in\mathbb{C})$ is defined by (see [5] and [6]):

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}\,(\alpha\in\mathbb{C},\Re\{\alpha\}>0). \tag{1.3}$$

Srivastava and Tomovski [7] introduced the generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,k}(z)\,(z\in\mathbb{C})$ in the form:

$$E_{\alpha,\beta}^{\gamma,k}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{nk}\ z^n}{\Gamma(\alpha n+\beta)\ n!} \tag{1.4}$$

where $\beta, \gamma \in \mathbb{C}$, $\Re\{\alpha\} > \max\{0; \Re\{k\} - 1\}; \Re\{k\} > 0$, $\Re\{\alpha\} = 0$ when $\Re\{k\} = 1$ with $\beta \neq 0$ and $(v)_m$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(\nu)_m := \begin{cases} 1, & \text{if} \quad m = 0, \\ \nu(\nu+1)(\nu+2)\dots(\nu+m-1), & \text{if} \quad m \in \mathbb{N}. \end{cases}$$

Srivastava and Tomovski [7] proved that the function $E_{\alpha,\beta}^{\gamma,k}(z)$ defined by (1.4) is an entire function in the complex $z-$plane .

We now define the function $\mathcal{B}_{p,\alpha,\beta}^{\gamma,k}(z)$ by

$$\mathcal{B}_{p,\alpha,\beta}^{\gamma,k}(z) = z^{-p}\Gamma(\beta) E_{\alpha,\beta}^{\gamma,k}(z) \tag{1.5}$$

Corresponding to the function $\mathcal{B}_{p,\alpha,\beta}^{\gamma,k}(z)$ defined by (1.5), we introduce a linear operator $\mathcal{T}_{p,\alpha,\beta}^{\gamma,k} f(z) : \sum_p \to \sum_p$ by

$$\mathcal{T}_{p,\alpha,\beta}^{\gamma,k} f(z) = \mathcal{B}_{p,\alpha,\beta}^{\gamma,k}(z) * f(z) = z^{-p} + \sum_{n=1}^{\infty} \frac{\Gamma(\gamma + nk)\ \Gamma(\beta)}{\Gamma(\gamma)\Gamma(\beta + \alpha n)\ n!} a_n z^{n-p} \tag{1.6}$$

$$(\Re\{\alpha\} > \max\{0; \Re\{k\} - 1\}; \Re\{k\} > 0, \Re\{\alpha\} = 0 \ \textit{when}\ \Re\{k\} = 1 \ \textit{with}\ \beta \neq 0).$$

We note that

$$\mathcal{T}_{p,0,\beta}^{1,1} f(z) = f(z) \quad \text{and} \quad \mathcal{T}_{p,0,\beta}^{2,1} f(z) = zf'(z) + (p+1) f(z)$$

Also, it is easily verified from (1.6) that

$$k\ z\left(\mathcal{T}_{p,\alpha,\beta}^{\gamma,k} f(z)\right)' = \gamma\ \mathcal{T}_{p,\alpha,\beta}^{\gamma+1,k} f(z) - (\gamma + pk)\ \mathcal{T}_{p,\alpha,\beta}^{\gamma,k} f(z) \tag{1.7}$$

and

$$\alpha\ z\left(\mathcal{T}_{p,\alpha,\beta+1}^{\gamma,k} f(z)\right)' = \beta\ \mathcal{T}_{p,\alpha,\beta}^{\gamma,k} f(z) - (\beta + p\alpha)\ \mathcal{T}_{p,\alpha,\beta+1}^{\gamma,k} f(z). \tag{1.8}$$

For $-1 \leq A < B \leq 1$ and $z \in \mathbb{U}^*$, Mogra [8] defined the following subclass of $\sum_p$ as follows

$$\sum S_p[A,B] = \left\{ f \in \sum\nolimits_p : -\frac{zf'(z)}{f(z)} \prec p\frac{1+Az}{1+Bz}, z \in \mathbb{U} \right\} \tag{1.9}$$

and Srivastava et al. [9] defined the following subclass of $\sum_p$ as follows

$$\sum K_p[A,B] = \left\{ f \in \sum\nolimits_p : -\left[1 + \frac{zf''(z)}{f'(z)}\right] \prec p\frac{1+Az}{1+Bz}, z \in \mathbb{U} \right\}. \tag{1.10}$$

It is clear that

$$f(z) \in \sum K_p[A,B] \Leftrightarrow -\frac{zf'(z)}{p} \in \sum S_p[A,B]. \tag{1.11}$$

We note that

(i) $\sum S_1[2\alpha - 1, 1] = \sum S^*(\alpha)$ $(0 \leq \alpha < 1)$ (Juneja and Reddy [10]),
(ii) $\sum K_1[2\alpha - 1, 1] = \sum K(\alpha)$ $(0 \leq \alpha < 1)$ (Srivastava et al. [9]),
(iii) $\sum S_p[\frac{2\alpha}{p} - 1, 1] = \sum S_p^*(\alpha)$ $(0 \leq \alpha < p)$ (Aouf and Hossen [11]),
(iv) $\sum K_p[\frac{2\alpha}{p} - 1, 1] = \sum K_p(\alpha)$ $(0 \leq \alpha < p)$ (Aouf and Srivastava [12]).

Using the operator $\mathcal{T}_{p,\alpha,\beta}^{\gamma,k}$, we define the classes $\sum S_{p,\alpha,\beta}^{\gamma,k}(A,B)$ and $\sum K_{p,\alpha,\beta}^{\gamma,k}(A,B)$ as follows:

$$\sum S_{p,\alpha,\beta}^{\gamma,k}(A,B) = \left\{ f \in \sum\nolimits_p : \mathcal{T}_{p,\alpha,\beta}^{\gamma,k} f(z) \in \sum S_p[A,B], z \in \mathbb{U}^* \right\}, \tag{1.12}$$

and

$$\sum K_{p,\alpha,\beta}^{\gamma,k}(A,B)=\left\{f\in\sum\nolimits_p:\mathcal{T}_{p,\alpha,\beta}^{\gamma,k}f(z)\in\sum K_p[A,B],z\in\mathbb{U}^*\right\}. \tag{1.13}$$

We notice that

$$f\in\sum K_{p,\alpha,\beta}^{\gamma,k}(A,B)\Leftrightarrow-\frac{zf'}{p}\in\sum S_{p,\alpha,\beta}^{\gamma,k}(A,B). \tag{1.14}$$

To prove our results we need the following lemmas.

Lemma 1.1. *[1, 2] The function $f(z)$ defined by (1.1) is in the class $\sum S_p[A,B]$ if and only if*

$$z^p\left[f(z)*\frac{1+(D-1)z}{z^p(1-z)^2}\right]\neq 0, \tag{1.15}$$

where

$$D=\frac{e^{-i\theta}+B}{p(A-B)}. \tag{1.16}$$

Lemma 1.2. *[1, 2] The function $f(z)$ defined by (1.1) is in the class $\sum K_p[A,B]$ if and only if*

$$z^p\left\{f(z)*\left[\frac{p-\{2+p-(p-1)(D-1)\}z-(p+1)(D-1)z^2}{pz^p(1-z)^3}\right]\right\}\neq 0.$$

Lemma 1.3. *[13] Let h be convex (univalent) in $\mathbb{U}$, with $Re[\delta h(z)+\eta]>0$ for all $z\in\mathbb{U}$. If g is analytic in $\mathbb{U}$, with $g(0)=h(0)$, then*

$$g(z)+\frac{zg'(z)}{\delta g(z)+\eta}\prec h(z)\Rightarrow g(z)\prec h(z). \tag{1.17}$$

In this paper, we obtain some interesting results for the families $\sum S_{p,\alpha,\beta}^{\gamma,k}(A,B)$ and $\sum K_{p,\alpha,\beta}^{\gamma,k}(A,B)$ of meromorphic $p-$valent functions defined by the operator $\mathcal{T}_{p,\alpha,\beta}^{\gamma,k}$.

2 Main results

Unless otherwise mentioned, we shall assume in this paper that $\Re\{\alpha\}>\max\{0;\Re\{k\}-1\}$, $\Re\{k\}>0$, $\Re\{\alpha\}=0$ when $\Re\{k\}=1$ with $\beta\neq 0$, $-1\leq A<B\leq 1$, $0<\theta<2\pi$, $p\in\mathbb{N}$ and $z\in\mathbb{U}^*$.

Theorem 2.1. *The function $f(z)$ defined by (1.1) is in the class $\sum S_{p,\alpha,\beta}^{\gamma,k}(A,B)$ if and only if*

$$1+\sum_{n=1}^{\infty}\left[\frac{ne^{-i\theta}+pA+(n-p)B}{p(A-B)}\right]\frac{\Gamma(\gamma+nk)\ \Gamma(\beta)}{\Gamma(\gamma)\Gamma(\beta+\alpha n)\ \ n!}a_nz^n\neq 0. \tag{2.1}$$

Proof. From Lemma 1.1, we find that $f\in\sum S_{p,\alpha,\beta}^{\gamma,k}(A,B)$ if and only if

$$z^p\left[\mathcal{T}_{p,\alpha,\beta}^{\gamma,k}f(z)*\frac{1+(D-1)z}{z^p(1-z)^2}\right]\neq 0.$$

Expanding $\frac{1+(D-1)z}{z^p(1-z)^2}$, we have (2.1) which completes the proof of Theorem 2.1. □

Theorem 2.2. *The function $f(z)$ defined by (1.1) is in the class $\sum K_{p,\alpha,\beta}^{\gamma,k}(A,B)$ if and only if*

$$1-\sum_{n=1}^{\infty}\left[\frac{n\left[ne^{-i\theta}+pA+(n-p)B\right]}{p^2(A-B)}\right]\frac{\Gamma(\gamma+nk)\ \Gamma(\beta)}{\Gamma(\gamma)\Gamma(\beta+\alpha n)\ \ n!}a_nz^n\neq 0. \tag{2.2}$$

Proof. From Lemma 1.2, we find that $f \in \sum K^{\gamma,k}_{p,\alpha,\beta}(A,B)$ if and only if

$$z^p\left\{\mathcal{T}^{\gamma,k}_{p,\alpha,\beta}f(z) * \left[\frac{p-\{2+p-(p-1)(D-1)\}z-(p+1)(D-1)z^2}{pz^p(1-z)^3}\right]\right\} \neq 0. \tag{2.3}$$

Now it can be easily shown that

$$z^{-p}(1-z)^{-3} = z^{-p} + \sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{2}z^{n-p}, \tag{2.4}$$

$$z^{1-p}(1-z)^{-3} = \sum_{n=1}^{\infty}\frac{n(n+1)}{2}z^{n-p}, \tag{2.5}$$

$$z^{2-p}(1-z)^{-3} = \sum_{n=1}^{\infty}\frac{n(n-1)}{2}z^{n-p}. \tag{2.6}$$

Using (2.4)-(2.6) and (1.16) in (2.3), we have the desired result which completes the proof of Theorem 2.2. □

Theorem 2.3. *If the function $f(z)$ defined by (1.1) belongs to the class $\sum S^{\gamma,k}_{p,\alpha,\beta}(A,B)$, then*

$$\sum_{n=1}^{\infty}[n+pA+(n-p)B]\left|\frac{\Gamma(\gamma+nk)\ \Gamma(\beta)}{\Gamma(\gamma)\Gamma(\beta+\alpha n)\,n!}a_n\right| \leq p(A-B). \tag{2.7}$$

Proof. Since

$$\left|\frac{ne^{-i\theta}+pA+(n-p)B}{p(A-B)}\right| = \frac{\left|ne^{-i\theta}+pA+(n-p)B\right|}{p(A-B)} \leq \frac{n+pA+(n-p)B}{p(A-B)}$$

and

$$\left|1+\sum_{n=1}^{\infty}\frac{\left[ne^{-i\theta}+pA+(n-p)B\right]}{p(A-B)}\frac{\Gamma(\gamma+nk)\ \Gamma(\beta)}{\Gamma(\gamma)\Gamma(\beta+\alpha n)\ n!}|a_nz^n|\right|$$

$$> 1-\sum_{n=1}^{\infty}\left|\frac{\left[ne^{-i\theta}+pA+(n-p)B\right]}{p(A-B)}\right|\left|\frac{\Gamma(\gamma+nk)\ \Gamma(\beta)}{\Gamma(\gamma)\Gamma(\beta+\alpha n)\ n!}a_k\right|.$$

The result follows from Theorem 2.1. □

Using the same technique, we can also prove the following theorem.

Theorem 2.4. *If the function $f(z)$ defined by (1.1) belongs to the class $\sum K^{\gamma,k}_{p,\alpha,\beta}(A,B)$, then*

$$\sum_{n=1}^{\infty}n\,[n+pA+(n-p)B]\left|\frac{\Gamma(\gamma+nk)\ \Gamma(\beta)}{\Gamma(\gamma)\Gamma(\beta+\alpha n)\ n!}a_k\right| \leq p^2(A-B). \tag{2.8}$$

Theorem 2.5. *Let the function $f(z)$ be defined by (1.1). If*

$$\frac{1+AB+(A+B)\cos\theta}{1+B^2+2B\cos\theta} \leq \frac{\gamma}{k}+p \tag{2.9}$$

and $f \in \sum S^{\gamma+1,k}_{p,\alpha,\beta}(A,B)$ with $\gamma,k>0$ and $\mathcal{T}^{\gamma,k}_{p,\alpha,\beta}f(z) \neq 0$, then $f \in \sum S^{\gamma,k}_{p,\alpha,\beta}(A,B)$.

Proof. Let $f \in \sum S^{\gamma+1,k}_{p,\alpha,\beta}(A,B)$ and define the function

$$g(z) = -\frac{z\left(\mathcal{T}^{\gamma,k}_{p,\alpha,\beta}f(z)\right)'}{\mathcal{T}^{\gamma,k}_{p,\alpha,\beta}f(z)}, \tag{2.10}$$

we see that $g(z)$ is analytic in $\mathbb{U}$ with $g(0)=1$. Using the identity (1.7) in (2.10) we have

$$\frac{\gamma}{k}\frac{\mathcal{T}_{p,\alpha,\beta}^{\gamma+1,k}f(z)}{\mathcal{T}_{p,\alpha,\beta}^{\gamma,k}f(z)}=-g(z)+\frac{\gamma}{k}+p. \tag{2.11}$$

Differentiating (2.11) logarithmically and using (2.10), we have

$$-\frac{z\left(\mathcal{T}_{p,\alpha,\beta}^{\gamma+1,k}f(z)\right)'}{\mathcal{T}_{p,\alpha,\beta}^{\gamma+1,k}f(z)}=g(z)+\frac{zg'(z)}{-g(z)+\frac{\gamma}{k}+p}\prec\frac{1+Az}{1+Bz}=h(z). \tag{2.12}$$

Simple computations show that the inequality $\Re\{-h(z)+\frac{\gamma}{k}+p\}>0$ can be written in the form

$$\Re\frac{1+Az}{1+Bz}<\frac{\gamma}{k}+p,$$

which is equivalent to (2.9). Since the function $h(z)$ is a convex function, then applying Lemma 1.3, we see that the subordination (2.12) implies $g(z)\prec h(z)$. This completes the proof of Theorem 2.5. □

Theorem 2.5 yields the following theorem.

Theorem 2.6. *Let the function $f(z)$ be defined by (1.1). If (2.9) holds and $f\in\sum K_{p,\alpha,\beta}^{\gamma+1,k}(A,B)$ with $\gamma,k>0$ and $\mathcal{T}_{p,\alpha,\beta}^{\gamma,k}f(z)\neq 0$, then $f\in\sum K_{p,\alpha,\beta}^{\gamma,k}(A,B)$.*

Proof. Let $f\in\sum K_{p,\alpha,\beta}^{\gamma,k}(A,B)$ we have

$$\begin{aligned} f\in\sum K_{p,\alpha,\beta}^{\gamma,k}(A,B) \quad &\Leftrightarrow \quad -\tfrac{zf'}{p}\in\sum S_{p,\alpha,\beta}^{\gamma,k}(A,B) \qquad \text{(From (1.14))}\\ &\Rightarrow \quad -\tfrac{zf'}{p}\in\sum S_{p,\alpha,\beta}^{\gamma+1,k}(A,B) \quad \text{(by Theorem 2.5)}\\ &\Leftrightarrow \quad f\in\sum K_{p,\alpha,\beta}^{\gamma+1,k}(A,B)\end{aligned}$$

this proves Theorem 2.6 □

Similarly we can prove the following theorems

Theorem 2.7. *Let the function $f(z)$ be defined by (1.1). If*

$$\frac{1+AB+(A+B)\cos\theta}{1+B^2+2B\cos\theta}\leq\frac{\beta}{\alpha}+p \tag{2.13}$$

and $f\in\sum S_{p,\alpha,\beta}^{\gamma,k}(A,B)$ with $\alpha,\beta>0$ and $\mathcal{T}_{p,\alpha,\beta+1}^{\gamma,k}f(z)\neq 0$, then $f\in\sum S_{p,\alpha,\beta+1}^{\gamma,k}(A,B)$.

Theorem 2.8. *Let the function $f(z)$ be defined by (1.1). If (2.13) holds and $f\in\sum K_{p,\alpha,\beta}^{\gamma,k}(A,B)$ with $\alpha,\beta>0$ and $\mathcal{T}_{p,\alpha,\beta+1}^{\gamma,k}f(z)\neq 0$, then $f\in\sum K_{p,\alpha,\beta+1}^{\gamma,k}(A,B)$.*

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