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A NEW GENERALIZED $\mathbb{S}$-ALGORITHM VIA ADMISSIBLE APPROACH TO COMMON FIXED POINTS OF GENERAL-TYPE CONTRACTION MAPPINGS

Ali Mohamed Saddeek[1] and Sayed Attia Ahmed[1,2] *

[1]Department of Mathematics Faculty of Science, Assiut University, Assiut, Egypt

[2]Umm Al Qura University, Al-Jamom University College, Makka, Saudi Arabia

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Abstract. In this paper, a new generalized $\mathbb{S}$-algorithm in terms of admissible and generalized admissible mappings for approximating common fixed points of three mappings satisfying general contraction type conditions is introduced and its strong convergence is proved in uniformly convex Banach spaces. The results improve and generalize the main results of several authors.

Keywords: Generalized $\mathbb{S}$-algorithm; Generalized weakly contraction mapping; Common fixed point; Uniformly convex Banach space

AMS 2000 Subject Classification: 41A65; 39B12; 37C25; 47H10; 65Q30

1 Introduction

Let X be a Banach space and C a nonempty convex subset of X, and let R, S, T be self mappings on C. The sequence $\{x_n\}_{n=0}^{\infty}$ in C defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n^{(1)} R x_n + \alpha_n^{(2)} S x_n + \alpha_n^{(3)} T y_n, \\ y_n = \beta_n^{(1)} R x_n + \beta_n^{(2)} S x_n + \beta_n^{(3)} T x_n, \quad n \geq 0, \end{cases} \tag{1}$$

is called the Ishikawa iteration process for R, S, and T, in the sense of Ghosh and Debnath [1], where, (i) $0 < a \leq \alpha_n^{(i)} \leq b < 1$, $n \geq 0$ and $\sum_{i=1}^{3} \alpha_n^{(i)} = 1$; (ii) $0 \leq \beta_n^{(i)} \leq \beta < 1$, and $\sum_{i=1}^{3} \beta_n^{(i)} = 1$; (iii) $\limsup_{n\to\infty} \beta_n^{(i)} \leq 1$.
The sequence $\{x_n\}_{n=0}^{\infty}$ in C defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n^{(1)} R x_n + \alpha_n^{(2)} S x_n + \alpha_n^{(3)} T x_n, \ n \geq 0, \end{cases} \tag{2}$$

is called the Mann iteration process for R, S and T in the light of [1], where, $\{\alpha_n^{(i)}\}_{n=0}^{\infty}$, $i = 1, 2, 3$ satisfies the condition (i). It is clear that the standard Ishikawa and Mann iteration processes (see, for example, [2-3]) are special cases (i.e., $R = S = I$, the identity mapping on C) of (1.1) and (1.2), respectively.
It should be noted that unless $R = S = I$, process (1.1) will not reduce to (1.2) when $\beta_n^{(i)} = 0$ for all $n \geq 0$.

*s_a_ahmed2003@yahoo.com, corresponding author

If $R = S$, then the iteration processes (1.1) and (1.2) reduce to the following iteration processes (see, Huang and Jeng [4]):

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n^{(3)})Sx_n + \alpha_n^{(3)}Ty_n, \\ y_n = (1-\beta_n^{(3)})Sx_n + \beta_n^{(3)}Tx_n, \quad n \geq 0, \end{cases} \tag{3}$$

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n^{(3)})Sx_n + \alpha_n^{(3)}Tx_n, \ n \geq 0. \end{cases} \tag{4}$$

In [5], Agarwal et al. introduced the following iteration ($\mathbb{S}$-iteration) process:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n^{(3)})Tx_n + \alpha_n^{(3)}Ty_n, \\ y_n = (1-\beta_n^{(3)})x_n + \beta_n^{(3)}Tx_n, \quad n \geq 0, \end{cases} \tag{5}$$

where, $\{\alpha_n^{(3)}\}_{n=0}^{\infty}$ and $\{\beta_n^{(3)}\}_{n=0}^{\infty}$ are real sequences in $(0,1)$.

If $\beta_n^{(3)} = 0$, $\forall n \geq 0$, then the $\mathbb{S}$-iteration process (1.5) reduces to the well known Picard iteration process.

Inspired and motivated by the iteration processes (1.1) and (1.2), we now introduce the following modified iteration process for R, S and T:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n^{(1)}Rx_n + \alpha_n^{(2)}Sx_n + \alpha_n^{(3)}Ty_n, \\ y_n = (1-\beta_n^{(3)})x_n + \beta_n^{(3)}Tx_n, \quad n \geq 0, \end{cases} \tag{6}$$

where, $0 \leq \beta_n^{(3)} \leq \beta < 1$, $\forall n \geq 0$, $\limsup_{n\to\infty} \beta_n^{(3)} \leq 1$, and $\{\alpha_n^{(i)}\}_{n=0}^{\infty}$, $i = 1, 2, 3$ satisfies the condition (i).

The iteration process (1.2) is a special case of the iteration process (1.6) when $\beta_n^{(3)} = 0$, $\forall n \geq 0$.

If we set $R = S = T$, then the iteration process (1.6) reduces to the iteration process (1.5).

The standard Ishikawa (resp., Mann) iteration process is a special case of (1.6) when $R = S = I$ (resp., $R = S = I$, $\beta_n^{(3)} = 0$, $\forall n \geq 0$).

Given $x, y \in C$, let

$$M_T^3(x,y) = \{\|x-y\|, \ \frac{[\|x-Tx\| + \|y-Ty\|]}{2}, \ \frac{[\|x-Ty\| + \|y-Tx\|]}{2}\},$$

$$N_S^2(x,y) = \{\|Sx-y\|, \ \|x-Sy\|\},$$

$$P_{R,S}^2(x,y) = \{\|Sx-y\|, \ \|x-Ry\ |\}.$$

Suppose that $m(x,y)$ is the maximum of $M_T^3(x,y)$ and $n(x,y)$ (resp., $p(x,y)$ is the minimum of $N_S^2(x,y)$ (resp., $P_{R,S}^2(x,y)$).

Let us now consider the following contraction conditions: For all $x, y \in C$,

$$\|Tx - Ty\| \leq m(x,y), \tag{7}$$

$$\|Sx - Sy\| \leq n(x,y), \tag{8}$$

$$\|Sx - Ry\| \leq p(x,y). \tag{9}$$

The set of fixed points of T is denoted by $\mathfrak{F}(T)$.

In [6, Theorem 2], Rhoades established the following convergence theorem which in turn is a generalization of theorem 4.1

of Ganguly and Bandyopadhyay [7]:

Theorem 1.1 ([6]) Let T be a self mapping on a nonempty closed convex subset C of a uniformly convex Banach space X satisfying (1.7) and such that $T(C)$ is relatively compact. If $\mathfrak{F}(T) \neq \phi$, then the Mann iteration $\{x_n\}_{n=0}^{\infty}$ given by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1 - \alpha_n^{(3)})x_n + \alpha_n^{(3)} T x_n, \ n \geq 0, \end{cases} \tag{10}$$

with $0 < a \leq \alpha_n^{(3)} \leq 1$, $\forall n \geq 0$, converges to a fixed point of T.

In [8], Osilike extended this result to the Ishikawa iteration process; see also Tiwary and Debnath [9].

In [4], Huang and Jeng remedied and generalized some results of Rhoades [6], Osilike [8], and Tiwary and Debenath [9] and gave the following generalization:

Theorem 1.2 ([4]) Let S and T be two self mappings on a closed convex subset C of a uniformly convex Banach space X such that T satisfies condition (1.7) and S satisfies condition (1.8). Suppose one of $S(C)$ and $T(C)$ is relatively compact. Let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.3), where $\{\alpha_n^{(3)}\}_{n=0}^{\infty}$ and $\{\beta_n^{(3)}\}_{n=0}^{\infty}$ are two sequences in [0,1] satisfying the conditions $0 < a \leq \alpha_n^{(3)} \leq b < 1$, $0 \leq \beta_n^{(3)} \leq \beta < 1$, $\forall n \geq 0$ and $\limsup_{n\to\infty} \beta_n^{(3)} < 1$. If $\mathfrak{F}(S) \cap \mathfrak{F}(T) \neq \phi$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to some common fixed point of S and T.

In [10], Rashwan and Saddeek obtained the following generalization of Huang and Jeng's result [4]:

Theorem 1.3 ([10]) Let R, S and T be self mappings on a closed convex subset C of a uniformly convex Banach space X such that T satisfies condition (1.7) and R, S satisfying the condition (1.9). Suppose one of $R(C)$, $S(C)$ and $T(C)$ is relatively compact. Let $\{x_n\}_{n=0}^{\infty}$ be defined by (1.1) with (i), (ii) and (iii). If $\mathfrak{F}(R) \cap \mathfrak{F}(S) \cap \mathfrak{F}(T) \neq \phi$, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to some common fixed point of R, S and T, provided that

$$\lim_{n\to\infty} \|Rx_n - Sx_n\| = 0. \tag{11}$$

Let $\mathbb{R}^+ = [0, \infty)$. A real valued function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a comparison function if it is monotone increasing and $\lim_{n\to\infty} \varphi^n(t) = 0$, $\forall\ t \geq 0$. It is known that if φ is a comparison function, then $\varphi(t) < t$, $\forall\ t > 0$ and φ is continuous at zero. For examples and further details on such functions, we refer to Rus [11] and Berinde [12].

Let us suppose that there exists a comparison function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|Tx - Ty\| \leq \varphi(m(x, y)), \ \forall x, y \in C, \tag{12}$$

where $m(x, y)$ is defined as above.

This class of mappings is called generalized weakly contraction. (see, Hammache et al. [13]). **Remark 1.1** Obviously, (1.12) reduces to (1.7), when $\varphi(t) = t$.

In [14], Rus introduced the concepts of admissible mapping and admissible perturbation mapping as follows:

Definition 1.1 Let V_1 be a nonempty set and let $G : V_1 \times V_1 \to V_1$ be a mapping. Then G is said to be admissible if the following conditions hold:

(A_1) $G(x, x) = x, \ \forall x \in V_1$;

(A_2) $G(x, y) = x \Rightarrow x = y, \forall x, y \in V_1$.

Definition 1.2 Let V_1 be a nonempty set, $T : V_1 \to V_1$ be a given mapping and $G : V_1 \times V_1 \to V_1$ be an admissible mapping. The admissible perturbation of T, denoted by T_G, is defined as

$$T_G(x) = G(x, Tx), \ \forall x \in V_1. \tag{13}$$

Note that, T_G maps V_1 into itself.

Remark 1.2 The admissible perturbation mapping $T_G : V_1 \to V_1$ of the given mapping $T : V_1 \to V_1$ has the property that

$\mathfrak{F}(T_G) = \mathfrak{F}(T)$.

Example 1.2 (see, [15]) Let V_2 be a real vector space, V_1 a convex subset of V_2, T a self mapping of V_1 into itself and $G_n^1 : V_1 \times V_1 \to V_1$. Then the mapping G_n^1 defined by

$$G_n^1(T(\cdot), T(\diamond)) = (1 - \alpha_n^3)T(\cdot) + \alpha_n^3 T(\diamond), \quad \forall n \geq 0,$$

where $\alpha_n^3 \in (0, 1]$, is an admissible perturbation of T.

Example 1.3 (see, [15]) Let V_1, V_2, T be as in example 1.2 and let $G_n^2 : V_1 \times V_1 \to V_1$. Then the mapping $T_n : V_1 \to V_1$ defined by

$$T_n(\cdot) = T_{G_n^2}(\cdot) = G_n^2(\cdot, T(\cdot)) = (1 - \beta_n^{(3)})I(\cdot) + \beta_n^{(3)} T(\cdot), \quad \forall n \geq 0,$$

is an admissible perturbation of T and $\mathfrak{F}(T_n) = \mathfrak{F}(T)$, $\forall\ \beta_n^{(3)} \in (0, 1]$.

Such a perturbation T_n is called the Mann perturbation of T.

For other interesting examples of such mappings, we can refer to Rus [14], Bunlue et al. [15] and Berinde et al. [16].

In terms of admissible mappings, Bunlue et al. [15] introduced the following representation of iteration algorithms:

Algorithm 1.1 (GM-algorithm) Let $T : V_2 \to V_2$ be a nonlinear mapping and G_n^2 be an admissible mapping from $V_2 \times V_2$ to V_2 for $n \geq 0$. The Mann algorithm corresponding to G_n^2 (GM-algorithm) is defined by, $x_0 \in V_2$,

$$x_{n+1} = G_n^2(x_n, Tx_n), \ \forall n \geq 0. \tag{14}$$

For V_1 a nonempty convex subset of a real Banach space V_2, T a nonlinear mapping of V_1 into itself and $G_n^2(x_n, Tx_n) = (1 - \beta_n^{(3)})x_n + \beta_n^{(3)} Tx_n, \quad n \geq 0$, where $\beta_n^{(3)} \in (0, 1]$, and $\{x_n\}_{n=0}^{\infty} \subset V_1$, we obtain the standard Mann iteration algorithm.

Algorithm 1.2 (G$\mathbb{S}$-algorithm) Let $T : V_2 \to V_2$ be a nonlinear mapping and G_n^1 and G_n^2 be admissible mappings from $V_2 \times V_2$ to V_2 for $n \geq 0$. The $\mathbb{S}$-algorithm corresponding to G_n^1 and G_n^2 (G$\mathbb{S}$-algorithm) is defined by, $x_0 \in V_2$,

$$\begin{cases} x_{n+1} = G_n^1(Tx_n, Ty_n), \\ y_n = G_n^2(x_n, Tx_n), \ n \geq 0, \end{cases} \tag{15}$$

For V_1 a nonempty convex subset of a real Banach space V_2, T a nonlinear mapping of V_1 into itself,

$$G_n^2(x_n, Tx_n) = (1 - \beta_n^{(3)})x_n + \beta_n^{(3)} Tx_n,$$

and

$$G_n^1(Tx_n, T(G_n^2(x_n, Tx_n)) = (1 - \alpha_n^{(3)})Tx_n + \alpha_n^{(3)} T(G_n^2(x_n, Tx_n)), \ n \geq 0,$$

where $\{\alpha_n^{(3)}\}_{n=0}^{\infty}$ and $\{\beta_n^{(3)}\}_{n=0}^{\infty}$ are real sequences in $[0, 1)$ and $\{x_n\}_{n=0}^{\infty} \subset V_1$, we obtain the $\mathbb{S}$-iteration process defined by (1.5).

We shall now present the following generalized definitions of admissible and admissible perturbations mappings:

Definition 1.3 Let $G' : V_1 \times V_1 \times V_1 \to V_1$ be a mapping. We say that G' is generalized admissible if the following conditions hold:

(A_1') $G'(x, x, x) = x, \ \forall x \in V_1$;

(A_2') $G'(x, y, z) = x \Rightarrow x = y = z, \forall x, y, z \in V_1$.

Definition 1.4 Let $G' : V_1 \times V_1 \times V_1 \to V_1$ be generalized admissible mapping and let R, S, T be nonlinear mappings from V_1 into itself. The generalized admissible perturbation of R, S and T, denoted by $T_{G'}^{R,S}$, is defined by

$$T_{G'}^{R,S}(x, y) = G'(Rx, Sx, Ty), \forall x, y \in V_1. \tag{16}$$

Note that $T_{G'}^{R,S}$ maps $V_1 \times V_1 \times V_1$ into V_1.

Example 1.4 Let V_1, V_2 be as in example 1.2 and R, S and T be nonlinear mappings from V_1 into itself. Then the mapping $G_n^{'} : V_1 \times V_1 \times V_1 \to V_1$, $n \geq 0$ defined by

$$G_n^{'}(R(\cdot), S(\cdot), T(\diamond)) = \alpha_n^{(1)} R(\cdot) + \alpha_n^{(2)} S(\cdot) + \alpha_n^{(3)} T(\diamond), \quad \forall n \geq 0,$$

where $\alpha_n^{(3)} \in [0, 1)$, is a generalized admissible perturbation of R, S and T.

Now, let us generalize the (G$\mathbb{S}$-algorithm) for three mappings as follows:

Algorithm 1.3 (Generalized $G^{'}\mathbb{S}$-algorithm) Let R, S and T be nonlinear mappings from V_2 into itself, $G_n^{'1} : V_2 \times V_2 \times V_2 \to V_2$ be a generalized admissible mapping and $G_n^{'2}$ be an admissible mapping from $V_2 \times V_2$ to V_2 for $n \geq 0$. The generalized $\mathbb{S}$-algorithm corresponding to $G_n^{'1}$ and $G_n^{'2}$ (generalized $G^{'}\mathbb{S}$-algorithm) is defined by, $x_0 \in V_2$,

$$\begin{cases} x_{n+1} = G_n^{'1}(Rx_n, Sx_n, Ty_n), \\ y_n = G_n^{'2}(x_n, Tx_n). \ n \geq 0. \end{cases} \tag{17}$$

When V_1 is a nonempty convex subset of a real Banach space V_2, R, S and T are nonlinear mappings on V_1 and

$$G_n^{'1}(Rx_n, Sx_n, Ty_n) = \alpha_n^{(1)} Rx_n + \alpha_n^{(2)} Sx_n + \alpha_n^{(3)} Ty_n,$$

$$G_n^{'2}(x_n, Tx_n) = (1 - \beta_n^{(3)})x_n + \beta_n^{(3)} Tx_n, \ \forall n \geq 0,$$

where $\{\alpha_n^{(i)}\}_{n=0}^{\infty}$, $i = 1, 2, 3$ and $\{\beta_n^{(3)}\}_{n=0}^{\infty}$ are real sequences in $[0, 1)$ and $\{x_n\}_{n=0}^{\infty} \subset V_1$, for $n \geq 0$, then we get the modified iteration process (1.6).

The following definition is due to Bunlue et al. [15]:

Definition 1.5 Let $(V_2, \|.\|)$ be a real normed space and let $G_n^{'2} : V_2 \times V_2 \to V_2$ be an admissible mapping. Then $\{G_n^{'2}\}_{n=0}^{\infty}$ is called sequentially affine Lipschitzian if there exists a real sequence $\{\beta_n^{(3)}\}_{n=0}^{\infty}$ in $[0, 1]$ such that, for all x_1, x_2, y_1, y_2 in V_2,

$$\|G_n^{'2}(x_1, y_1) - G_n^{'2}(x_2, y_2)\| \leq \|(1 - \beta_n^{(3)})(x_1 - x_2) + \beta_n^{(3)}(y_1 - y_2)\|. \tag{18}$$

If $\beta_n^{(3)} = \mu \in [0, 1]$, $\forall n \geq 0$, then Definition 1.5 reduces to the definition of affine Lipschizity introduced by Berinde et al. [17].

Definition 1.6 Let $(V_2, \|.\|)$ be a real normed space and let $G_n^{'1} : V_2 \times V_2 \times V_2 \to V_2$ be a generalized admissible mapping. We say that $\{G_n^{'1}\}_{n=0}^{\infty}$ is generalized sequentially affine Lipschitzian if there exists a real sequence $\{\alpha_n^{(i)}\}_{n=0}^{\infty}$, $i = 1, 2, 3$ in $[0, 1]$ such that, for all $x_1, x_2, y_1, y_2, z_1, z_2$ in V_2,

$$\|G_n^{'1}(x_1, y_1, z_1) - G_n^{'1}(x_2, y_2, z_2)\| \leq \|\alpha_n^{(1)}(x_1 - x_2) + \alpha_n^{(2)}(y_1 - y_2) + \alpha_n^{(3)}(z_1 - z_2)\|. \tag{19}$$

In this paper, we prove the convergence of the generalized $\mathbb{S}$-algorithm (1.17) to common fixed points of three mappings satisfying the general contractions (1.9) and (1.12) in uniformly convex Banach spaces.

The following lemma will be used to establish our main results:

Lemma 1.1 (see, [18]) Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two sequences in the closed unit ball in a uniformly convex Banach space X and let $\{z_n\}_{n=0}^{\infty} = \{(1 - t_n)u_n + t_n v_n\}_{n=0}^{\infty}$, where $0 < a \leq t_n \leq b < 1$. If $\lim_{n\to\infty} \|z_n\| = 1$, then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

2 Main result

Theorem 2.1. Let X be a uniformly convex Banach space and let C be a nonempty closed convex subset of X. Let R, S and T be three mappings on C such that (1.9), (1.11) and (1.12) hold and $\mathfrak{F}(R) \cap \mathfrak{F}(S) \cap \mathfrak{F}(T) \neq \phi$. Suppose one of the mappings $R(C)$, $S(C)$ and $T(C)$ is relatively compact. Let $G_n^{'1} : C \times C \times C \to C$ be a generalized admissible generalized sequentially affine Lipschitzian mapping with a sequence $\{\alpha_n^{(i)}\}_{n=0}^{\infty}$, $i = 1, 2, 3$ in $[0, 1]$. Let $G_n^{'2} : C \times C \to C$ be an admissible sequentially affine Lipschitzian mapping with a sequence $\{\beta_n^{(3)}\}_{n=0}^{\infty}$ in $[0, 1]$. Assume that the sequences $\{\alpha_n^{(i)}\}_{n=0}^{\infty}$, $i = 1, 2, 3$ and $\{\beta_n^{(3)}\}_{n=0}^{\infty}$ satyisfying the conditions (i) and $\limsup_{n\to\infty} \beta_n^{(3)} < \frac{1}{3}$, respectively. Then, for any x_0 in C, the generalized generalized $G'\mathbb{S}$-algorithm (1.17) converges strongly to some common fixed point of R, S and T.

Proof. Setting $x^* \in \mathfrak{F}(R) \cap \mathfrak{F}(S) \cap \mathfrak{F}(T)$. Then, from the admissiblity of $\{G_n^{'2}\}_{n=0}^{\infty}$, it follows that $G_n^{'2}(x^*, x^*) = x^*$ for $n \geq 0$. Since $\{G_n^{'2}\}_{n=0}^{\infty}$ is sequentially affine Lipschitzian with $\{\beta_n^{(3)}\}_{n=0}^{\infty}$, then

$$\begin{aligned} \|y_n - x^*\| &= \|G_n^{'2}(x_n, Tx_n) - G_n^{'2}(x^*, x^*)\| \\ &\leq \|(1 - \beta_n^{(3)})(x_n - x^*) + \beta_n^{(3)}(Tx_n - x^*)\| \\ &\leq (1 - \beta_n^{(3)})\|x_n - x^*\| + \beta_n^{(3)}\|Tx_n - x^*\|. \end{aligned} \tag{20}$$

From (1.12), we obtain

$$\|Tx_n - x^*\| \leq \varphi(m(x_n, x^*)) = \varphi(\|x_n - x^*\|), \tag{21}$$

where

$$m(x_n, x^*) = \max\{\|x_n - x^*\|, \ \frac{\|x_n - Tx_n\|}{2}, \ \frac{[\|x_n - x^*\| + \|x^* - Tx_n\|]}{2}\} = \|x_n - x^*\|.$$

Since $\varphi(\|x_n - x^*\|) < \|x_n - x^*\|$, then by (2.2), we deduce that

$$\|Tx_n - x^*\| \leq \|x_n - x^*\|. \tag{22}$$

Furthermore, it follows from (1.9) that

$$\|Sx - x^*\| \leq \|x - x^*\|, \tag{23}$$

$$\|Rx - x^*\| \leq \|x - x^*\|. \tag{24}$$

Substituting (2.3) into (2.1), we have

$$\|y_n - x^*\| \leq \|x_n - x^*\|. \tag{25}$$

Since the mapping $\{G_n^{'1}\}_{n=0}^{\infty}$ is generalized admissible and generalized sequentially affine Lipschitzian with $\{\alpha_n^{(i)}\}_{n=0}^{\infty}$, $i = 1, 2, 3$, from (1.17), (2.3)-(2.6), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|G_n^{'1}(Rx_n, Sx_n, Ty_n) - G_n^{'1}(x^*, x^*, x^*)\| \\ &\leq \|\alpha_n^{(1)}(Rx_n - x^*) + \alpha_n^{(2)}(Sx_n - x^*) + \alpha_n^{(3)}(Ty_n - x^*)\| \\ &\leq \alpha_n^{(1)}\|Rx_n - x^*\| + \alpha_n^{(2)}\|Sx_n - x^*\| + \alpha_n^{(3)}\|Ty_n - x^*\| \\ &\leq \alpha_n^{(1)}\|x_n - x^*\| + \alpha_n^{(2)}\|x_n - x^*\| + \alpha_n^{(3)}\|x_n - x^*\|. \end{aligned}$$

This implies that $\|x_{n+1} - x^*\| \leq \|x_n - x^*\|$, $\forall n \geq 0$ and so $\lim_{n\to\infty} \|x_n - x^*\| = l \geq 0$.
If $l = 0$, then $\lim_{n\to\infty} x_n = x^*$. Assume $l > 0$. If we set

$$u_n = \frac{\alpha_n^{(1)}}{1 - \alpha_n^{(3)}} \frac{Rx_n - x^*}{\|x_n - x^*\|} + \frac{\alpha_n^{(2)}}{1 - \alpha_n^{(3)}} \frac{Sx_n - x^*}{\|x_n - x^*\|}, \quad v_n = \frac{Ty_n - x^*}{\|y_n - x^*\|},$$

then it follows from (2.3)-(2.5) that $\|u_n\| \leq 1$, $\|v_n\| \leq 1$, and $\lim_{n\to\infty} \|(1 - \alpha_n^{(3)})u_n + \alpha_n^{(3)}v_n\| = 1$. Applying Lemma 1.1, it results

$$\lim_{n\to\infty} \|\frac{1}{1 - \alpha_n^{(3)}}(\alpha_n^{(1)}Rx_n + \alpha_n^{(2)}Sx_n) - Ty_n\| = 0. \tag{26}$$

If $T(C)$ is relatively compact, then there is a subsequence $\{Ty_{n_j}\}_{j=0}^{\infty}$ of $\{Ty_n\}_{n=0}^{\infty}$ and $\bar{x} \in C$ such that

$$\lim_{j\to\infty} Ty_{n_j} = \bar{x}. \tag{27}$$

Then, by (2.7), we obtain

$$\lim_{j\to\infty} \frac{1}{1-\alpha_{n_j}^{(3)}}(\alpha_{n_j}^{(1)} Rx_{n_j} + \alpha_{n_j}^{(2)} Sx_{n_j}) = \bar{x}. \tag{28}$$

By using (1.11) and (2.7), we can obtain

$$\lim_{n\to\infty} \|Sx_n - Ty_n\| = 0, \tag{29}$$

and

$$\lim_{n\to\infty} \|Rx_n - Ty_n\| = 0. \tag{30}$$

From (2.8), (2.10) and (2.11), it follows that

$$\lim_{j\to\infty} Sx_{n_j} = \bar{x}, \tag{31}$$

and

$$\lim_{j\to\infty} Rx_{n_j} = \bar{x}. \tag{32}$$

Further, since for each n, we have

$$\|Sx_{n+1} - x_{n+1}\| \leq \|x_{n+1} - Rx_n\| + \|Rx_n - Sx_{n+1}\| \leq 2\|x_{n+1} - Rx_n\|,$$

and since $\{G_n^{'1}\}_{n=0}^{\infty}$ is sequentially affine Lipschitzian, then

$$\begin{aligned}
\|Sx_{n+1} - x_{n+1} \quad \| \leq & \quad 2\|G_n^{'1}(Rx_n, Sx_n, Ty_n) - G_n^{'1}(Rx_n, Rx_n, Rx_n)\| \\
\leq & \quad 2[\alpha_n^{(2)}\|Sx_n - Rx_n\| + \alpha_n^{(3)}\|Ty_n - Rx_n\|] \\
\leq & \quad 2b[\|Sx_n - Rx_n\| + \|Ty_n - Rx_n\|].
\end{aligned}$$

Hence, by (1.11) and (2.11), we obtain

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0, \tag{33}$$

and

$$\lim_{j\to\infty} x_{n_j} = \bar{x}. \tag{34}$$

Now, we prove that $\lim_{j\to\infty} Tx_{n_j} = \bar{x}$.
Since

$$\begin{aligned}
\|x_{n_j} - Tx_{n_j}\| & \leq \|x_{n_j} - Ty_{n_j}\| + \|Ty_{n_j} - Tx_{n_j}\| \\
& \leq \varphi(m(x_{n_j}, y_{n_j})) + \|x_{n_j} - Ty_{n_j}\|,
\end{aligned} \tag{35}$$

where

$$m(x_{n_j}, y_{n_j}) = \max\{\|y_{n_j} - x_{n_j}\|,\ \frac{[\|y_{n_j} - Ty_{n_j}\| + \|x_{n_j} - Tx_{n_j}\|]}{2},\ \frac{[\|y_{n_j} - Tx_{n_j}\| + \|x_{n_j} - Ty_{n_j}\|]}{2}\},$$

and

$$\|y_{n_j} - x_{n_j}\| = \|G_{n_j}^{'2}(x_{n_j}, Tx_{n_j}) - G_{n_j}^{'2}(x_{n_j}, x_{n_j})\| \leq \beta_{n_j}^{(3)}\|x_{n_j} - Tx_{n_j}\|,$$

$$\begin{aligned}\|y_{n_j}-Ty_{n_j}\| &\leq \|y_{n_j}-x_{n_j}\|+\|x_{n_j}-Ty_{n_j}\| \\ &= \|G^{'2}_{n_j}(x_{n_j},Tx_{n_j})-G^{'2}_{n_j}(x_{n_j},x_{n_j})\|+\|x_{n_j}-Ty_{n_j}\| \\ &\leq \beta^{(3)}_{n_j}\|x_{n_j}-Tx_{n_j}\|+\|x_{n_j}-Ty_{n_j}\|,\end{aligned}$$

$$\|y_{n_j}-Tx_{n_j}\| \leq \quad \|y_{n_j}-x_{n_j}\|+\|x_{n_j}-Tx_{n_j}\| \leq \quad (1+\beta^{(3)}_{n_j})\|x_{n_j}-Tx_{n_j}\|,$$

so that

$$\begin{aligned}m(x_{n_j},y_{n_j}) &\leq \max\{\beta^{(3)}_{n_j}\|x_{n_j}-Tx_{n_j}\|,\ \frac{[[1+\beta^{(3)}_{n_j}]\|x_{n_j}-Tx_{n_j}\|+\|x_{n_j}-Ty_{n_j}\|]}{2}\} \\ &\leq \frac{[[1+3\beta^{(3)}_{n_j}]\|x_{n_j}-Tx_{n_j}\|+\|x_{n_j}-Ty_{n_j}\|]}{2}.\end{aligned}$$

This, together with (2.16) and $\varphi(t)<t\ \forall\ t>0$ implies that

$$\|x_{n_j}-Tx_{n_j}\| \leq \frac{3}{1-3\beta^{(3)}_{n_j}}\|x_{n_j}-Ty_{n_j}\|,$$

which by (2.8) and $\limsup_{j\to\infty}\beta^{(3)}_{n_j}<\frac{1}{3}$ implies

$$\lim_{j\to\infty}\|x_{n_j}-Tx_{n_j}\|=0, \tag{36}$$

that is

$$\lim_{j\to\infty}Tx_{n_j}=\bar{x}. \tag{37}$$

Now, we prove that $\bar{x}\in\mathfrak{F}(R)\cap\mathfrak{F}(S)\cap\mathfrak{F}(T)$. Indeed

$$\begin{aligned}\|T\bar{x}-\bar{x}\| &\leq \|T\bar{x}-Tx_{n_j}\|+\|Tx_{n_j}-\bar{x}\| \\ &\leq \varphi(m(\bar{x},x_{n_j}))+\|Tx_{n_j}-\bar{x}\|,\end{aligned}$$

where

$$m(\bar{x},x_{n_j})=\max\{\|x_{n_j}-\bar{x}\|,\ \frac{[\|x_{n_j}-Tx_{n_j}\|+\|\bar{x}-T\bar{x}\|]}{2},\frac{[\|x_{n_j}-T\bar{x}\|+\|\bar{x}-Tx_{n_j}\|]}{2}\}.$$

Taking $j\to\infty$ and using the continuity of φ at zero, (2.17), (2.18) and $\varphi(t)<t$, we obtain $\|\bar{x}-T\bar{x}\|=0$, that is $\bar{x}\in\mathfrak{F}(T)$. Using (1.9), we have

$$\begin{aligned}\|S\bar{x}-\bar{x}\| &\leq \|S\bar{x}-Rx_{n_j}\|+\|Rx_{n_j}-\bar{x}\| \\ &\leq \min\{\|S\bar{x}-x_{n_j}\|,\|x_{n_j}-Rx_{n_j}\|\}+\|Rx_{n_j}-\bar{x}\|.\end{aligned}$$

Taking $j\to\infty$ and using (2.12), (2.13) and (2.15), we obtain $\|\bar{x}-S\bar{x}\|=0$, that is $\bar{x}\in\mathfrak{F}(S)$.
Similarly, using (1.9), (2.12) and (2.15), we can prove that $\bar{x}\in\mathfrak{F}(T)$.
Hence, $\bar{x}\in\mathfrak{F}(R)\cap\mathfrak{F}(S)\cap\mathfrak{F}(T)$.
Finally, since the sequence $\{\|x_n-\bar{x}\|\}$ is decreasing and $\lim_{j\to\infty}x_{n_j}=\bar{x}$, it follows that $\lim_{n\to\infty}x_n=\bar{x}$.
In the case when $S(C)$ or $R(C)$ is relatively compact can be proved directly in a similar manner. This ends the proof.
Remark 2.1 Significantly, our approach ameliorates and broadness the key findings of Rashwan and Saddeek [10], Bunlue and Suantai [15]. Further, it extends the results of others such as Huang and Jeng [4], Rhoades [6], Ganguly and Bandyopadhyay [7], Osilike [8] and Tiwary and Debnath [9].

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